Stress & Heat Flux II

Course Codes: MEG 416, SSG 431

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Topics to be Covered

STRESS DECOMPOSITIONS & INVARIANTS

STATES OF STRESS

PIOLA TRANSFORMATION

PRINCIPAL STRESSES & PRINCIPAL PLANES

HEAT FLUX

Invariants of the stress tensor

- As is the case with any second-order tensor, Cauchy Stress Tensor has associated scalar-valued functions called invariants because their values are dependent only on the stress fields alone and independent of coordinate systems chosen to represent them. From equations 1S7.24, we easily see that, for a set of arbitrarily selected linearly independent vectors, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$,

$$I_1(\boldsymbol{\sigma}) = \text{tr}\,\boldsymbol{\sigma} = \sigma_{ii}$$

$$= \frac{[\boldsymbol{\sigma}\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \sigma_{11} + \sigma_{22} + \sigma_{33}$$

Invariants of the stress tensor

$$
\begin{aligned}
I_2(\boldsymbol{\sigma}) &= \operatorname{tr} \boldsymbol{\sigma}^{\mathrm{c}} \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) \\
&= \sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12} + \sigma_{22}\sigma_{33} - \sigma_{32}\sigma_{23} + \sigma_{11}\sigma_{33} - \sigma_{31}\sigma_{13} \\
&= \frac{[\boldsymbol{\sigma}\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \boldsymbol{\sigma}\mathbf{c}] + [\boldsymbol{\sigma}\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \text{ and} \\
I_3(\boldsymbol{\sigma}) &= \det \boldsymbol{\sigma} \\
&= e_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} \\
&= \frac{[\boldsymbol{\sigma}\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}
\end{aligned}
$$

Additive Decompositions: Skew & Symmetric Parts

- Again, as is the case with any second order tensor, the stress tensor admits two additive decompositions. It can be broken into its symmetric and skew parts:

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{1}{2}\left(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}\right) + \frac{1}{2}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{T}}\right) \\ &= \operatorname{sym} \boldsymbol{\sigma} + \operatorname{skw} \boldsymbol{\sigma}\end{aligned}$$

- A more important additive decomposition is the separation into spherical and deviatoric parts. Let

$$s \equiv \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$$

Additive Decompositions: Spherical & Deviatoric Parts

- The spherical and deviatoric parts of $\boldsymbol{\sigma}$ are:
$$\boldsymbol{\sigma} = \text{sph}\,\boldsymbol{\sigma} + \text{dev}\,\boldsymbol{\sigma}$$
- where $\text{sph}\,\boldsymbol{\sigma} = s\mathbf{I}, s = \frac{1}{3}\text{tr}\,\boldsymbol{\sigma}$ and $\text{dev}\,\boldsymbol{\sigma} = \boldsymbol{\sigma} - s\mathbf{I}$. The second principal invariant of the deviatoric stress tensor,
$$I_2(\text{dev}\,\boldsymbol{\sigma})$$
- Plays a crucial role in the design of metallic elements and is usually computed in the various design software as the Von-Mises or Equivalent Stress.
- In a static fluid, shear stresses vanish; and with it, the deviatoric part of the stress. Consequently, for a fluid at rest,
$$\boldsymbol{\sigma} = \text{sph}\,\boldsymbol{\sigma} = \frac{1}{3}\text{tr}(\boldsymbol{\sigma})\mathbf{I} = -p\mathbf{I}$$
- so that the pressure is the negative of the third of trace of the stress tensor,
$$p = -\frac{1}{3}\text{tr}(\boldsymbol{\sigma}).$$

States of Stress

- The law of Cauchy states that the state of stress at any point in a body is completely determined by the Cauchy Stress tensor. We therefore only need to specify the surface of interest once this tensor is known.
- Equation S7.27 provides the stress at any given surface as the operation of the tensor on the unit normal to the surface.
- In this section, we see examples of stress tensors for simple states of stress. Uniaxial, biaxial and triaxial stresses, pure shear as well as hydrostatic stress. Each will be computed by the product of Cauchy stress on the surface unit normal.

Uniaxial Stress

- Given any vector $\mathbf{v}$ and a scalar stress intensity σ, the Cauchy stress tensor field for uniaxial stress in the same direction as $\mathbf{v}$ is given byintensity σ, the Cauchy stress tensor field for uniaxial stress in the same direction as $\mathbf{v}$ is given by:

$$\boldsymbol{\sigma}(\mathbf{x}) = \left(\frac{1}{||\mathbf{v}||}\right)^2 \sigma(\mathbf{v} \otimes \mathbf{v})$$

- Uniaxial stress of intensity σ in direction of the unit vector, $\mathbf{e}_1$, is therefore given by the stress tensor field,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1).$$

- In figure S8.9, a bar is aligned to direction $\mathbf{e}_1$ as shown. Consider a surface with unit outward drawn normal $\mathbf{n}$ at an angle θ as shown. The resultant traction on this surface can be found, using equation S7.27:

$$\begin{aligned} \mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}\mathbf{n} = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{n} \\ &= \sigma\mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{n}) \\ &= (\sigma \cos\theta)\mathbf{e}_1 \end{aligned}$$

Uniaxial Stress

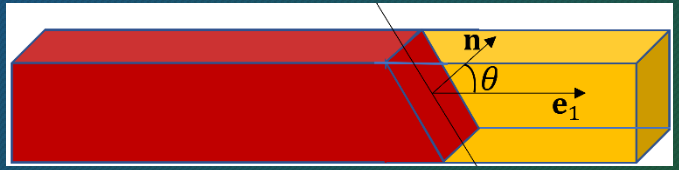
- From the last expression, uniaxial stress tensor creates traction in only one direction no matter the orientation of the plane on which it acts. When the plane is oriented at right angles, $\theta = \frac{\pi}{2}$, to the direction of uniaxial stress, the surface traction is

$$\mathbf{t}^{(\mathbf{n}_{\theta=\pi/2})} = \left(\sigma \cos\frac{\pi}{2}\right)\mathbf{e}_1 = \mathbf{o}.$$

- On a surface with normal in the direction of uniaxial stress, $\theta = 0$, and the surface traction,

$$\mathbf{t}^{(\mathbf{n}_{\theta=0})} = \sigma\mathbf{e}_1.$$

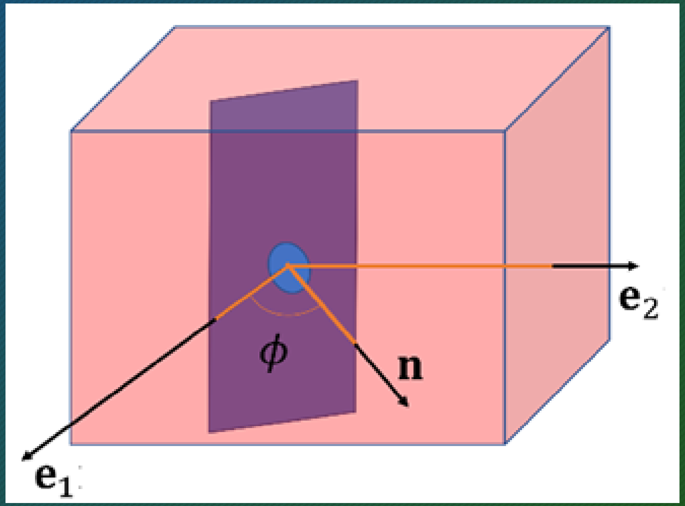
- For all other surfaces, it is a non-zero vector of magnitude less than σ and always in the same direction $\mathbf{e}_1$.



Biaxial Stress

- Given two perpendicular unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, the stress tensor field,
  $\boldsymbol{\sigma}(\mathbf{x}) = \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_1 \otimes \mathbf{e}_2 + \sigma_4 \mathbf{e}_2 \otimes \mathbf{e}_1$
- is a bi-axial stress tensor. The traction on an arbitrary plane oriented with a unit normal, outwardly drawn, $\mathbf{n}$

$$\mathbf{t}^{(n)} = \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}$$
$$= (\sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_1 \otimes \mathbf{e}_2 + \sigma_4 \mathbf{e}_2 \otimes \mathbf{e}_1)\mathbf{n}$$
$$= \sigma_1 \mathbf{e}_1(\mathbf{n} \cdot \mathbf{e}_1) + \sigma_2 \mathbf{e}_2(\mathbf{n} \cdot \mathbf{e}_2) + \sigma_3 \mathbf{e}_1(\mathbf{n} \cdot \mathbf{e}_2) + \sigma_4 \mathbf{e}_2(\mathbf{n}$$
$$= \sigma_1 \mathbf{e}_1 \cos\phi + \sigma_2 \mathbf{e}_2 \sin\phi + \sigma_3 \mathbf{e}_1 \sin\phi + \sigma_4 \mathbf{e}_2 \cos\phi$$

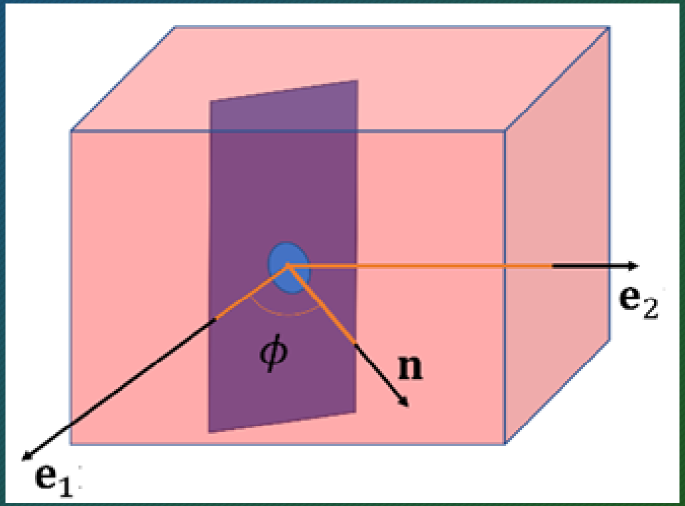


Biaxial Stress

- If the eigenvalues (principal stresses) of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{s_1, s_2, 0\}$. $\boldsymbol{\sigma}(\mathbf{x})$ in this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = s_1 \mathbf{u}_1 \otimes \mathbf{u}_1 + s_2 \mathbf{u}_2 \otimes \mathbf{u}_2$$

- where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the corresponding eigenvectors (principal planes)



Biaxial Stress

- The most general stress tensor in a triaxial field will be in the form,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- By the balance of angular momentum, to be shown in the next chapter, only six of them will be independent. In spectral form (using principal directions), the number of independent terms becomes 3. If $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are also principal directions, then,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3\mathbf{e}_3 \otimes \mathbf{e}_3$$

- is a triaxial stress field. σ_1, σ_2 and σ_3 are, as we shall soon see, the principal stresses or eigenvalues of the stress tensor.

Pure Shear

- The stress tensor field,
$$\boldsymbol{\sigma}(\mathbf{x}) = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$$
- state of pure shear of intensity τ.
- The traction on an arbitrary plane $\mathbf{n}$

$$\begin{aligned}\mathbf{t}^{(n)} &= \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{n} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\mathbf{n} \\ &= \tau\mathbf{e}_1(\mathbf{n}\cdot\mathbf{e}_2) + \tau\mathbf{e}_2(\mathbf{n}\cdot\mathbf{e}_1) \\ &= \tau\mathbf{e}_1 \sin\phi + \tau\mathbf{e}_2 \cos\phi\end{aligned}$$

- At the $\mathbf{e}_1$**plane (that is, perpendicular to** $\mathbf{e}_1$) $\phi = 0$.
$$\mathbf{t}^{(\mathbf{n})} = \tau(\mathbf{e}_1 \sin 0 + \tau\mathbf{e}_2 \cos 0) = \tau\mathbf{e}_2$$
- Similarly, at $\phi = \frac{\pi}{2}$,
$$\mathbf{t}^{(\mathbf{n})} = \tau\left(\mathbf{e}_1 \sin\frac{\pi}{2} + \tau\mathbf{e}_2 \cos\frac{\pi}{2}\right) = \tau\mathbf{e}_1$$
- Showing that the resultant stresses on the coordinate planes are shear of the same intensity as τ.

Pure Shear

- This situation is depicted by the diagram in figure here. When $\phi = \frac{\pi}{4}$,

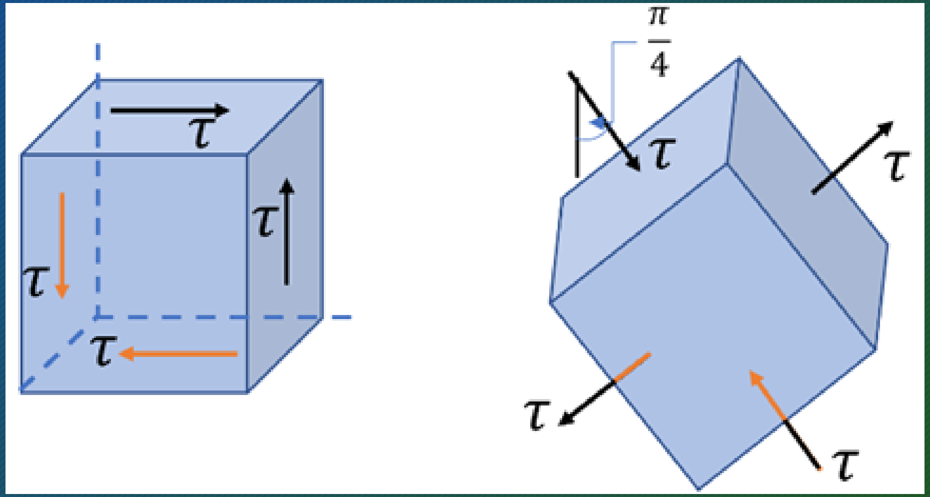
$$\mathbf{t}^{(n)} = \tau\left(\frac{\mathbf{e}_1}{\sqrt{2}} + \frac{\mathbf{e}_2}{\sqrt{2}}\right) = \tau\mathbf{n}$$

- where $\mathbf{n}$ is the surface with normal at 45^o to $\mathbf{e}_1$ axis. When $\phi = \frac{3\pi}{4}$,

$$\mathbf{t}^{(\mathbf{n})} = \tau\left(\frac{\mathbf{e}_1}{\sqrt{2}} - \frac{\mathbf{e}_2}{\sqrt{2}}\right) = -\tau\mathbf{m}$$

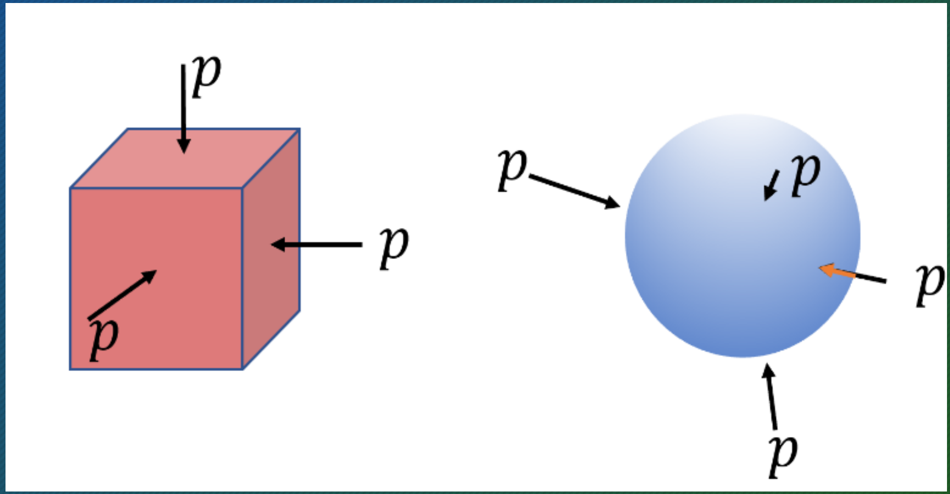
- where $\mathbf{n}$ is the surface normal at 135^o to $\mathbf{e}_1$ axis. The tensile and compressive stresses on these planes are shown in the figure (b). The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{\tau, -\tau, 0\}$. Furthermore, $\boldsymbol{\sigma}$ in this case has the spectral form,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \tau\mathbf{u}_1 \otimes \mathbf{u}_1 - \tau\mathbf{u}_2 \otimes \mathbf{u}_2$$



Hydrostatic Stress

- It is customary to characterize the state of stress called "hydrostatic pressure" in terms of a cuboid with coordinate compressive normal stresses.
  - This way of description only tells part of the story.
  - While it is true that hydrostatic state of stress is usually compressive, it is possible to create a tensile case.
- More important is the fact that hydrostatic pressure is a state of stress that creates the same traction at every surface, no matter what the orientation of the surface is, at that point.



Hydrostatic Stress

- The spherical stress tensor,

$$\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{I}$$

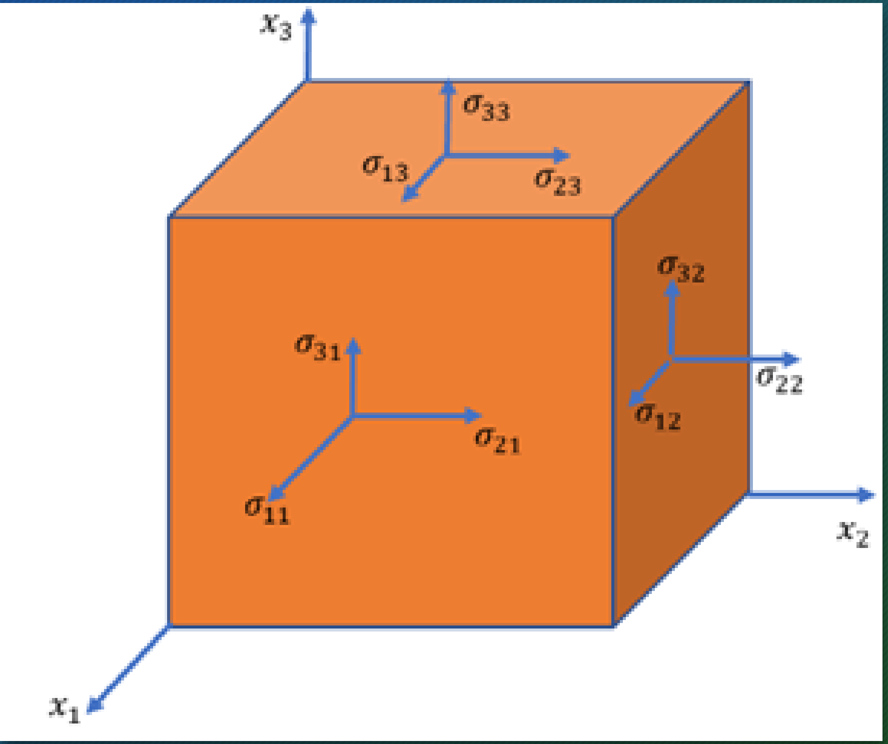
- Is hydrostatic pressure of intensity p. In fact, every spherical stress tensor creates hydrostatic pressure. The traction $\mathbf{t}^{(\mathbf{n})}$ on a surface with unit outward normal $\mathbf{n}$ is

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\mathbf{x})\mathbf{n} \\ &= -p\mathbf{I}\mathbf{n} \\ &= -p\mathbf{n}.\end{aligned}$$

- This always produces a traction, normal to the surface of the same magnitude, no matter the orientation of the surface. For this reason, the stress intensity of a fluid at rest is independent of the orientation of the surface at that point. While it may vary from point to point, it is constant for every surface orientation at a given point.

The Stress Eigenvalue Problem

- Principal stresses are the eigenvalues of the Cauchy Stress Tensor. The surfaces in which they act, are the eigenvectors. We know from Cauchy's stress law that the components of the stress tensor can be depicted as shown in the figure below.
- We also know that by suitable rotations, we are able to transform the components of the stress tensor into other orthonormal systems of coordinates using rotation tensors



The Eigenvalue Problem

- The eigenvalue problem for the stress tensor is simply this:
  Can we find suitable rotations such that the only stress components we have to deal with are the normal stresses?
  - If so, what will those normal stresses be? What will those directions be? Given any tensor $\boldsymbol{\sigma}$ and a vector $\mathbf{n}$, the product $\boldsymbol{\sigma}\mathbf{n}$ is obviously a vector.
  - When can we have the new vector to be such that,

$$\boldsymbol{\sigma}\mathbf{n} = \alpha\mathbf{n}$$

- Or, expressed in component form, $\sigma_{ij}n_j = \alpha n_i$. This will happen only when we can solve the equations,

$$\left|\sigma_{ij} - \alpha\delta_{ij}\right| = 0$$

- Opening the determinant gives us the equation,

$$-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3 = 0$$

- This is the characteristic equation of the stress tensor. A most important equation in mechanical design as we shall see.

Principal Stresses

- The three solutions to this equation are the principal stresses, σ_1, σ_2 and σ_3. The principal directions can be obtained by substituting each back in the equation, $\sigma_{ij} n_j = \alpha n_i$ and solving for the three vector directions in each case.
- $I_1, I_2,$ and I_3 are called the **Principal Invariants** of the stress tensor. They are extremely important scalars in design.

Principal Stresses

- We obtain the stresses and the directions as shown. As usual, we can rotate to this new set of coordinates after the transforming rotation tensor
- Given any stress tensor, you can use Mathematica to obtain the eigenvalues and principal directions as the following examples show.

Plane Rotations

- Let the coordinate system be rotated to point in the direction of the polar coordinates shown so that the normal stresses are now σ_r and σ_θ, while the shear stresses are $\tau_{r\theta}$. The unit vectors are rotated from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ such that,

$$\mathbf{e}_r = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \mathbf{e}_\theta = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 \mathbf{e}_z = \mathbf{e}_3$$

- For this transformation of coordinates, the <u>rotation tensor</u> is, $\mathbf{e}_1 \otimes \mathbf{e}_r + \mathbf{e}_2 \otimes \mathbf{e}_\theta + \mathbf{e}_3 \otimes \mathbf{e}_z$. Note that this is the transpose of the tensor for vector transformation.
- Let the Cartesian unit vectors be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Let σ_x, σ_y and σ_z be the normal stresses and τ_{xy} the only non-vanishing shear stress.

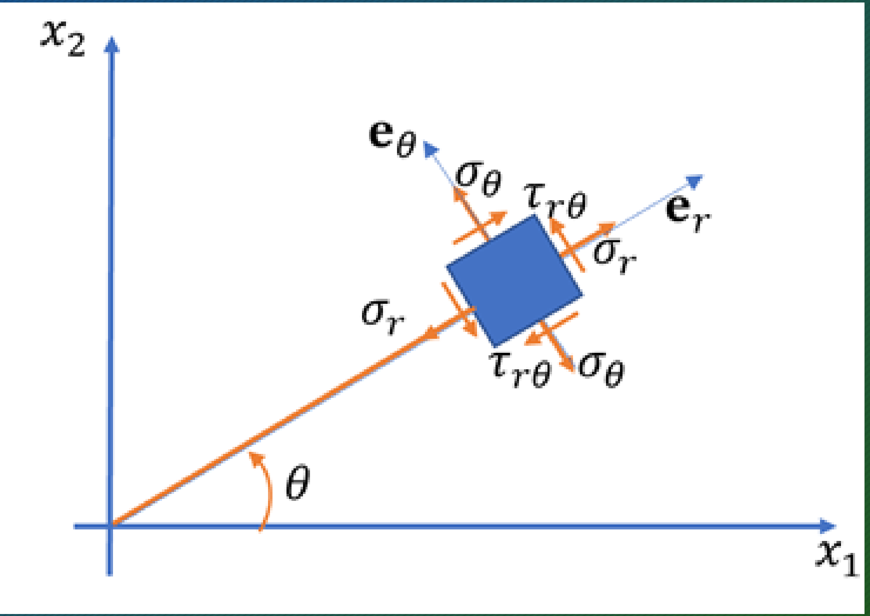
Plane Rotations

- The tensor components after the rotation to plane polar system shown given below is derived from the following Mathematica Notebook:

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



```
CauchyStr = {{σx, τxy, 0}, {τxy, σy, 0}, {0, 0, σz}};
e1 = {1, 0, 0};
e2 = {0, 1, 0};
e3 = {0, 0, 1};
er[α_] := Cos[α] e1 + Sin[α] e2;
eθ[α_] := -Sin[α] e1 + Cos[α] e2;
ez[α_] := e3;
Rot[α_] := TensorProduct[e1, er[α]] + TensorProduct[e2, eθ[α]] + TensorProduct[e3, ez[α]]
Rot[θ]

{{Cos[θ], Sin[θ], 0}, {-Sin[θ], Cos[θ], 0}, {0, 0, 1}}

MatrixForm[%]
```

xForm=

$$\begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] & 0 \\ -\text{Sin}[\theta] & \text{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
RotStr[θ] = Rot[θ].CauchyStr.Transpose[Rot[θ]]

{{Sin[θ] (Sin[θ] σy + Cos[θ] τxy) + Cos[θ] (Cos[θ] σx + Sin[θ] τxy),
  Cos[θ] (Sin[θ] σy + Cos[θ] τxy) - Sin[θ] (Cos[θ] σx + Sin[θ] τxy), 0},
 {Cos[θ] (-Sin[θ] σx + Cos[θ] τxy) + Sin[θ] (Cos[θ] σy - Sin[θ] τxy),
  -Sin[θ] (-Sin[θ] σx + Cos[θ] τxy) + Cos[θ] (Cos[θ] σy - Sin[θ] τxy), 0}, {0, 0, σz}}

MatrixForm[Simplify[%]]
```

xForm=

$$\begin{pmatrix} \cos^2(\theta)\,\sigma_x + \sin(2\,\theta)\,\tau_{xy} + \sin^2(\theta)\,\sigma_y & \frac{1}{2}(-\sin(2\,\theta)\,\sigma_x + 2\cos(2\,\theta)\,\tau_{xy} + \sin(2\,\theta)\,\sigma_y) & 0 \\ \frac{1}{2}(-\sin(2\,\theta)\,\sigma_x + 2\cos(2\,\theta)\,\tau_{xy} + \sin(2\,\theta)\,\sigma_y) & \sin^2(\theta)\,\sigma_x + \cos(\theta)\,(\cos(\theta)\,\sigma_y - 2\sin(\theta)\,\tau_{xy}) & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

Heat Flux

- The rate at which heat flows through a surface is quantified by Fourier-Stokes law of Heat Fluxes. This is the fundamental law in heat flow. Cauchy's postulated the existence of a stress tensor on the basis of which the load intensity arising from mechanical forces (body and surface forces) can be elegantly quantified in a consistent manner.
- The counterpart of this for thermal exchanges with the surroundings is the Fourier-Stokes heat flux theorem. In this section, after stating this law, we shall examine its implications for material (reference) configuration and the transformation of the heat flux from spatial (current-configuration).
- Consider a spatial volume $\mathbf{B}_t$ with boundary $\partial\mathbf{B}_t$. Let the outwardly drawn normal to the surface be the unit vector $\mathbf{n}$. Fourier Stokes heat Flux Principle states that

Heat Flux

- $\exists \mathbf{q}(\mathbf{x}, t)$ – vector field such that, the heat flow out of the volume is
$$h(\mathbf{x}, t, \partial \mathbf{B}_t) = h(\mathbf{x}, t, \mathbf{n}) = -\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}\, \mathbf{q}(\mathbf{x}, t)$$
- is called the heat flux through the surface.
- Heat flow into the spatial volume $\mathbf{B}_t$ volume is
$$\int_{\partial \mathbf{B}_t} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}\, da = \int_{\partial \mathbf{B}} J \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{F}^{-\mathrm{T}} \mathbf{N}\, dA = \int_{\partial \mathbf{B}} J \mathbf{F}^{-1} \mathbf{q}(\mathbf{x}, t) \cdot d\mathbf{A} = \int_{\partial \mathbf{B}} \mathbf{Q} \cdot d\mathbf{A}$$
- $\mathbf{Q}(\mathbf{X}, t)$is a Piola transformation of the spatial heat flux. That is,
$$\mathbf{Q}(\mathbf{X}, t) = J \mathbf{F}^{-1} \mathbf{q}(\mathbf{x}, t)$$
- Hence to obtain the material heat flux from the spatial heat flux, we do a Piola transformation

Piola Transformation

- Recall that, in order to obtain the Piola-Kirchhoff's stress from Cauchy Stress earlier, we also took the latter through the same Piola transformation.
- Begin with Cauchy stress law,

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}^{\mathbf{n}}$$

- we see that the Cauchy stress tensor is a spatial tensor because it transforms a spatial vector (in this case, the normal to a spatial area) to a spatial vector - the traction in spatial configuration. Recall, in the chapter on Kinematics, we saw the following tensors: Deformation gradient, $\mathbf{F}$, its transpose,$\mathbf{F}^{\mathrm{T}}$, inverse $\mathbf{F}^{-1}$,and cofactor, $\mathbf{F}^{\mathrm{c}}$,the left $\mathbf{V}$,and right $\mathbf{U}$, stretch tensors, as well as the various strain tensors. By the same consideration, we have the following table:

Tensor	Typical Transformation	Type
Deformation gradient, $\mathbf{F}$	$d\mathbf{x} = \mathbf{F}d\mathbf{X}$	Material to Spatial
Transpose of Deformation gradient, $\mathbf{F}^{\mathrm{T}}$		Spatial to material
Inverse of Deformation gradient, $\mathbf{F}^{-1}$	$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$	Spatial to Material
Cofactor of Deformation gradient, $\mathbf{F}^{\mathrm{c}}$	$d\mathbf{a} = \mathbf{F}^{\mathrm{c}}d\mathbf{X}$	Material to Spatial
Rotation Tensor, $\mathbf{R}$		Material to Spatial
Transpose of Rotation Tensor, $\mathbf{R}^{\mathrm{T}}$		Spatial to Material
Left Stretch Tensor, $\mathbf{V}$		Spatial
Right Stretch Tensor, $\mathbf{V}$		Material
Rotation Tensor, $\mathbf{R}$		Material to Spatial
Lagrange Strain Tensor, $\mathbf{E}$	$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$	Material
Eulerian Strain Tensor, $\mathbf{e}$	$\mathbf{E} = \frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2})$	Spatial

Tensor Transformations

- The list is not exhaustive. From the above we can see that some tensors transform across the configurations while others transform within the configuration. We therefore have material tensors, spatial tensors and two-toe tensors that transform across configurations. In this latter group, we observe that some transform one way (for example material to spatial) while the other may reverse the transformation as can be seen in the pair of the deformation gradient and its inverse; or the rotation tensor and its transpose.

Piola Transformation

- In this light, we see that, while Cauchy stress is a spatial tensor, the first Piola Kirchhoff stress is a two-toe tensor, transforming from material to spatial. To see this, observe that, given an elemental material area, $d\mathbf{A}$,

$$\begin{aligned}\mathbf{S}d\mathbf{A} &= \boldsymbol{\sigma}\mathbf{F}^{\mathrm{c}}d\mathbf{A}\\ &= \boldsymbol{\sigma}d\mathbf{a} = \boldsymbol{\sigma}\mathbf{n}da\\ &= \mathbf{t}^{\mathbf{n}}da\end{aligned}$$

- the last expression being the spatial traction on the elemental spatial area. The tensor basis of the first Piola Kirchhoff tensor is therefore from Spatial to Material.
- For a spatial tensor $\mathbf{A}$, Piola transformation, $\mathbf{A}\mathbf{F}^{\mathrm{c}}$ creates a two-toe tensor from material to spatial configurations.
- For a material vector, $d\mathbf{X}$, Piola Transformation, $\mathbf{F}^{\mathrm{cT}}d\mathbf{X} = J\mathbf{F}^{-1}d\mathbf{X}$ creates a spatial vector