Tensor Analysis II

Gradient, Divergence and Curl of Vectors

MEG 324 SSG 321 Introduction to Continuum Mechanics

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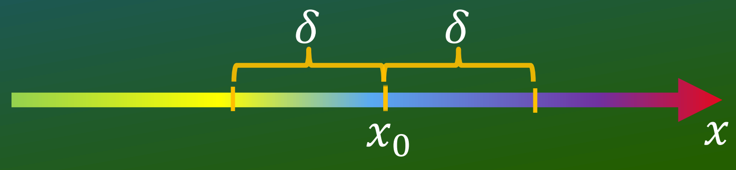
Scope of Today's Lecture

Slides	Topic
3-5	Limits, Continuity & Directional Derivatives: A Review
6-10	Gateaux Differential: A Generalization of the Directional Derivative
11-14	Fréchet Derivative: Grad - Confusion of notation in the Literature
15-23	Illustrative Examples: Practice these to understand concepts
24-36	Differentiation of Fields: Gradient, Divergence & Curl of Vectors
Beyond	Omitted: Gradient, Divergence & Curl of Tensors. Its in the text. Integral Theorems will be done in Practical Class. Plenty tedium; Low rigor. The things to memorize are few. Understand Principles, OK!

Limit & Continuity for Real Scalar Domains

A Review

- Let $x, x_0, w_0 \in \mathbb{R}$ we can say that the limit of a scalar-valued real function,
$$\lim_{x \to x_0} f(x) = w_0$$
- if for any pre-assigned real number $\epsilon > 0$, no matter how small, we can always find a real number $\delta > 0$ such that $|f(x) - w_0| \leq \epsilon$ whenever $|x - x_0| < \delta$. The function is said to be continuous at x_0 if $f(x_0)$ exists and $f(x_0) = w_0$
- In less fanciful words, we are simply saying that we can find a neighborhood where the absolute value of the difference between $f(x)$ and w_0 can be made as small as we choose a-priori.



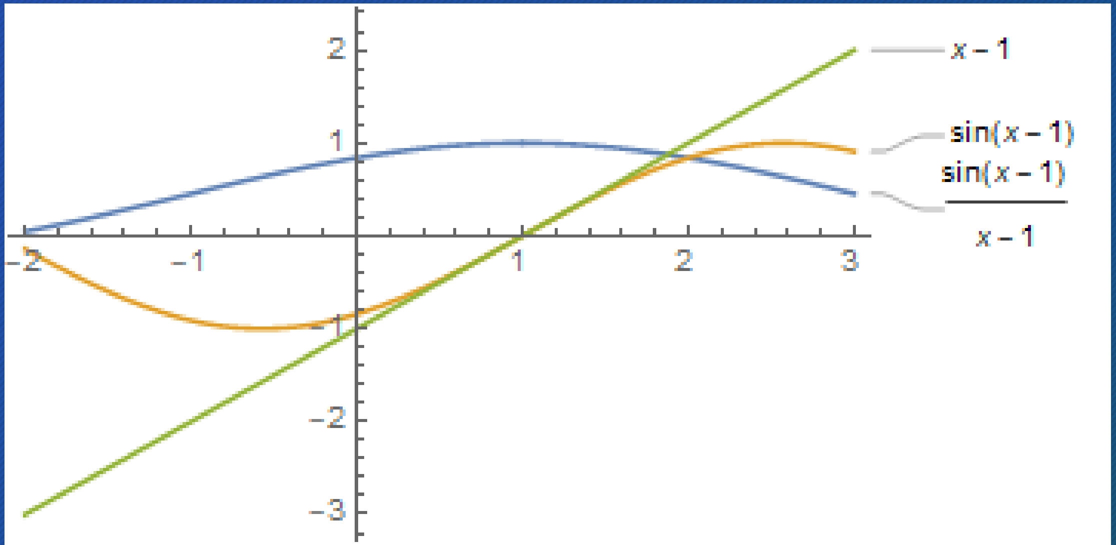
Meaning of Limit & Continuity

What do we mean when we say:

$$\lim_{x \to 1} \frac{\sin(x-1)}{x-1} = 1$$

Mere substitution gives $\frac{\sin(x-1)}{x-1} = \frac{0}{0}$ a NAN

Consider the plot:



x	$\sin x$	$(\delta), x-1$	Quotient	ϵ
0.00000	−0.841470985	−1.0000	0.841470985	0.158529015
0.40000	−0.564642473	−0.6000	0.941070789	0.058929211
0.90000	−0.099833417	−0.1000	0.998334166	0.001665834
0.93000	−0.069942847	−0.0700	0.999183533	0.000816467
0.95000	−0.049979169	−0.0500	0.999583385	0.000416615
0.99000	−0.009999833	−0.0100	0.999983333	0.000020166
0.99900	−0.001000000	−0.0010	0.999999833	0.000000166
0.99990	−0.000100000	−0.0001	0.999999998	0.000000002
1.00010	0.000100000	0.0001	0.999999998	0.000000002
1.00100	0.001000000	0.0010	0.999999833	0.000000166
1.01100	0.010999778	0.0110	0.999979833	0.000020166
1.05000	0.049979169	0.0500	0.999583385	0.000416615

The Directional Derivative

- Consider a scalar function $f(\mathbf{x})$ in the 2-D Euclidean domain shown as a surface. Let $\mathbf{u}$ and $\mathbf{v}$ be two perpendicular directions at a point

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

- Directional derivative is achieved by taking cutting planes (for direction $\mathbf{u}$, plane normal is $\mathbf{v}$, and vice versa) through the point as shown, and computing the regular differential quotient:

$$D_{\mathbf{u}}f = \lim_{\alpha\to 0}\frac{f(\mathbf{x}+\alpha\mathbf{u}) - f(\mathbf{x})}{\alpha}$$

$$D_{\mathbf{v}}f = \lim_{\alpha\to 0}\frac{f(\mathbf{x}+\alpha\mathbf{v}) - f(\mathbf{x})}{\alpha}$$

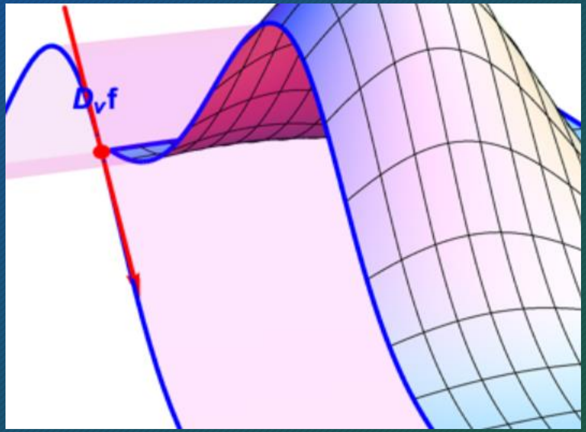
- Gateaux Differential generalizes this simple idea and applies to tensor-valued functions in tensor domains.

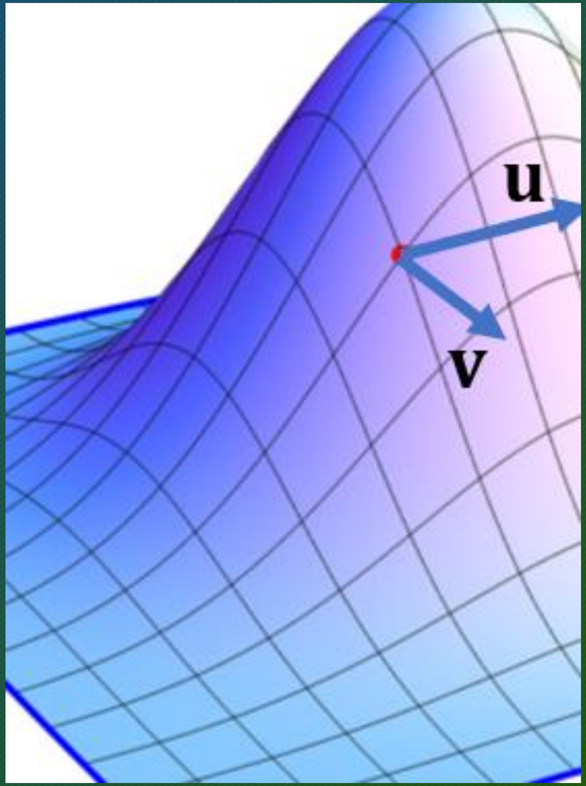
What do these become if you select $\mathbf{u}\to\mathbf{e}_1; \mathbf{v}\to\mathbf{e}_2$? What are

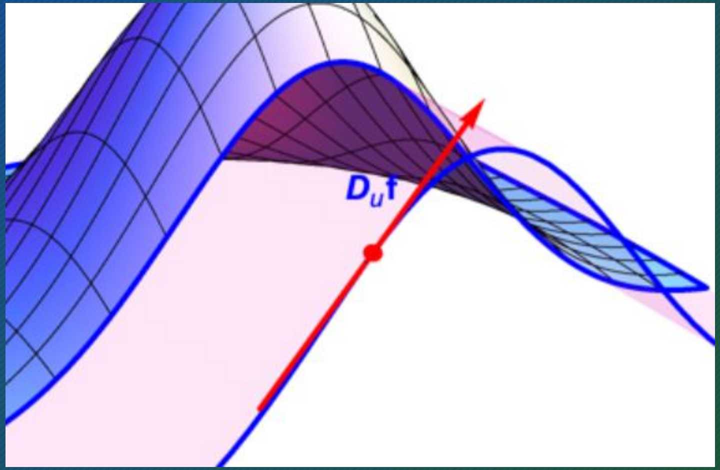
$$\lim_{\alpha\to 0}\frac{f(x_1+\alpha, x_2) - f(x_1, x_2)}{\alpha}$$

$$\lim_{\alpha\to 0}\frac{f(x_1, x_2+\alpha) - f(x_1, x_2)}{\alpha}$$

Is it clear that, in that case, $D_{\mathbf{u}}f = \frac{\partial f}{\partial x_1}$ and $D_{\mathbf{v}}f = \frac{\partial f}{\partial x_2}$







Directional & Partial Derivatives

What does $D_{\mathbf{u}}f$ become if you select $\mathbf{u} \to \mathbf{e}_1$?

$$\mathbf{x} + \alpha\mathbf{u} \ \to (x_1 + \alpha)\mathbf{e}_1 + x_2\mathbf{e}_2$$

so that the neighboring point is selected with x_2 kept fixed:

$$D_{\mathbf{e}_1}f = \lim_{\alpha\to 0}\frac{f(x_1+\alpha, x_2) - f(x_1, x_2)}{\alpha} = \frac{\partial f(\mathbf{x})}{\partial x_1}$$

selecting $\mathbf{v} \to \mathbf{e}_2$, we can similarly argue that,

$$D_{\mathbf{e}_2}f = \lim_{\alpha\to 0}\frac{f(x_1, x_2+\alpha) - f(x_1, x_2)}{\alpha} = \frac{\partial f(\mathbf{x})}{\partial x_2}$$

From which we can see clearly that our partial differentiation is simply the directional derivative taken along the coordinate axis!

Limit & Continuity for Normed Vector Spaces

- Let $\mathbf{v}_0 \in \mathbb{V}$ and $\mathbf{w}_0 \in \mathbb{W}$, as usual we can say that the limit
$$\lim_{\mathbf{v} \to \mathbf{v}_0} \mathbf{F}(\mathbf{v}) = \mathbf{w}_0$$
- if for any pre-assigned real number $\epsilon > 0$, no matter how small, we can always find a real number $\delta > 0$ such that $||\mathbf{F}(\mathbf{v}) - \mathbf{w}_0|| \leq \epsilon$ whenever $||\mathbf{v} - \mathbf{v}_0|| < \delta$. The function is said to be continuous at $\mathbf{v}_0$ if $\mathbf{F}(\mathbf{v}_0)$ exists and $\mathbf{F}(\mathbf{v}_0) = \mathbf{w}_0$
- Again, we are simply saying that we can find a neighborhood where the norm of the difference between $\mathbf{F}(\mathbf{v})$ and $\mathbf{w}_0$ can be made as small as we choose a-priori.

- Specifically, for $\alpha \in \mathbb{R}$, $\mathbf{x}, \mathbf{h}$, $\mathbf{F}$ are Euclidean Vector Spaces, we have:

$$D\mathbf{F}(\mathbf{x}, \mathbf{h}) \equiv \lim_{\alpha \to 0} \frac{\mathbf{F}(\mathbf{x} + \alpha\mathbf{h}) - \mathbf{F}(\mathbf{x})}{\alpha} = \frac{d}{d\alpha} \mathbf{F}(\mathbf{x} + \alpha\mathbf{h})\Big|_{\alpha=0}$$

- We focus attention on the second variable $\mathbf{h}$ while we allow the dependency on $\mathbf{x}$ to be as general as possible. We shall show that while the above function can be any given function of $\mathbf{x}$ (linear or nonlinear), the above map is always linear in $\mathbf{h}$ irrespective of what kind of Euclidean space we are mapping from or into. It is called the ***Gateaux Differential***.

The Gateaux Differential

Survival Strategy:
If you are confused, go to the simple, scalar definition (Slide 3) let the geometrical interpretation lead you to the algebraic statement. Remember that this definition is a simple extension of the algebra!

- Let us make the Gateaux differential a little more familiar: First, we move the function value $F(x)$, and domain variable x, to the real space. Let $h \to dx$ and we obtain,

$$DF(x, dx) = \lim_{\alpha \to 0} \frac{F(x + \alpha dx) - F(x)}{\alpha} = \frac{d}{d\alpha} F(x + \alpha dx)\bigg|_{\alpha=0}$$

- And let $\alpha dx \to \Delta x$, the middle term becomes,

$$\left(\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \right) dx = \boxed{\frac{dF}{dx} dx = dF}$$

> Always, the goal is simple: To compute a meaningful value for the quantity, $\frac{dF}{dx}$, the way dependent variable changes wrt the domain variable, when we have no way of defining a division or quotient, and when we shall always know, from Gateaux, what dF is, and what dx is. If we are successful, what we compute as $\frac{dF}{dx}$ no matter what size the domain or the function value, is what we mean by grad, or Fréchet derivative.

- from which it is obvious that the Gateaux derivative is a generalization of the well-known differential from elementary calculus. The Gateaux differential helps to compute a local linear approximation of any function (linear or nonlinear).

Real functions in Real Domains

9 "Kill all mosquitoes" Ad

Linearity

- Gateaux differential is linear in its second argument, i.e., for $a \in \mathbb{R}$,

$$D\mathbf{F}(\mathbf{x}, a\mathbf{h}) = aD\mathbf{F}(\mathbf{x}, \mathbf{h})$$

How can you demonstrate this?
Write the expression for $D\mathbf{F}(\mathbf{x}, a\mathbf{h})$ and allow $\beta \to a\alpha$. Substitute for α

- Furthermore,

$$\begin{aligned} D\mathbf{F}(\mathbf{x}, \mathbf{g}+\mathbf{h}) &= \lim_{\alpha \to 0} \frac{\mathbf{F}\big(\mathbf{x}+\alpha(\mathbf{g}+\mathbf{h})\big) - \mathbf{F}(\mathbf{x})}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{\mathbf{F}\big(\mathbf{x}+\alpha(\mathbf{g}+\mathbf{h})\big) - \mathbf{F}(\mathbf{x}+\alpha\mathbf{g}) + \mathbf{F}(\mathbf{x}+\alpha\mathbf{g}) - \mathbf{F}(\mathbf{x})}{\alpha} \\ &= \lim_{\alpha \to 0} \frac{\mathbf{F}(\mathbf{y}+\alpha\mathbf{h}) - \mathbf{F}(\mathbf{y})}{\alpha} + \lim_{\alpha \to 0} \frac{\mathbf{F}(\mathbf{x}+\alpha\mathbf{g}) - \mathbf{F}(\mathbf{x})}{\alpha} \\ &= D\mathbf{F}(\mathbf{x}, \mathbf{h}) + D\mathbf{F}(\mathbf{x}, \mathbf{g}) \end{aligned}$$

as the variable $\mathbf{y} \equiv \mathbf{x} + \alpha\mathbf{g} \to \mathbf{x}$ as $\alpha \to 0$; For $a, b \in \mathbb{R}$, using similar arguments, we can also show that,

$$D\mathbf{F}(\mathbf{x}, a\mathbf{g} + b\mathbf{h}) = aD\mathbf{F}(\mathbf{x}, \mathbf{g}) + bD\mathbf{F}(\mathbf{x}, \mathbf{h})$$

How can you demonstrate this?
2 steps: the addition, then the scalar multiply

Points to Note:

- The Gateaux differential is not unique to the point of evaluation.
  - Rather, at each point $\mathbf{x}$ there is a Gateaux differential for each “vector” $\mathbf{h}$. If the domain is a vector space, then we have a Gateaux differential for each of the infinitely many directions at each point. In two of more dimensions, there are infinitely many Gateaux differentials at each point!
  - $\mathbf{h}$ may not even be a vector, but second- or higher-order tensor.
  - It does not matter, as the tensors themselves are in a Euclidean space that define magnitude and direction as a result of the embedded inner product.
- The Gateaux differential is a one-dimensional calculation along a specified direction $\mathbf{h}$. Because it’s one-dimensional, you can use ordinary one-dimensional calculus to compute it. Product rules and other constructs for the differentiation in real domains apply.

Gradient or Fréchet Derivatives

- A scalar, vector or tensor valued function in a scalar, vector or tensor valued domain is said to be Fréchet differentiable if a subdomain exists in which we can find $\text{grad}\,\mathbf{F}(\mathbf{x})$ such that,

$$(\text{grad}\,\mathbf{F}(\mathbf{x})) \bullet \mathbf{h} = D\mathbf{F}(\mathbf{x}, \mathbf{h})$$

- This equation defines the gradient of a function in terms of its operating on the domain object to obtain the Gateaux differential.
- The nature of,

$$\text{grad}\,\mathbf{F}(\mathbf{x})$$

- as well as the kind of product, " $\bullet$ ", between the gradient and the differential depend on the value type of the function and the type of argument involved.
  - In the simple case of a scalar valued function of a scalar argument, we are back to the regular derivative as can be seen in the first row of the table, and the product involved is simply multiplication of two scalars.
  - The Gateaux differential here is your regular differential.

- For a scalar valued function, when the argument type is a vector, we get, for the Fréchet derivative, the familiar gradient operation

$$\text{grad}\ \phi(\mathbf{x})$$

- The Gateaux differential here is the directional derivative,

$$(\text{grad}\ \phi(\mathbf{x}))\cdot d\mathbf{x}$$

in the direction given by the differential, $d\mathbf{x}$. Notice two things here:

- The function value is a scalar; The function differential, $D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad}\ \phi(\mathbf{x}))\cdot d\mathbf{x}$ is also a scalar.
- The product between $\text{grad}\ \phi(\mathbf{x})$ and the vector differential is a scalar product so that the gradient of a scalar valued function of a vector argument is itself a vector. In the table, product of the Fréchet derivative in column 2 with the argument in column 3 is a scalar hence the correct product here is the scalar product.

Gradient, Grad or Fréchet Derivatives

No	grad $\mathbf{F}(\mathbf{x})$	Argument	Product •	$\mathbf{F}(\mathbf{x})$	Gateaux-Fréchet Example
1	Scalar	Scalar	Multiply	Scalar	$DF(x) = \frac{dF(x)}{dx} dx$
2	Vector	Vector	Scalar product	Scalar	$D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad } \phi(\mathbf{x})) \cdot d\mathbf{x}$
3	Tensor	Tensor	Scalar product	Scalar	$Df(\mathbf{T}, d\mathbf{T}) = \frac{df(\mathbf{T})}{d\mathbf{T}} : d\mathbf{T}$
4	Tensor	Vector	Contraction	Vector	$D\boldsymbol{\psi}(\mathbf{x}, d\mathbf{x}) = (\text{grad } \boldsymbol{\psi}(\mathbf{x}))d\mathbf{x}$
5	Tensor	Scalar	Scalar multiply	Tensor	$D\mathbf{T}(x) = \frac{d\mathbf{T}(x)}{dx} dx$
6	Tensor (3)	Vector	Contraction	Tensor	$D\mathbf{F}(\mathbf{x}, d\mathbf{x}) = (\text{grad } \mathbf{F}(\mathbf{x}))d\mathbf{x}$
7	Tensor (4)	Tensor	Contraction	Tensor	$D\mathbf{F}(\mathbf{S}, d\mathbf{S}) = (\text{grad } \mathbf{F}(\mathbf{S}))d\mathbf{S}$

Fréchet Derivatives

- Line 1 is our familiar differentiation result. Line 2 contains our chain rule for a function of several variables. Line 3 gives the analogous result for tensors.
- For a scalar valued function of a tensor argument, row 3, Gateaux differential is a scalar
- The gradient here is a second-order tensor. The proper product to recover the scalar value from the product of these tensors is the tensor scalar product. On rows six and seven, the tensor order for the Fréchet derivative is higher than two and so stated.

Scalar-Valued Function of Tensors

- Some of the most important functions you will differentiate are scalar-valued functions that take tensor arguments. Here are some examples:
  - Principal Invariants of Tensors and related functions. This could include invariants of other tensors such as the deviatoric parts of the original tensor, etc.
  - Traces of powers of the tensor. Traces of products and transposes, etc. These results are powerful because we often able to convert scalar valued functions to the sums and products of traces and their powers.
  - Magnitudes of tensors.

Example:
Direct Application of Gateaux Differential

- Show that $\frac{d}{d\mathbf{S}} \operatorname{tr} \mathbf{S} = \mathbf{I}$, and that $\frac{d}{d\mathbf{S}} \operatorname{tr} \mathbf{S}^2 = \mathbf{S}^{\mathrm{T}}$.
- Compute the Gateaux differential directly here:

$$D f(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha} f(\mathbf{S} + \alpha d\mathbf{S})\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \operatorname{tr}(\mathbf{S} + \alpha d\mathbf{S})\Big|_{\alpha=0} = \operatorname{tr} \frac{d}{d\alpha}(\mathbf{S} + \alpha d\mathbf{S})\Big|_{\alpha=0} = \operatorname{tr}(\mathbf{I}\, d\mathbf{S})$$

$$= \mathbf{I}: d\mathbf{S} = \frac{d f(\mathbf{S})}{d\mathbf{S}} : d\mathbf{S}$$

Example:
Direct Application of Gateaux Differential

$$= \mathbf{I} : d\mathbf{S} = \frac{df(\mathbf{S})}{d\mathbf{S}} : d\mathbf{S}$$

- So that, we have found a function that, multiplies the differential argument to give us the Gateaux differential. That is the Fréchet derivative, or gradient. As you can see here, it is the Identity tensor:

$$\frac{d}{d\mathbf{S}} \mathrm{tr}(\mathbf{S}) = \mathbf{I}.$$

Example: Direct Application of Gateaux Differential

- For $\frac{d}{d\mathbf{S}} \operatorname{tr} \mathbf{S}^2$, the Gateaux differential in this case,

$$Df(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha} f(\mathbf{S} + \alpha d\mathbf{S})\bigg|_{\alpha=0} = \frac{d}{d\alpha} \operatorname{tr}(\mathbf{S} + \alpha d\mathbf{S})^2\bigg|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \operatorname{tr}\{(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\}\bigg|_{\alpha=0}$$

$$= \operatorname{tr}\left[\frac{d}{d\alpha}(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\right]\bigg|_{\alpha=0}$$

$$= \operatorname{tr}[d\mathbf{S}(\mathbf{S} + \alpha d\mathbf{S}) + (\mathbf{S} + \alpha d\mathbf{S})d\mathbf{S}]\big|_{\alpha=0}$$

$$= \operatorname{tr}[d\mathbf{S}\,\mathbf{S} + \mathbf{S}\,d\mathbf{S}] = 2\mathbf{S}^{\mathrm{T}}: d\mathbf{S}$$

$$= \frac{df(\mathbf{S})}{d\mathbf{S}}: d\mathbf{S}$$

So that,

$$\frac{d}{d\mathbf{S}} \operatorname{tr} \mathbf{S}^2 = 2\mathbf{S}^{\mathrm{T}}$$

Is the answer to $\frac{d}{d\mathbf{S}^{\mathrm{T}}} \operatorname{tr} \mathbf{S}^2$ obvious?
Remember that $\operatorname{tr} \mathbf{S}^2 = \mathbf{S}: \mathbf{S}^{\mathrm{T}}$

Example: Differentiate the Second Invariant

- Using these two results and the linearity of the trace operation, we can proceed to find the derivative of the second principal invariant of the tensor $\boldsymbol{S}$:

$$\frac{d}{d\mathbf{S}} I_2(\mathbf{S}) = \frac{1}{2}\frac{d}{d\mathbf{S}}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)]$$
$$= \frac{1}{2}[2\mathrm{tr}(\mathbf{S})\mathbf{I} - 2\,\mathbf{S}^{\mathrm{T}}]$$
$$= \mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}$$

- using the fact that differentiating $\mathrm{tr}^2(\boldsymbol{S})$ with respect to $\mathrm{tr}(\boldsymbol{S})$ is a scalar derivative of a scalar argument.

Example:
Differentiate the Third Invariant

- To find the derivative of the third principal invariant of the tensor $\mathbf{S}$, we appeal to the Cayley-Hamilton theorem, which expresses the determinant in terms of traces only,

$$I_3(\mathbf{S}) = \frac{1}{6}[\mathrm{tr}^3(\mathbf{S}) - 3\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{S}^2) + 2\mathrm{tr}(\mathbf{S}^3)]$$

$$\begin{aligned}\frac{d}{d\mathbf{S}} I_3(\mathbf{S}) &= \frac{1}{6}\frac{d}{d\mathbf{S}}[\mathrm{tr}^3(\mathbf{S}) - 3\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{S}^2) + 2\mathrm{tr}(\mathbf{S}^3)] \\ &= \frac{1}{6}[3\mathrm{tr}^2(\mathbf{S})\mathbf{I} - 3\mathrm{tr}(\mathbf{S}^2)\mathbf{I} - 3\mathrm{tr}(\mathbf{S})2\mathbf{S}^{\mathrm{T}} + 2\times 3(\mathbf{S}^2)^{\mathrm{T}}] \\ &= I_2\mathbf{I} - I_1\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{2\mathrm{T}}.\end{aligned}$$

Is it obvious that $\frac{d}{d\mathbf{S}} I_3(\mathbf{S}) = \mathbf{S}^{\mathrm{c}}$?

Example: Differentiate the trace of a product with a constant tensor

- Given that $\mathbf{A}$ is a constant tensor, show that
$$\frac{d}{d\mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \mathbf{A}^{\mathrm{T}}.$$
- For this scalar-valued function, the Gateaux differential,
$$Df(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha}\mathrm{tr}(\mathbf{AS} + \alpha\mathbf{A}d\mathbf{S})\Big|_{\alpha=0} = \frac{df(\mathbf{S})}{d\mathbf{S}} : d\mathbf{S}$$
$$= \frac{d}{d\alpha}\mathrm{tr}(\mathbf{AS})\Big|_{\alpha=0} + \frac{d}{d\alpha}\alpha\mathrm{tr}(\mathbf{A}d\mathbf{S})\Big|_{\alpha=0}$$
$$= \mathrm{tr}(\mathbf{A}d\mathbf{S}) = \mathbf{A}^{\mathrm{T}} : d\mathbf{S}$$

Giving us,
$$\frac{df(\mathbf{S})}{d\mathbf{S}} = \mathbf{A}^{\mathrm{T}}.$$
A result that should not be surprising since
$$\mathrm{tr}(\mathbf{AS}) = \mathbf{A}^{\mathrm{T}} : \mathbf{S}$$

Compare this result to $\frac{d}{dx}ax$. What will the answer be for $\frac{d}{d\mathbf{S}^{\mathrm{T}}}\mathrm{tr}(\mathbf{AS})$?

Example:
Differentiate the Magnitude of a Tensor

- Finally, we look at the derivative of magnitude. We have found this by a simpler means earlier. It is instructive to look more rigorously, using the Gateaux differential. Given a scalar α variable, the derivative of a scalar function of a tensor $f(\mathbf{A})$ is

$$\frac{df(\mathbf{A})}{d\mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{d}{d\alpha} f(\mathbf{A} + \alpha \mathbf{B})$$

- for any arbitrary tensor $\mathbf{B}$. In the case of $f(\mathbf{A}) = \|\mathbf{A}\|$,

$$\frac{d\|\mathbf{A}\|}{d\mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{d}{d\alpha} \|\mathbf{A} + \alpha \mathbf{B}\|$$

$$\|\mathbf{A} + \alpha \mathbf{B}\| = \sqrt{\mathrm{tr}(\mathbf{A} + \alpha \mathbf{B})(\mathbf{A} + \alpha \mathbf{B})^T} = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}} + \alpha \mathbf{B}\mathbf{A}^{\mathrm{T}} + \alpha \mathbf{A}\mathbf{B}^{\mathrm{T}} + \alpha^2 \mathbf{B}\mathbf{B}^{\mathrm{T}})}$$

- Note that the expression under the root sign here is scalar and that the trace operation is linear.

Example:
Differentiate the Magnitude of a Tensor

- Consequently, we can write,

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}\|\mathbf{A}+\alpha\mathbf{B}\|$$

$$=\lim_{\alpha\to 0}\frac{\operatorname{tr}\left(\mathbf{BA}^{\mathrm{T}}\right)+\operatorname{tr}\left(\mathbf{AB}^{\mathrm{T}}\right)+2\alpha\operatorname{tr}\left(\mathbf{BB}^{\mathrm{T}}\right)}{2\sqrt{\operatorname{tr}(\mathbf{AA}^{\mathrm{T}}+\alpha\mathbf{BA}^{\mathrm{T}}+\alpha\mathbf{AB}^{\mathrm{T}}+\alpha^2\mathbf{BB}^{\mathrm{T}})}}$$

$$=\frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}}=\frac{\mathbf{A}}{\|\mathbf{A}\|}:\mathbf{B}$$

So that,

$$\frac{d\|\mathbf{A}\|}{d\mathbf{A}}:\mathbf{B}=\frac{\mathbf{A}}{\|\mathbf{A}\|}:\mathbf{B}$$

or,

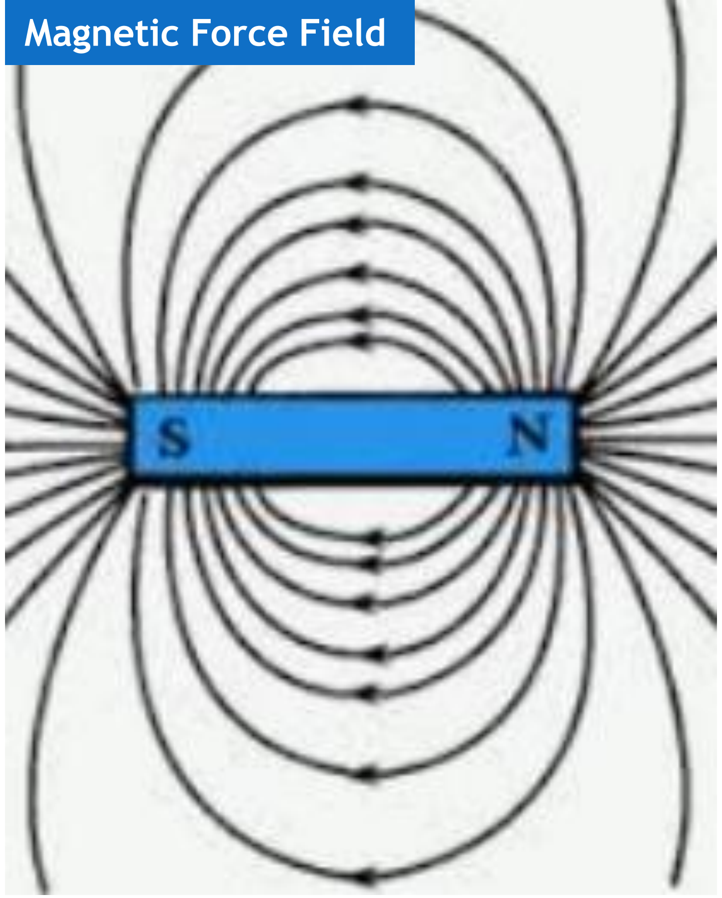
$$\frac{d\|\mathbf{A}\|}{d\mathbf{A}}=\frac{\mathbf{A}}{\|\mathbf{A}\|}$$

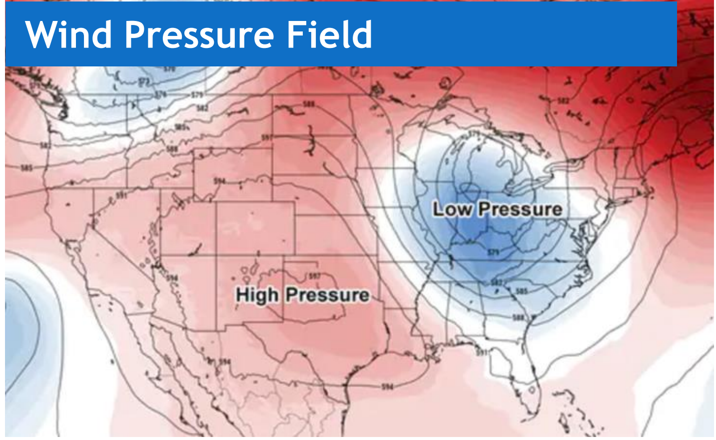
as required since $\mathbf{B}$ is arbitrary.

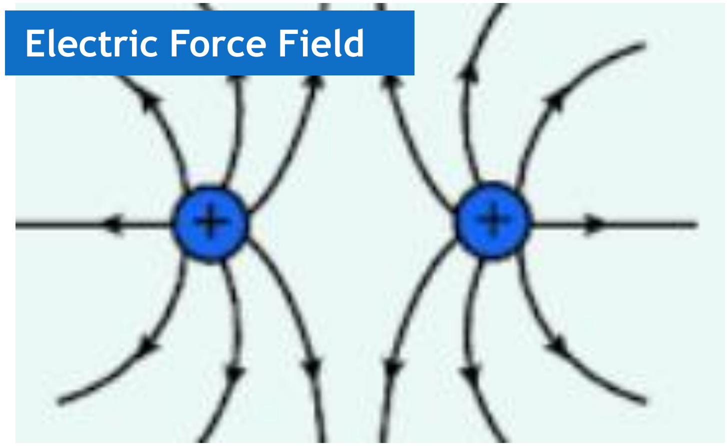
What is a Field?

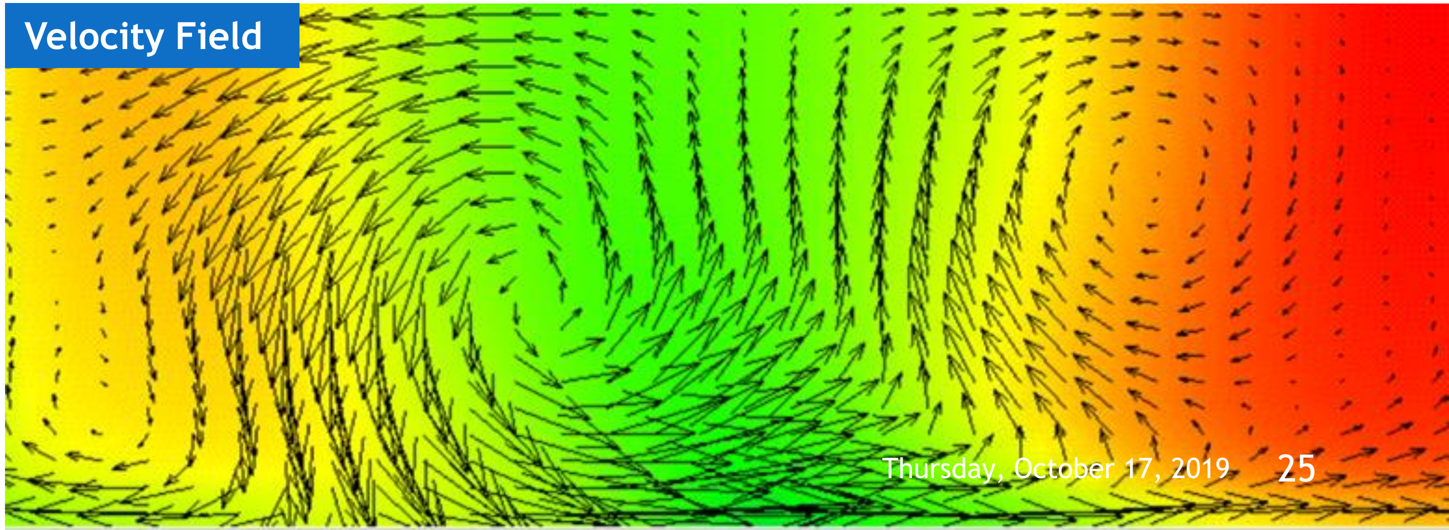
Euclidean Point Space in Concrete Terms

- If at each point of a Euclidean point space, a scalar, a vector or a tensor is defined, then we call such a point space a field of the defined object.
- On this page we have a magnetic force field, a pressure field, a repulsive electric force field because of like poles and a velocity field.
- These are scalar and vector fields. On the next page, you will see stress and strain fields: tensor fields.


Magnetic Force Field

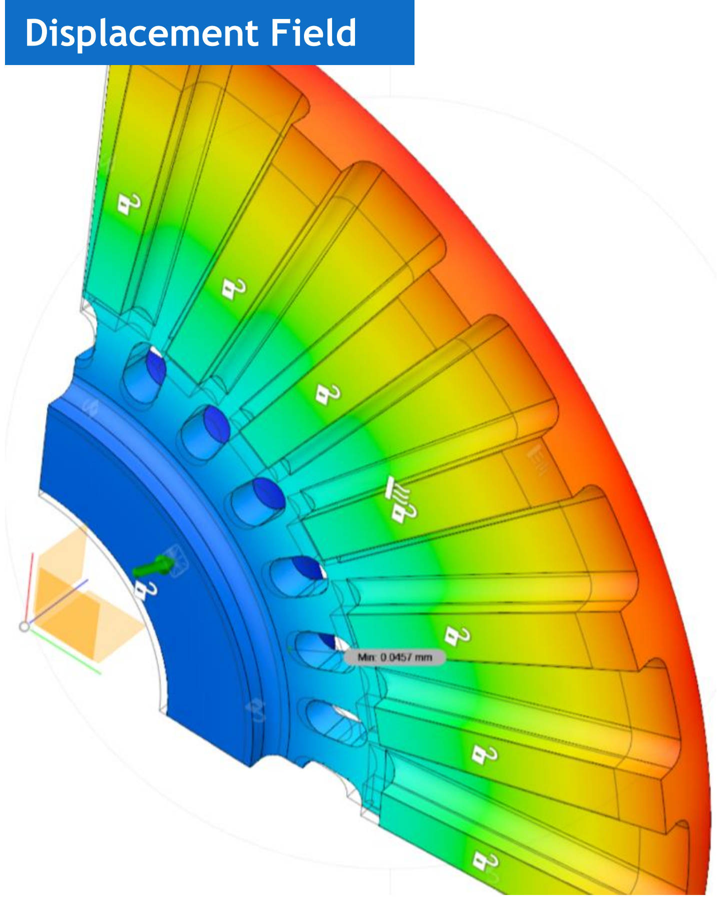

Wind Pressure Field

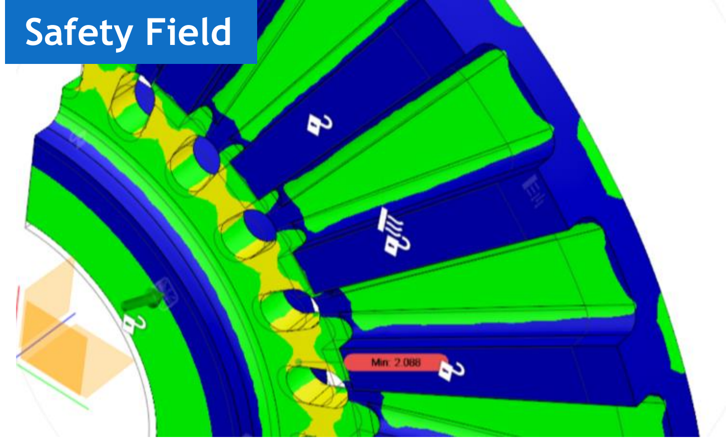

Electric Force Field

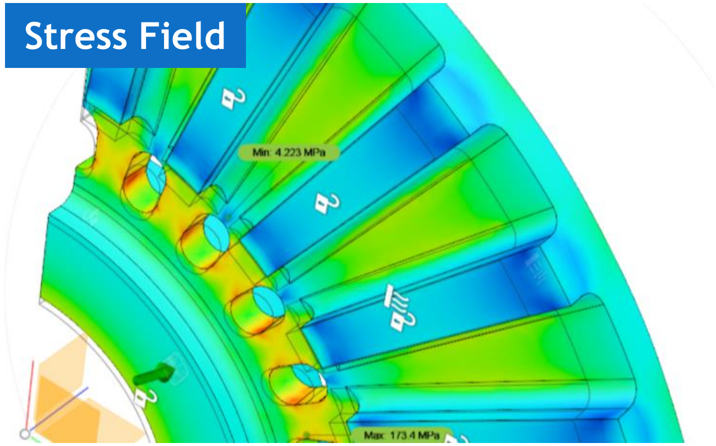

Velocity Field

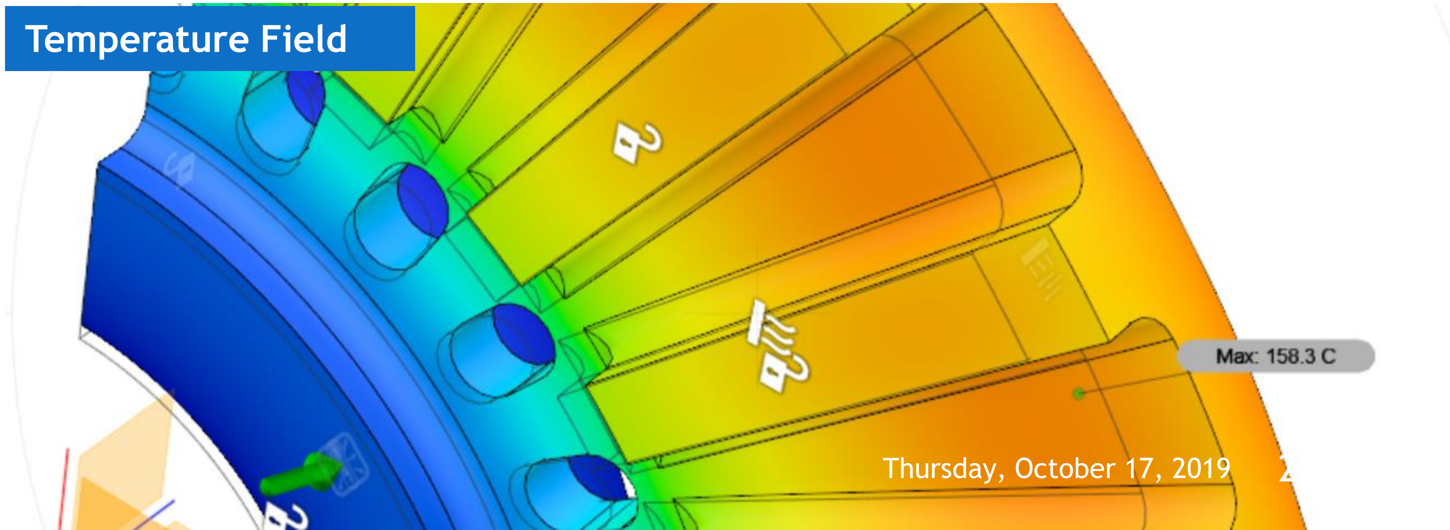
Example:
The Quarter Brake Rotor

- Temperature, Stress, Displacement and Strain fields in the analysis of the optimal design of a brake rotor after solving the governing equations by Finite Element Analysis.
- Because of symmetry, we know that the same result is replicated in the four quarters. This fact is used to reduce the computational load in the analysis.
- This was done in Fusion 360 on a Windows 10 Computer equipped with a graphics processor, using the cloud services of Autodesk.









Fields in Design Optimization

Engineers encounter Fields when optimizing design. The goal is to create objects that will not fail in service and still be economical to manufacture.

Fake product is a synonym of inadequate engineering and production skills.

It is your burden to design, prototype and manufacture real products for Nigeria and Africa.

This set of courses are in place to stimulate your minds and provide you with skills and trade tools to do these!

Components of Gradient in ONB Systems

- We now express the gradients measuring changes in the fields beginning with
$$D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad}\ \phi(\mathbf{x})) \cdot d\mathbf{x}$$
- With a scalar-valued function with vector arguments that are now the position vectors in the Euclidean Point space. For the point $\{x_1, x_2, x_3\}$ consider the neighboring point is at $\{x_1 + dx_1, x_2 + dx_2, x_3 + dx_3\}$

$$\begin{aligned} D\phi(\mathbf{x}, \mathrm{d}\mathbf{x}) &= \lim_{\alpha \to 0} \frac{\phi(\mathbf{x} + \alpha d\mathbf{x}) - \phi(\mathbf{x})}{\alpha} \\ &= \left(\frac{\partial \phi(\mathbf{x})}{\partial x_1} \mathbf{e}_1 + \frac{\partial \phi(\mathbf{x})}{\partial x_2} \mathbf{e}_2 + \frac{\partial \phi(\mathbf{x})}{\partial x_3} \mathbf{e}_3 \right) \cdot d\mathbf{x} \end{aligned}$$

So that, for a scalar-valued function, in an Orthonormal system,

$$\text{grad}\ \phi(\mathbf{x}) = \frac{\partial \phi(\mathbf{x})}{\partial x_1} \mathbf{e}_1 + \frac{\partial \phi(\mathbf{x})}{\partial x_2} \mathbf{e}_2 + \frac{\partial \phi(\mathbf{x})}{\partial x_3} \mathbf{e}_3$$

After a simple proof of the above relationship, we shall generalize this expression to provide the results for more general cases. We use the comma notation to denote partial derivatives and apply it post.

The proof first:

Proof of the Components of the Gradient

- Remember that the Gateaux Differential is linear in its second variable. Consequently, if we write,

$$d\mathbf{x} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3$$

- Linearity leads to:

$$D\phi(\mathbf{x}, d\mathbf{x}) = dx_1 D\phi(\mathbf{x}, \mathbf{e}_1) + dx_2 D\phi(\mathbf{x}, \mathbf{e}_2) + dx_3 D\phi(\mathbf{x}, \mathbf{e}_3)$$
$$= \frac{\partial\phi(\mathbf{x})}{\partial x_1} dx_1 + \frac{\partial\phi(\mathbf{x})}{\partial x_2} dx_2 + \frac{\partial\phi(\mathbf{x})}{\partial x_3} dx_3$$
$$= \left(\frac{\partial\phi(\mathbf{x})}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi(\mathbf{x})}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi(\mathbf{x})}{\partial x_3}\mathbf{e}_3\right) \cdot (dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3)$$
$$= (\mathrm{grad}\,\phi(\mathbf{x})) \cdot d\mathbf{x}$$

$$D\mathbf{F}(\mathbf{x}, \mathbf{h}) \equiv \lim_{\alpha \to 0} \frac{\mathbf{F}(\mathbf{x} + \alpha\mathbf{h}) - \mathbf{F}(\mathbf{x})}{\alpha}$$
$$\Rightarrow D\phi(\mathbf{x}, \mathbf{e}_i) \equiv \lim_{\alpha \to 0} \frac{\phi(\mathbf{x} + \alpha\mathbf{e}_i) - \phi(\mathbf{x})}{\alpha}$$
$$= \frac{\partial\phi(\mathbf{x})}{\partial x_i}$$

Puzzle: Why is $\mathrm{grad}\,\phi(\mathbf{x})$ said to be maximum slope? Examine the result for all unit vectors at the point

Components of Gradient in ONB Systems

$$\text{grad } \phi(\mathbf{x}) = \frac{\partial \phi(\mathbf{x})}{\partial x_i} \mathbf{e}_i = \phi_{,i} \, \mathbf{e}_i$$

- Addition, product and other rules apply to Gradient in the comma notation as follows:

$$\begin{aligned} \text{grad } (\phi\psi) &= (\phi\psi)_{,i} \, \mathbf{e}_i \\ &= (\phi_{,i} \, \psi + \phi\psi_{,i} \,)\mathbf{e}_i \\ &= \psi(\mathbf{x})\text{grad } \phi(\mathbf{x}) \end{aligned}$$

If we define the gradient operator as a post fix operator,

$$\text{grad } (\blacksquare) = (\blacksquare)_{,\alpha} \otimes \mathbf{e}_\alpha$$

Where (as long as the coordinate system of reference is ONB) the comma signifies partial derivative, with respect to the α coordinate. The tensor product applying in all cases except for the scalar function where there is no existing basis vector to take the product with. It therefore stands for ordinary product in this case.

Gradient of a Vector Function

- Consider a vector function, $\mathbf{v}(x)$, defined in a Euclidean space spanned by ONB. This can be written in terms of its components,

$$\mathbf{v}(\mathbf{x}) = v_i(\mathbf{x})\mathbf{e}_i$$

- Then,

$$\begin{aligned}\text{grad}\,\mathbf{v}(\mathbf{x}) &= (v_i(x)\mathbf{e}_i)_{,\alpha} \otimes \mathbf{e}_\alpha \\ &= v_{i,\alpha}(x)\mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}\end{aligned}$$

Gradient of a Tensor Function

- For a tensor field $\mathbf{T}(\mathbf{x})$, the gradient can be obtained in the same way:

$$\begin{aligned} \text{grad } \mathbf{T}(\mathbf{x}) &= \left(T_{ij}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j\right)_{,\alpha} \otimes \mathbf{e}_\alpha \\ &= T_{ij,\alpha}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_\alpha \end{aligned}$$

- which is a third-order tensor containing 27 terms. Each set of nine terms can be written out as we have done for the tensor gradient of a vector-valued field above.
- Now, a temperature field is a scalar field. The forgoing shows that the gradient of such a field is a vector field on its own. A velocity field is a vector field. Gradient of such a field is a second-order tensor. This is a well-known field called Velocity Gradient.

The Divergence

- Gradients of objects larger than scalar are at least second-order tensors. Such derived fields can be contracted in the following way by taking the trace of the last two bases (when they are more than two.) Such a contraction is called the divergence, not of the derived field, but of the original field.
- Temperature and other scalar fields cannot have divergence because their gradients are vectors and therefore cannot be contracted (you cannot take the trace). The gradient of a vector field such as the Velocity Gradient, can only be contracted in one way (has only one possible trace). Gradients of larger objects such as tensor fields can be contracted (traced) in more than one way: In that case, the disambiguation rule for contraction to obtain a divergence is to contract with the basis that came from the derivative. For example,

The Divergence

For a vector field $\mathbf{v}(\mathbf{x})$, the gradient,

$$\text{grad}\,\mathbf{v}(\mathbf{x}) = v_{i,\alpha}(x)\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

The divergence of the same field is the trace,

$$\begin{aligned}\text{div}\,\mathbf{v}(\mathbf{x}) &= \text{tr grad}\,\mathbf{v}(\mathbf{x}) \\ &= v_{i,\alpha}(x)\mathbf{e}_i \cdot \mathbf{e}_\alpha = v_{i,\alpha}(x)\delta_{i\alpha} \\ &= v_{i,i}(x) = \frac{\partial v_i(\mathbf{x})}{\partial x_i} \\ &= \frac{\partial v_1(\mathbf{x})}{\partial x_1} + \frac{\partial v_2(\mathbf{x})}{\partial x_2} + \frac{\partial v_3(\mathbf{x})}{\partial x_3}\end{aligned}$$

Curl of a Vector Field

- **The Levi-Civita Tensor.** The third-order alternating tensor, $\mathcal{E} \equiv e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ was introduced in the last chapter. Compositions of this tensor with vectors and other tensors yield some useful constructs in Continuum Mechanics. We have already seen its action resulting in the axial vector for a skew tensor. We are about see that the well-known curl of vectors and tensors can be neatly defined by the divergence of products with this tensor.
- **Curl of a vector**. Given any vector $\mathbf{u}(x) = u_\alpha(x)\mathbf{e}_\alpha$, the second-order tensor, the composition $\mathcal{E}\mathbf{u}$ is skew and is the transpose of the vector cross of $\mathbf{u}$.

Curl of a Vector Field

- $$\begin{aligned}\mathcal{E}\mathbf{u} &= e_{ijk}u_\alpha(x)\left(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\right)\mathbf{e}_\alpha \\ &= e_{ijk}u_\alpha(x)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_k \cdot \mathbf{e}_\alpha \\ &= e_{ijk}u_\alpha(x)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\delta_{k\alpha} = e_{ijk}u_k(x)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\end{aligned}$$
- Gradient of $\mathcal{E}\mathbf{u}$ is,

$$\operatorname{grad}(\mathcal{E}\mathbf{u}) = e_{ijk}u_{k,\alpha}(x)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_\alpha$$

Curl of a Vector Field

- And the trace gives us the divergence,

$$\text{curl}\,\mathbf{u} \equiv \text{div}(\boldsymbol{\mathcal{E}}\mathbf{u}) = \text{tr grad}(\boldsymbol{\mathcal{E}}\mathbf{u})$$

$$= e_{ijk} u_{k,\alpha}\, \mathbf{e}_i \left(\mathbf{e}_j \cdot \mathbf{e}_\alpha\right)$$

$$= e_{ijk} u_{k,j}\, \mathbf{e}_i$$

$$= \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{pmatrix}$$

which is the curl of the vector field $\mathbf{u}(x)$. The curl of a vector has zero divergence:

$$\text{grad curl}\,\mathbf{u} = e_{ijk} u_{k,jl}\, \mathbf{e}_i \otimes \mathbf{e}_l$$

The trace of this expression,

$$\text{div curl}\,\mathbf{u} = \text{tr grad curl}\,\mathbf{u}$$

$$= e_{ijk} u_{k,jl}\, \mathbf{e}_i \cdot \mathbf{e}_l$$

$$= e_{ijk} u_{k,jl}\, \delta_{il}$$

$$= 0$$