Tutorial Three

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Given that $\alpha(t) \in \mathbb{R}, \mathbf{u}(t) \in \mathbb{E}$ are both functions of a scalar variable t, show that $\frac{d}{dt}(\alpha\mathbf{u}) = \alpha\frac{d\mathbf{u}}{dt} + \frac{d\alpha}{dt}\mathbf{u}$

- We find,

$$\frac{d}{dt}(\alpha\mathbf{u}) = \lim_{h\to 0}\frac{\alpha(t+h)\mathbf{u}(t+h) - \alpha(t)\mathbf{u}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\alpha(t+h)\mathbf{u}(t+h) - \alpha(t)\mathbf{u}(t+h) + \alpha(t)\mathbf{u}(t+h) - \alpha(t)\mathbf{u}(t)}{h}$$

$$= \lim_{h\to 0}\frac{\alpha(t+h)\mathbf{u}(t+h) - \alpha(t)\mathbf{u}(t+h)}{h} + \lim_{h\to 0}\frac{\alpha(t)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t)}{h}$$

$$= \left(\lim_{h\to 0}\frac{\alpha(t+h) - \alpha(t)}{h}\right)\left(\lim_{h\to 0}\mathbf{u}(t+h)\right) + \alpha(t)\lim_{h\to 0}\frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{h}$$

$$= \frac{d}{dt}(\alpha\mathbf{T}) = \alpha\frac{d\mathbf{u}}{dt} + \frac{d\alpha}{dt}\mathbf{u}$$

Given that $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$ are both functions of a scalar variable t, show that $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}$

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\cdot\mathbf{v}(t+h) - \mathbf{u}(t)\cdot\mathbf{v}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\cdot\mathbf{v}(t+h) - \mathbf{u}(t)\cdot\mathbf{v}(t+h) + \mathbf{u}(t)\cdot\mathbf{v}(t+h) - \mathbf{u}(t)\cdot\mathbf{v}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\cdot\mathbf{v}(t+h) - \mathbf{u}(t)\cdot\mathbf{v}(t+h)}{h} + \lim_{h\to 0}\frac{\mathbf{u}(t)\cdot\mathbf{v}(t+h) - \mathbf{u}(t)\cdot\mathbf{v}(t)}{h} \\
&= \left(\lim_{h\to 0}\frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h}\right)\cdot\left(\lim_{h\to 0}\mathbf{v}(t+h)\right) + \mathbf{u}(t)\cdot\lim_{h\to 0}\frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} \\
&= \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}.
\end{aligned}$$

Given that $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$, are both functions of a scalar variable t, show that $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$

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$$\begin{aligned}
\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \lim_{h \to 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h} \\
&= \lim_{h \to 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t+h) + \mathbf{u}(t) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h} \\
&= \lim_{h \to 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t+h)}{h} + \lim_{h \to 0} \frac{\mathbf{u}(t) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h} \\
&= \left(\lim_{h \to 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} \right) \times \left(\lim_{h \to 0} \mathbf{v}(t+h) \right) + \mathbf{u}(t) \times \lim_{h \to 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h} \\
&= \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}.
\end{aligned}$$

Given that $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$ are both functions of a scalar variable t, show that $\frac{d}{dt}(\mathbf{u} \otimes \mathbf{v}) = \frac{d\mathbf{u}}{dt} \otimes \mathbf{v} + \mathbf{u} \otimes \frac{d\mathbf{v}}{dt}$

$$\frac{d}{dt}(\mathbf{u} \otimes \mathbf{v}) = \lim_{h\to 0} \frac{\mathbf{u}(t+h) \otimes \mathbf{v}(t+h) - \mathbf{u}(t) \otimes \mathbf{v}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\mathbf{u}(t+h) \otimes \mathbf{v}(t+h) - \mathbf{u}(t) \otimes \mathbf{v}(t+h) + \mathbf{u}(t) \otimes \mathbf{v}(t+h) - \mathbf{u}(t) \otimes \mathbf{v}(t)}{h}$$

$$= \lim_{h\to 0} \frac{\mathbf{u}(t+h) \otimes \mathbf{v}(t+h) - \mathbf{u}(t) \otimes \mathbf{v}(t+h)}{h} + \lim_{h\to 0} \frac{\mathbf{u}(t) \otimes \mathbf{v}(t+h) - \mathbf{u}(t) \otimes \mathbf{v}(t)}{t}$$

$$= \left(\lim_{h\to 0} \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h}\right) \otimes \left(\lim_{h\to 0} \mathbf{v}(t+h)\right) + \mathbf{u}(t) \otimes \lim_{h\to 0} \frac{\mathbf{v}(t+h) - \mathbf{v}(t)}{h}$$

$$= \frac{d\mathbf{u}}{dt} \otimes \mathbf{v} + \mathbf{u} \otimes \frac{d\mathbf{v}}{dt}.$$

Given that and $\mathbf{S}(t), \mathbf{T}(t) \in \mathbb{L}$ are both functions of a scalar variable t, show that $\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}$

$$\begin{aligned}
\frac{d}{dt}(\mathbf{T}+\mathbf{S}) &= \lim_{h\to 0} \frac{\mathbf{T}(t+h)+\mathbf{S}(t+h)-\big(\mathbf{T}(t)+\mathbf{S}(t)\big)}{h} \\
&= \lim_{h\to 0} \frac{\mathbf{T}(t+h)-\mathbf{T}(t)+\mathbf{S}(t+h)-\mathbf{S}(t)}{h} \\
&= \lim_{h\to 0} \frac{\mathbf{T}(t+h)-\mathbf{T}(t)}{h} + \lim_{h\to 0} \frac{\mathbf{S}(t+h)-\mathbf{S}(t)}{h} \\
&= \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}.
\end{aligned}$$

Given that $\mathbf{S}(t), \mathbf{T}(t) \in \mathbb{L}$, are both functions of a scalar variable t, show that $\frac{d}{dt}\mathbf{ST} = \frac{d\mathbf{S}}{dt}\mathbf{T} + \mathbf{S}\frac{d\mathbf{T}}{dt}$

$$\begin{aligned}
\frac{d}{dt}(\mathbf{ST}) &= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{T}(t+h) - \mathbf{S}(t)\mathbf{T}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{T}(t+h) - \mathbf{S}(t)\mathbf{T}(t+h) + \mathbf{S}(t)\mathbf{T}(t+h) - \mathbf{S}(t)\mathbf{T}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{T}(t+h) - \mathbf{S}(t)\mathbf{T}(t+h)}{h} \\
&= \left(\lim_{h\to 0}\frac{\mathbf{S}(t+h) - \mathbf{S}(t)}{h}\right)\left(\lim_{h\to 0}\mathbf{T}(t+h)\right) + \mathbf{S}(t)\lim_{h\to 0}\frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{h} \\
&= \frac{d}{dt}(\mathbf{ST}) = \frac{d\mathbf{S}}{dt}\mathbf{T} + \mathbf{S}\frac{d\mathbf{T}}{dt}.
\end{aligned}$$

Given that $\mathbf{u}(t) \in \mathbb{E}$ and $\mathbf{S}(t) \in \mathbb{L}$, are both functions of a scalar variable t, show that $\frac{d}{dt}(\mathbf{Su}) = \frac{d\mathbf{S}}{dt}\mathbf{u} + \mathbf{S}\frac{d\mathbf{u}}{dt}$

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$$\begin{aligned}
\frac{d}{dt}(\mathbf{Su}) &= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{u}(t+h) - \mathbf{S}(t)\mathbf{u}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{u}(t+h) - \mathbf{S}(t)\mathbf{u}(t+h) + \mathbf{S}(t)\mathbf{u}(t+h) - \mathbf{S}(t)\mathbf{u}(t)}{h} \\
&= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{u}(t+h) - \mathbf{S}(t)\mathbf{u}(t+h)}{h} + \lim_{h\to 0}\frac{\mathbf{S}(t)\mathbf{u}(t+h) - \mathbf{S}(t)\mathbf{u}(t)}{h} \\
&= \left(\lim_{h\to 0}\frac{\mathbf{S}(t+h) - \mathbf{S}(t)}{h}\right)\left(\lim_{h\to 0}\mathbf{u}(t+h)\right) + \mathbf{S}(t)\lim_{h\to 0}\frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} \\
&= \frac{d}{dt}(\mathbf{Su}) = \frac{d\mathbf{S}}{dt}\mathbf{u} + \mathbf{S}\frac{d\mathbf{u}}{dt}.
\end{aligned}$$

Given that $\mathbf{S}(t) \in \mathbb{L}$, is a function of a scalar variable t, show that $\frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \left(\frac{d\mathbf{S}}{dt}\right)^{\mathrm{T}}$

$$\frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \lim_{h\to 0}\frac{\mathbf{S}^{\mathrm{T}}(t+h) - \mathbf{S}^{\mathrm{T}}(t)}{h}$$
$$= \lim_{h\to 0}\frac{\left(\mathbf{S}(t+h) - \mathbf{S}(t)\right)^{\mathrm{T}}}{h}$$

Because the transpose of a sum is the sum of the transposes (proved S8.17).

$$\frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \lim_{h\to 0}\left(\frac{\mathbf{S}(t+h) - \mathbf{S}(t)}{h}\right)^{\mathrm{T}}$$

In the equation $\mathbf{a}\cdot\mathbf{Sb} = \mathbf{b}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{a}$, multiply both sides by $\alpha \in \mathbb{R}$, and we have,

$$\alpha\mathbf{a}\cdot\mathbf{Sb} = \mathbf{a}\cdot(\alpha\mathbf{S})\mathbf{b} = \mathbf{b}\cdot(\alpha\mathbf{S})^{\mathrm{T}}\mathbf{a} = \mathbf{b}\cdot\alpha\mathbf{S}^{\mathrm{T}}\mathbf{a}$$

As the transpose of a scalar multiple equals the scalar multiple of the transpose

As transpose of a scalar multiple equals the scalar multiple of the transpose. The scalar multiplier here is $\frac{1}{h}$. Consequently,

$$\frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \left(\frac{d\mathbf{S}}{dt}\right)^{\mathrm{T}}.$$

Given that $\mathbf{Q}(t)\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{I}$, the identity tensor, show (a) that $\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}$ is an antisymmetric tensor, and (b) that $\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}$ is an antisymmetric tensor

Differentiating $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$,

$$\frac{d}{dt}\left(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\right) = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = \frac{d\mathbf{I}}{dt} = \mathbf{O}$$

Consequently,

$$\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = -\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}$$

So we have that the tensor $\mathbf{T} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}$ is negative of its own transpose, hence it is skew.

$$\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$$

Differentiating as before,

$$\frac{d}{dt}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) = \frac{d\mathbf{Q}^{\mathrm{T}}}{dt}\mathbf{Q} + \mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt} = \mathbf{O}$$

So that,

$$\frac{d\mathbf{Q}^{\mathrm{T}}}{dt}\mathbf{Q} = \left(\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}} = -\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}.$$

Tensor $\mathbf{T} = \mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}$ is negative of its own transpose, hence skew as we are required to show.

Show that $\frac{d}{d\mathbf{S}}\,\mathrm{tr}\left(\mathbf{S}^k\right) = k\left(\mathbf{S}^{k-1}\right)^{\mathrm{T}}$

- The cases, $k = 1, k = 2$ are already provided in the text. When $k = 3$,

$$Df(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha} f(\mathbf{S} + \alpha d\mathbf{S})\bigg|_{\alpha=0} = \frac{d}{d\alpha}\mathrm{tr}\{(\mathbf{S} + \alpha d\mathbf{S})^3\}\bigg|_{\alpha=0}$$

$$= \frac{d}{d\alpha}\mathrm{tr}\{(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\}\bigg|_{\alpha=0}$$

$$= \mathrm{tr}\left[\frac{d}{d\alpha}(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\right]\bigg|_{\alpha=0}$$

$$= \mathrm{tr}[d\mathbf{S}(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S}) + (\mathbf{S} + \alpha d\mathbf{S})d\mathbf{S}(\mathbf{S} + \alpha d\mathbf{S}) + (\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})d\mathbf{S}]\bigg|_{\alpha=0}$$

$$= \mathrm{tr}[d\mathbf{S}\,\mathbf{S}\,\mathbf{S} + \mathbf{S}\,d\mathbf{S}\,\mathbf{S} + \mathbf{S}\,\mathbf{S}\,d\mathbf{S}] = 3(\mathbf{S}^2)^{\mathrm{T}}: d\mathbf{S}$$

- It easily follows by induction that, $\frac{d}{d\mathbf{S}} f(\mathbf{S}) = k\left(\mathbf{S}^{k-1}\right)^{\mathrm{T}}$.

Define the magnitude of tensor $\mathbf{A}$ as, $\|\mathbf{A}\| = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}})}$ Show that $\frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{\|\mathbf{A}\|}$

- Given a scalar α variable, the derivative of a scalar function of a tensor $f(A)$ is $\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}}:\mathbf{B} = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha} f(\mathbf{A}+\alpha\mathbf{B})$ for any arbitrary tensor $\mathbf{B}$.
- In the case of $f(\mathbf{A}) = \|\mathbf{A}\|$,

$$\frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}}:\mathbf{B} = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\|\mathbf{A}+\alpha\mathbf{B}\|$$

$$\|\mathbf{A}+\alpha\mathbf{B}\| = \sqrt{\mathrm{tr}(\mathbf{A}+\alpha\mathbf{B})(\mathbf{A}+\alpha\mathbf{B})^{\mathrm{T}}}$$

$$= \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{B}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{A}\mathbf{B}^{\mathrm{T}} + \alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}$$

Continued T3.6

- Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write,

$$\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\|\mathbf{A}+\alpha\mathbf{B}\| = \lim_{\alpha\to 0}\frac{\mathrm{tr}\left(\mathbf{B}\mathbf{A}^{\mathrm{T}}\right)+\mathrm{tr}\left(\mathbf{A}\mathbf{B}^{\mathrm{T}}\right)+2\alpha\,\mathrm{tr}\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right)}{2\sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{B}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{A}\mathbf{B}^{\mathrm{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}} = \frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}} = \frac{\mathbf{A}}{\|\mathbf{A}\|}:\mathbf{B}$$

- So that,

$$\frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}}:\mathbf{B} = \frac{\mathbf{A}}{\|\mathbf{A}\|}:\mathbf{B} \text{ or, } \frac{\partial\|\mathbf{A}\|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{\|\mathbf{A}\|}$$

- as required since $\mathbf{B}$ is arbitrary.

Without breaking down into components, use Liouville's theorem, $\frac{\partial}{\partial\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\mathrm{tr}\big(\dot{\mathbf{T}}\mathbf{T}^{-1}\big)$, to establish the fact that $\frac{\partial\det(\mathbf{T})}{\partial\mathbf{T}} = \mathbf{T}^{\mathrm{c}}$

- Start from Liouville's Theorem, given a scalar parameter such that $\mathbf{T} = \mathbf{T}(\alpha)$,

$$\frac{\partial}{\partial\alpha}(\det\mathbf{T}) = \det\mathbf{T}\,\mathrm{tr}\left[\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)\mathbf{T}^{-1}\right] = [(\det\mathbf{T})\mathbf{T}^{-\mathrm{T}}]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$

- By the chain rule,

$$\frac{\partial}{\partial\alpha}(\det\mathbf{T}) = \left[\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T})\right]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$

- It therefore follows that, $\left[\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T}) - [(\det\mathbf{T})\mathbf{T}^{-\mathrm{T}}]\right]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right) = 0$. Hence

$$\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T}) = (\det\mathbf{T})\mathbf{T}^{-\mathrm{T}} = \mathbf{T}^{\mathrm{c}}$$

If $\mathbf{T}$ is invertible, show that $\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T})) = \mathbf{T}^{-\mathrm{T}}$

$$\frac{d}{d\mathbf{T}}(\log\det\mathbf{T}) = \frac{d(\log\det\mathbf{T})}{d\det\mathbf{T}}\frac{d\det\mathbf{T}}{d\mathbf{T}}$$

$$= \frac{1}{\det\mathbf{T}}\mathbf{T}^{\mathrm{c}}$$

$$= \frac{1}{\det\mathbf{T}}(\det\ \mathbf{T})\mathbf{T}^{-\mathrm{T}}$$

$$= \mathbf{T}^{-\mathrm{T}}$$

Use Caley-Hamilton theorem to express the third invariant in terms of traces only (b) Use this result and the fact that $\mathbf{S}^c = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})^T$ to show that the derivative of the determinant is the cofactor.

- Given a tensor **S**, by Cayley Hamilton theorem,

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3\mathbf{I} = 0$$

- Note that $I_1 = I_1(\mathbf{S}), I_2 = I_2(\mathbf{S})\ and\ I_3 = I_3(\mathbf{S})$ the first, second and third invariants of $\boldsymbol{S}$ are all scalar functions of the tensor **S**. Taking the trace of the above equation,

$$\begin{aligned} I_1(\mathbf{S}^3) &= I_1(\mathbf{S})I_1(\mathbf{S}^2) - I_2(\mathbf{S})I_1(\mathbf{S}) + 3I_3(\mathbf{S}) \\ &= I_1(\mathbf{S})\left(I_1^2(\mathbf{S}) - 2I_2(\mathbf{S})\right) - I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S}) \\ &= I_1^3(\mathbf{S}) - 3I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S}) \end{aligned}$$

Continued T3.10

- Or,

$$I_3(\mathbf{S}) = \frac{1}{3}\left(I_1(\mathbf{S}^3) - I_1^3(\mathbf{S}) + 3I_1(\mathbf{S})I_2(\mathbf{S})\right)$$
$$= \frac{1}{3}\left(I_1(\mathbf{S}^3) - I_1^3(\mathbf{S}) + \frac{3}{2}I_1(\mathbf{S})\left(I_1^2(\mathbf{S}) - I_1(\mathbf{S}^2)\right)\right)$$
$$= \frac{1}{6}\left(2I_1(\mathbf{S}^3) + I_1^3(\mathbf{S}) - 3I_1(\mathbf{S})I_1(\mathbf{S}^2)\right)$$

- which show that the third invariant is itself expressible in terms of traces only. It is therefore invariant in value as a result of coordinate transformation.

T3.10 Continued

- Taking the Fréchet derivative of the third invariant expression above, noting that $\frac{d}{dS}\,\mathrm{tr}\left(\mathbf{S}^k\right) = k\left(\mathbf{S}^{k-1}\right)^{\mathrm{T}}$

$$6\frac{\partial}{\partial\mathbf{S}}I_3(\mathbf{S}) = 6\mathbf{S}^{2\mathrm{T}} + 3I_1^2(\mathbf{S})\mathbf{I} - 6I_1(\mathbf{S})\mathbf{S}^{\mathrm{T}} - 3I_1(\mathbf{S}^2)\mathbf{I}$$

- so that

$$\begin{aligned}\frac{\partial}{\partial\mathbf{S}}I_3(\mathbf{S}) &= \mathbf{S}^{2\mathrm{T}} - I_1^2(\mathbf{S})\mathbf{I} + \mathbf{I}I_2(\mathbf{S}) + 3I_1(\mathbf{S})\left(I_1(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}\right)\\ &= (\mathbf{S}^2 - I_1(\mathbf{S})\mathbf{S} + I_2\,(\mathbf{S})\mathbf{I})^{\mathrm{T}}\\ &= \mathbf{S}^{\mathrm{c}}\end{aligned}$$

If $\mathbf{T}$ is invertible, show that $\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathrm{T}}$

- Apply the chain rule:

$$\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = \frac{d(\log\det(\mathbf{T}^{-1}))}{d\det(\mathbf{T}^{-1})}\frac{d\det(\mathbf{T}^{-1})}{d\mathbf{T}^{-1}}\frac{d\mathbf{T}^{-1}}{d\mathbf{T}}$$

$$= -\frac{1}{\det(\mathbf{T}^{-1})}\mathbf{T}^{-\mathrm{c}}\left(\mathbf{T}^{-1} \boxtimes \mathbf{T}^{-\mathrm{T}}\right)$$

$$= -\frac{1}{\det(\mathbf{T}^{-1})}\det(\mathbf{T}^{-1})\,\mathbf{T}^{\mathrm{T}}\left(\mathbf{T}^{-1} \boxtimes \mathbf{T}^{-\mathrm{T}}\right)$$

$$= -\mathbf{T}^{-\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{T}^{-\mathrm{T}} = -\mathbf{T}^{-\mathrm{T}}$$

If $\mathbf{T}$ is invertible, show that $\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathrm{T}}$

Again, using the chain rule after observing that the determinant of an inverse is the inverse of the determinant, we have a method that does not require the evaluation of a fourth-order tensor:

$$\begin{aligned}\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) &= \frac{d(\log\det(\mathbf{T}^{-1}))}{d\det(\mathbf{T}^{-1})}\frac{d\det(\mathbf{T}^{-1})}{d\det(\mathbf{T})}\frac{d\det(\mathbf{T})}{d\mathbf{T}} \\ &= \frac{1}{\det(\mathbf{T}^{-1})}\frac{d\big(1/\det(\mathbf{T})\big)}{d\det(\mathbf{T})}\mathbf{T}^{\mathrm{c}} \\ &= -\frac{\det(\mathbf{T})}{(\det\mathbf{T})^2}\mathbf{T}^{\mathrm{c}} \\ &= -\mathbf{T}^{-\mathrm{T}}\end{aligned}$$

If $\mathbf{T}$ is invertible, show that $\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathrm{T}}$

From slide T3.15, we know that,

$$\frac{d}{d\mathbf{T}}(\log\det\mathbf{T}) = \mathbf{T}^{-\mathrm{T}}$$

Observe that, $\det(\mathbf{T}^{-1}) = (\det\mathbf{T})^{-1}$ so that

$$\log\det(\mathbf{T}^{-1}) = \log\ (\det\mathbf{T})^{-1} = -\log\det\mathbf{T}$$

Consequently,

$$\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\frac{d}{d\mathbf{T}}(\log\det\mathbf{T})$$
$$= -\mathbf{T}^{-\mathrm{T}}$$

Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}} \mathrm{tr}\left(\mathbf{A}\mathbf{S}\mathbf{B}^{\mathrm{T}}\right) = \mathbf{A}^{\mathrm{T}}\mathbf{B}$

- First observe that $\mathrm{tr}\left(\mathbf{A}\mathbf{S}\mathbf{B}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{S}\right)$.
- If we write, $\mathbf{C} \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}$, it is obvious from S11.22 that $\frac{d}{d\mathbf{S}} \mathrm{tr}(\mathbf{C}\mathbf{S}) = \mathbf{C}^{\mathrm{T}}$.
- Therefore,

$$\frac{d}{d\mathbf{S}} \mathrm{tr}\left(\mathbf{A}\mathbf{S}\mathbf{B}^{\mathrm{T}}\right) = \left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$$

Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}}\operatorname{tr}\left(\mathbf{A}\mathbf{S}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right) = \mathbf{B}^{\mathrm{T}}\mathbf{A}$

- Observe that

$$\begin{aligned}\operatorname{tr}\left(\mathbf{A}\mathbf{S}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right) &= \operatorname{tr}\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{S}^{\mathrm{T}}\right)\\ &= \operatorname{tr}\left[\mathbf{S}\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}\right]\\ &= \operatorname{tr}\left[\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}\mathbf{S}\right]\end{aligned}$$

- [The transposition does not alter trace; neither does a cyclic permutation. Ensure you understand why each equality here is true.] Consequently,

$$\frac{d}{d\mathbf{S}}\operatorname{tr}\left(\mathbf{A}\mathbf{S}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\right) = \frac{d}{d\mathbf{S}}\operatorname{tr}\left[\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}\mathbf{S}\right] = \left[\left(\mathbf{B}^{\mathrm{T}}\mathbf{A}\right)^{\mathrm{T}}\right]^{\mathbf{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$$

Let $\mathbf{S}$ be a symmetric and positive definite tensor and let $I_1(\mathbf{S}), I_2(\mathbf{S})$ and $I_3(\mathbf{S})$ be the three principal invariants of $\mathbf{S}$ show, using components, that $\frac{\partial \boldsymbol{I}_1(\mathbf{S})}{\partial \mathbf{S}} = \mathbf{I}$ the identity tensor

- $\frac{\partial \boldsymbol{I}_1(\mathbf{S})}{\partial \mathbf{S}}$ can be written in the invariant component form as,

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j$$

- Recall that $I_1(\mathbf{S}) = \text{tr}\, \mathbf{S} = S_{\alpha\alpha}$ hence

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j = \frac{\partial S_{\alpha\alpha}}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j$$

$$= \delta_{i\alpha}\delta_{\alpha j} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{I}$$

- which is the identity tensor as expected.

Let $\mathbf{S}$ be a symmetric and positive definite tensor and let $I_1(\mathbf{S}), I_2(\mathbf{S})$ and $I_3(\mathbf{S})$ be the three principal invariants of $\mathbf{S}$ show, using components, that $\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$

- $\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}}$ in a similar way can be written in the invariant component form as, $\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2}\frac{\partial}{\partial S_{ij}}[S_{\alpha\alpha}S_{\beta\beta} - S_{\alpha\beta}S_{\beta\alpha}]\mathbf{e}_i \otimes \mathbf{e}_j$ where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}[\mathrm{tr}^2\, \mathbf{S} - \mathrm{tr}\, \mathbf{S}^2]$. Consequently,

$$\begin{aligned}
\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} &= \frac{1}{2}\frac{\partial}{\partial S_{ij}}[S_{\alpha\alpha}S_{\beta\beta} - S_{\alpha\beta}S_{\beta\alpha}]\mathbf{e}_i \otimes \mathbf{e}_j \\
&= \frac{1}{2}[\delta_{i\alpha}\delta_{\alpha j}S_{\beta\beta} + \delta_{i\beta}\delta_{\beta j}S_{\alpha\alpha} - \delta_{\beta i}\delta_{j\alpha}S_{\alpha\beta} - \delta_{\alpha i}\delta_{j\beta}S_{\beta\alpha}]\mathbf{e}_i \otimes \mathbf{e}_j \\
&= \frac{1}{2}[\delta_{ij}S_{\beta\beta} + \delta_{ij}S_{\alpha\alpha} - S_{ij} - S_{ji}]\mathbf{e}_i \otimes \mathbf{e}_j \\
&= I_1(\mathbf{S})\mathbf{I} - \mathbf{S}
\end{aligned}$$

Let $\mathbf{S}$ be a symmetric and positive definite tensor and let $I_1(\mathbf{S}), I_2(\mathbf{S})$ and $I_3(\mathbf{S})$ be the three principal invariants of $\mathbf{S}$ show, using components, that $\frac{\partial I_3(\mathbf{S})}{\partial \mathbf{S}} = I_3(\mathbf{S})\,\mathbf{S}^{-1}$

$$\det \mathbf{S} = \frac{1}{3!} e_{ijk} e_{rst} S_{ir} S_{js} S_{kt}$$

- Differentiating with respect to $S_{\alpha\beta}$, we obtain,

$$\frac{\partial \det \mathbf{S}}{\partial S_{\alpha\beta}} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \frac{1}{3!} e_{ijk} e_{rst} \left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}} S_{js} S_{kt} + S_{ir} \frac{\partial S_{js}}{\partial S_{\alpha\beta}} S_{kt} + S_{ir} S_{js} \frac{\partial S_{kt}}{\partial S_{\alpha\beta}} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$$= \frac{1}{3!} e_{ijk} e_{rst} \left[\delta_{i\alpha} \delta_{r\beta} S_{js} S_{kt} + S_{ir} \delta_{j\alpha} \delta_{s\beta} S_{kt} + S_{ir} S_{js} \delta_{k\alpha} \delta_{t\beta} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$$= \frac{1}{3!} e_{ijk} e_{rst} \left[S_{js} S_{kt} + S_{js} S_{kt} + S_{js} S_{kt} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$$= \frac{1}{2!} e_{ijk} e_{rst} S_{js} S_{kt} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \equiv [S^c]_{\alpha\beta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

- Which is the cofactor of $[S_{\alpha\beta}]$ or $\mathbf{S}^{\mathrm{c}} = I_3(\mathbf{S})\,\mathbf{S}^{-1}$ for a symmetric tensor where $\mathbf{S}^{-1} = \mathbf{S}^{-\mathrm{T}}$.

Given that φ is a scalar field and $\mathbf{v}$ a vector field, show that $\operatorname{div}(\varphi\mathbf{v}) = (\operatorname{grad}\varphi)\cdot\mathbf{v} + \varphi\operatorname{div}\mathbf{v}$

$$\begin{aligned}\operatorname{grad}(\varphi\mathbf{v}) &= (\varphi v_i)_{,j}\,\mathbf{e}_i\otimes\mathbf{e}_j\\ &= \varphi_{,j}\,v_i\mathbf{e}_i\otimes\mathbf{e}_j + \varphi v_{i,j}\,\mathbf{e}_i\otimes\mathbf{e}_j\\ &= \mathbf{v}\otimes(\operatorname{grad}\varphi) + \varphi\operatorname{grad}\mathbf{v}\end{aligned}$$

- Now, $\operatorname{div}(\varphi\mathbf{v}) = \operatorname{tr}\big(\operatorname{grad}(\varphi\mathbf{v})\big)$. Taking the trace of the above, we have:

$$\operatorname{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\operatorname{grad}\varphi) + \varphi\operatorname{div}\mathbf{v}$$

Show that $\text{grad}(\mathbf{u} \cdot \mathbf{v}) = (\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\text{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$

- $\mathbf{u} \cdot \mathbf{v} = u_i v_i$ is a scalar sum of components.

$$\text{grad}(\mathbf{u} \cdot \mathbf{v}) = (u_i v_i)_{,j}\, \mathbf{e}_j$$

$$= u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\, \mathbf{e}_j$$

Now $\text{grad}\,\mathbf{u} = u_{i,j}\, \mathbf{e}_i \otimes \mathbf{e}_j$ swapping the bases, we have that,

$$(\text{grad}\,\mathbf{u})^{\mathrm{T}} = u_{i,j}\, (\mathbf{e}_j \otimes \mathbf{e}_i).$$

- Writing $\mathbf{v} = v_k \mathbf{e}_k$, we have that,

$$(\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} = u_{i,j}\, v_k (\mathbf{e}_j \otimes \mathbf{e}_i)\mathbf{e}_k$$

$$= u_{i,j}\, v_k \mathbf{e}_j \delta_{ik} = u_{i,j}\, v_i \mathbf{e}_j$$

- It is easy to similarly show that $u_i v_{i,j}\, \mathbf{e}_j = (\text{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$. Clearly,

$$\text{grad}(\mathbf{u} \cdot \mathbf{v}) = (u_i v_i)_{,j}\, \mathbf{e}_j$$

$$= u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\, \mathbf{e}_j$$

$$= (\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v}$$

- As required

Show that $\text{grad}(\mathbf{u}\cdot\mathbf{v}) = (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{grad}\,\mathbf{v})\mathbf{u} + \mathbf{u}\times\text{curl}\,\mathbf{v} + \mathbf{v}\times\text{curl}\,\mathbf{u}$

- Observe from Earlier Proof that $\text{grad}(\mathbf{u}\cdot\mathbf{v}) = (\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\text{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}.$
- Note that

$$\begin{aligned}\mathbf{u}\times\text{curl}\,\mathbf{v} &= \mathbf{u}\times e_{ijk}v_{k,j}\,\mathbf{e}_i = e_{\alpha\beta\gamma}u_\beta e_{\gamma jk}v_{k,j}\,\mathbf{e}_\alpha\\ &= e_{\alpha\beta\gamma}e_{jk\gamma}u_\beta v_{k,j}\,\mathbf{e}_\alpha = \left(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\right)u_\beta v_{k,j}\,\mathbf{e}_\alpha\\ &= u_k v_{k,j}\,\mathbf{e}_j - u_j v_{k,j}\,\mathbf{e}_k\\ &= \left(v_{k,j}\,\mathbf{e}_j\otimes\mathbf{e}_k\right)u_\alpha\mathbf{e}_\alpha - \left(v_{k,j}\,\mathbf{e}_k\otimes\mathbf{e}_j\right)u_\beta\mathbf{e}_\beta\\ &= \left(\text{grad}^{\mathrm{T}}\mathbf{v}\right)\mathbf{u} - (\text{grad}\,\mathbf{v})\mathbf{u}\end{aligned}$$

- Similarly, $\mathbf{v}\times\text{curl}\,\mathbf{u} = \left(\text{grad}^{\mathrm{T}}\mathbf{u}\right)\mathbf{v} - (\text{grad}\,\mathbf{u})\mathbf{v}$ so that we substitute for the transposes of the gradients in our earlier equation and find,

$$\text{grad}(\mathbf{u}\cdot\mathbf{v}) = (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{grad}\,\mathbf{v})\mathbf{u} + \mathbf{u}\times\text{curl}\,\mathbf{v} + \mathbf{v}\times\text{curl}\,\mathbf{u}$$

Show that $\operatorname{div}\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = (\operatorname{div}\operatorname{grad}\mathbf{u})\cdot\mathbf{v} + (\operatorname{div}\operatorname{grad}\mathbf{v})\cdot\mathbf{u} + 2(\operatorname{grad}\mathbf{u}):(\operatorname{grad}\mathbf{v})$

- Observe that, in component form, $\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\,\mathbf{e}_j$.
- Note that

$$\begin{aligned}\operatorname{grad}\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= \operatorname{grad}\left(u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\,\mathbf{e}_j\right)\\ &= \left(u_{i,j}\, v_i\right)_{,k} \mathbf{e}_j \otimes \mathbf{e}_k + \left(u_i v_{i,j}\right)_{,k} \mathbf{e}_j \otimes \mathbf{e}_k\end{aligned}$$

- Taking the traces of both sides,

$$\begin{aligned}\operatorname{div}\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= \left(u_{i,j}\, v_i\right)_{,k} \mathbf{e}_j \cdot \mathbf{e}_k + \left(u_i v_{i,j}\right)_{,k} \mathbf{e}_j \cdot \mathbf{e}_k\\ &= \left(u_{i,j}\, v_i\right)_{,k} \delta_{jk} + \left(u_i v_{i,j}\right)_{,k} \delta_{jk}\\ &= u_{i,jj}\, v_i + u_{i,j}\, v_{i,j} + u_{i,j}\, v_{i,j} + u_i v_{i,jj}\\ &= (\operatorname{div}\operatorname{grad}\mathbf{u})\cdot\mathbf{v} + (\operatorname{div}\operatorname{grad}\mathbf{v})\cdot\mathbf{u} + 2(\operatorname{grad}\mathbf{u}):(\operatorname{grad}\mathbf{v})\end{aligned}$$

Some write this result as:
$\nabla^2(\mathbf{u}\cdot\mathbf{v}) = (\nabla^2\mathbf{u})\mathbf{v} + (\nabla^2\mathbf{v})\mathbf{u} + 2(\nabla\mathbf{u}):(\nabla\mathbf{v})$

Show that $\text{grad}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times)\text{grad}\,\mathbf{v} - (\mathbf{v} \times)\text{grad}\,\mathbf{u}$

$$\mathbf{u} \times \mathbf{v} = e_{ijk} u_j v_k \mathbf{e}_i$$

- Recall that the gradient of this vector is the tensor,

$$\begin{aligned}
\text{grad}(\mathbf{u} \times \mathbf{v}) &= \left(e_{ijk} u_j v_k\right)_{,l} \mathbf{e}_i \otimes \mathbf{e}_l \\
&= e_{ijk} u_{j,l}\, v_k \mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk} u_j v_{k,l}\, \mathbf{e}_i \otimes \mathbf{e}_l \\
&= e_{ikj} u_{j,l}\, v_k \mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk} u_j v_{k,l}\, \mathbf{e}_i \otimes \mathbf{e}_l \\
&= -(\mathbf{v} \times)\text{grad}\,\mathbf{u} + (\mathbf{u} \times)\text{grad}\,\mathbf{v}
\end{aligned}$$

Show that $\text{div}\,(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}$

We already have the expression for $\text{grad}(\mathbf{u}\times\mathbf{v})$ (T3.31); remember that

$$\begin{aligned}\text{div}\,(\mathbf{u}\times\mathbf{v}) &= \text{tr}[\text{grad}(\mathbf{u}\times\mathbf{v})]\\ &= -e_{ikj}u_{j,l}\,v_k\mathbf{e}_i\cdot\mathbf{e}_l + e_{ijk}u_jv_{k,l}\,\mathbf{e}_i\cdot\mathbf{e}_l\\ &= -e_{ikj}u_{j,l}\,v_k\delta_{il} + e_{ijk}u_jv_{k,l}\,\delta_{il}\\ &= -e_{ikj}u_{j,i}\,v_k + e_{ijk}u_jv_{k,i}\\ &= \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}\end{aligned}$$

Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that curl $(\phi\mathbf{v}) = \phi$ curl $\mathbf{v} + (\text{grad}\,\phi) \times \mathbf{v}$.

$$\begin{aligned}
\text{curl}\,(\phi\mathbf{v}) &= e_{ijk}(\phi v_k)_{,j}\,\mathbf{e}_i \\
&= e_{ijk}\left(\phi_{,j}\,v_k + \phi v_{k,j}\right)\mathbf{e}_i \\
&= e_{ijk}\phi_{,j}\,v_k\mathbf{e}_i + e_{ijk}\phi v_{k,j}\,\mathbf{e}_i \\
&= (\text{grad}\,\phi) \times \mathbf{v} + \phi\,\text{curl}\,\mathbf{v}
\end{aligned}$$

Show that $\text{div}\,(\mathbf{u} \otimes \mathbf{v}) = (\text{div}\,\mathbf{v})\mathbf{u} + (\text{grad}\,\mathbf{u})\mathbf{v}$

- $\mathbf{u} \otimes \mathbf{v}$ is the tensor, $u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$ in component form. The gradient of this is the third order tensor,

$$\text{grad}\,(\mathbf{u} \otimes \mathbf{v}) = \left(u_i v_j\right)_{,k} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

- And by divergence, we mean the contraction of the last basis vector:

$$\begin{aligned} \text{div}\,(\mathbf{u} \otimes \mathbf{v}) &= \left(u_i v_j\right)_{,k} \left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_k \\ &= \left(u_i v_j\right)_{,k} \mathbf{e}_i \delta_{jk} = \left(u_i v_j\right)_{,j} \mathbf{e}_i \\ &= u_{i,j}\, v_j \mathbf{e}_i + u_i v_{j,j}\, \mathbf{e}_i \\ &= (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{div}\,\mathbf{v})\mathbf{u} \end{aligned}$$

For a scalar field ϕ and a tensor field $\mathbf{T}$ show that $\text{grad}\,(\phi\mathbf{T}) = \phi\text{grad}\,\mathbf{T} + \mathbf{T} \otimes \text{grad}\phi$. Also show that $\text{div}\,(\phi\mathbf{T}) = \phi\,\text{div}\,\mathbf{T} + \mathbf{T}\text{grad}\phi$

- $$\begin{aligned}\text{grad}(\phi\mathbf{T}) &= (\phi T_{ij})_{,k}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \\ &= (\phi_{,k}\,T_{ij} + \phi T_{ij,k})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \\ &= \mathbf{T} \otimes \text{grad}\phi + \phi\text{grad}\,\mathbf{T}\end{aligned}$$
- Furthermore, we can contract the last two bases and obtain,
$$\begin{aligned}\text{div}(\phi\mathbf{T}) &= (\phi_{,k}\,T_{ij} + \phi T_{ij,k})\mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k \\ &= (\phi_{,k}\,T_{ij} + \phi T_{ij,k})\mathbf{e}_i\delta_{jk} \\ &= T_{ik}\,\phi_{,k}\,\mathbf{e}_i + \phi T_{ik,k}\,\mathbf{e}_i \\ &= \mathbf{T}\,\text{grad}\phi + \phi\,\text{div}\,\mathbf{T}\end{aligned}$$

For two arbitrary tensors **S** and **T**, show that $\text{grad}(\mathbf{ST}) = \left(\text{grad}\,\mathbf{S}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{T} + \mathbf{S}\,\text{grad}\,\mathbf{T}$

$$
\begin{aligned}
\text{grad}(\mathbf{ST}) &= \left(S_{ij}T_{jk}\right)_{,\alpha}\,\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\
&= \left(S_{ij,\alpha}T_{jk} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\
&= \left(T_{kj}S_{ji,\alpha} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\
&= \left(\text{grad}\,\mathbf{S}^{\mathrm{T}}\right)^{\mathrm{T}}\,\mathbf{T} + \mathbf{S}\,\text{grad}\,\mathbf{T}
\end{aligned}
$$

Show that $\text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}} = \text{grad}(\text{curl}\,\mathbf{v})$

- From previous derivation, we can see that,

$$\text{curl}\,\mathbf{T} = e_{ijk} T_{\alpha k,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha.$$

- Clearly,

$$\text{curl}\,\mathbf{T}^{\mathrm{T}} = e_{ijk} T_{k\alpha,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$$

- so that $\text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}} = e_{ijk} v_{k,\alpha j}\, \mathbf{e}_i \otimes \mathbf{e}_\alpha$. But $\text{curl}\,\mathbf{v} = e_{ijk} v_{k,j}\; \mathbf{e}_i$. The gradient of this is,

$$\begin{aligned} \text{grad}(\text{curl}\,\mathbf{v}) &= \left(e_{ijk} v_{k,j}\right)_{,\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= e_{ijk} v_{k,j\alpha}\, \mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= \text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}} \end{aligned}$$

For a second-order tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, write the explicit expression for $\text{curl}\,\mathbf{T}$ in Cartesian Coordinates

- $\text{curl}\,\mathbf{T} = e_{ijk}T_{k\alpha,j}\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$

i	α	$e_{ijk}T_{k\alpha,j}\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$
1	1	$\left(\frac{\partial T_{13}}{\partial x_2} - \frac{\partial T_{12}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_1$
1	2	$\left(\frac{\partial T_{23}}{\partial x_2} - \frac{\partial T_{22}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_3$
1	3	$\left(\frac{\partial T_{33}}{\partial x_2} - \frac{\partial T_{32}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_2$
2	1	$\left(\frac{\partial T_{11}}{\partial x_3} - \frac{\partial T_{13}}{\partial x_1}\right)\mathbf{e}_2 \otimes \mathbf{e}_1$
2	2	$\left(\frac{\partial T_{21}}{\partial x_3} - \frac{\partial T_{23}}{\partial x_1}\right)\mathbf{e}_2 \otimes \mathbf{e}_2$
2	3	$\left(\frac{\partial T_{31}}{\partial x_2} - \frac{\partial T_{33}}{\partial x_3}\right)\mathbf{e}_2 \otimes \mathbf{e}_3$
3	1	$\left(\frac{\partial T_{12}}{\partial x_1} - \frac{\partial T_{11}}{\partial x_2}\right)\mathbf{e}_3 \otimes \mathbf{e}_1$

For two arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$, show that $\mathrm{div}(\mathbf{ST}) = (\mathrm{grad}\,\mathbf{S}):\mathbf{T} + \mathbf{T}\,\mathrm{div}\,\mathbf{S}$

$$
\begin{aligned}
\mathrm{grad}(\mathbf{ST}) &= \left(S_{ij}T_{jk}\right)_{,\alpha}\,\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\
&= \left(S_{ij,\alpha}T_{jk} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\
\mathrm{div}(\mathbf{ST}) &= \left(S_{ij,\alpha}T_{jk} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i(\mathbf{e}_k \cdot \mathbf{e}_\alpha) \\
&= \left(S_{ij,\alpha}T_{jk} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i\delta_{k\alpha} \\
&= \left(S_{ij,k}T_{jk} + S_{ij}T_{jk,k}\right)\mathbf{e}_i \\
&= (\mathrm{grad}\,\mathbf{S}):\mathbf{T} + \mathbf{S}\,\mathrm{div}\,\mathbf{T}
\end{aligned}
$$

For a vector field $\mathbf{u}$, show that $\text{grad}(\mathbf{u}\times)$ is a third ranked tensor. Hence or otherwise show that $\text{div}(\mathbf{u}\times) = -\text{curl}\,\mathbf{u}$.

- The second-order tensor $(\mathbf{u}\times)$ is defined as $e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$. Taking the derivative with an independent base, we have

$$\text{grad}(\mathbf{u}\times) = e_{ijk}u_{j,l}\,\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

- This gives a third order tensor as we have seen. Contracting on the last two bases,

$$\begin{aligned}\text{div}(\mathbf{u}\times) &= e_{ijk}u_{j,l}\,\mathbf{e}_i \otimes \mathbf{e}_k \cdot \mathbf{e}_l \\ &= e_{ijk}u_{j,l}\,\mathbf{e}_i\delta_{kl} \\ &= e_{ijk}u_{j,k}\,\mathbf{e}_i \\ &= -\text{curl}\,\mathbf{u}\end{aligned}$$

Show that div $(\phi\mathbf{I}) = \text{grad}\ \phi$

- Note that $\phi\mathbf{I} = (\phi\delta_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$. Also note that
$$\text{grad}\ \phi\mathbf{I} = (\phi\delta_{\alpha\beta})_{,i}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_i$$
- The divergence of this third order tensor is the contraction of the last two bases:
$$\begin{aligned}\text{div}\ (\phi\mathbf{I}) &= \text{tr}(\text{grad}\ \phi\mathbf{I}) \\ &= (\phi\delta_{\alpha\beta})_{,i}\, (\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_i = (\phi\delta_{\alpha\beta})_{,i}\, \mathbf{e}_\alpha\delta_{\beta i} \\ &= \phi_{,i}\, \delta_{\alpha\beta}\mathbf{e}_\alpha\delta_{\beta i} \\ &= \phi_{,i}\, \delta_{\alpha i}\mathbf{e}_\alpha = \ \phi_{,i}\, \mathbf{e}_i \\ &= \text{grad}\ \phi\end{aligned}$$

Show that curl $(\phi\mathbf{I}) = (\text{ grad }\phi)\times$

- Note that $\phi\mathbf{I} = (\phi\delta_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, and
- That curl $\mathbf{T} = e_{ijk}T_{\alpha k},_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$ so that,

$$
\begin{aligned}
\text{curl }(\phi\mathbf{I}) &= e_{ijk}(\phi\delta_{\alpha k}),_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha \\
&= e_{ijk}(\phi,_j\ \delta_{\alpha k})\mathbf{e}_i \otimes \mathbf{e}_\alpha \\
&= e_{ijk}\phi,_j\ \mathbf{e}_i \otimes \mathbf{e}_k \\
&= (\text{ grad }\phi)\times
\end{aligned}
$$

Show that the dyad $\mathbf{u} \otimes \mathbf{v}$ is NOT, in general symmetric: $\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \times \mathbf{v}) \times$

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= e_{ijk} u_j v_k \, \mathbf{e}_i \\
\big((\mathbf{u} \times \mathbf{v}) \times\big) &= e_{\alpha i \beta} e_{ijk} u_j v_k \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= -\big(\delta_{\alpha j} \delta_{\beta k} - \delta_{\alpha k} \delta_{\beta j}\big) u_j v_k \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= \big(-u_\alpha v_\beta + u_\beta v_\alpha\big) \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}
\end{aligned}$$

Show that $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$

$$\text{grad}\,(\mathbf{u} \times \mathbf{v}) = \left(e_{ijk} u_j v_k\right)_{,l} \mathbf{e}_i \otimes \mathbf{e}_l$$

$$\text{div}\,(\mathbf{u} \times \mathbf{v}) = \left(e_{ijk} u_j v_k\right)_{,l} \delta_{il} = \left(e_{ijk} u_j v_k\right)_{,i}$$

- Noting that the tensor e_{ijk} behaves is a constant, we can write,

$$\begin{aligned} \text{div}\,(\mathbf{u} \times \mathbf{v}) &= \left(e_{ijk} u_j v_k\right)_{,i} \\ &= e_{ijk} u_{j,i} v_k + e_{ijk} u_j v_{k,i} \\ &= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \end{aligned}$$

Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that $\text{curl}\,(\phi\mathbf{v}) = \phi\,\text{curl}\,\mathbf{v} + (\text{grad}\,\phi) \times \mathbf{v}$.

$$\begin{aligned}
\text{curl}\,(\phi\mathbf{v}) &= e_{ijk}(\phi v_k)_{,j}\,\mathbf{e}_i \\
&= e_{ijk}\left(\phi_{,j}\,v_k + \phi v_{k,j}\right)\mathbf{e}_i \\
&= e_{ijk}\phi_{,j}\,v_k\mathbf{e}_i + e_{ijk}\phi v_{k,j}\,\mathbf{e}_i \\
&= (\text{grad}\,\phi) \times \mathbf{v} + \phi\,\text{curl}\,\mathbf{v}
\end{aligned}$$

Show that $\text{curl}\,(\mathbf{v}\,\times) = (\text{div}\,\mathbf{v})\mathbf{I} - \text{grad}\,\mathbf{v}$

$$(\mathbf{v}\,\times) = e_{\alpha\beta k} v_\beta\, \mathbf{e}_\alpha \otimes \mathbf{e}_k$$

$$\text{curl}\,\mathbf{T} = e_{ijk} T_{\alpha k,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$$

- so that

$$\begin{aligned}\text{curl}\,(\mathbf{v}\,\times) &= e_{ijk} e_{\alpha\beta k} v_{\beta,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= \left(\delta_{i\alpha}\delta_{j\beta} - \delta_{j\alpha}\delta_{i\beta}\right) v_{\beta,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= v_{j,j}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha - v_{i,j}\, \mathbf{e}_i \otimes \mathbf{e}_j \\ &= (\text{div}\,\mathbf{v})\mathbf{I} - \text{grad}\,\mathbf{v}\end{aligned}$$

Show that $\text{curl}\,(\text{grad}\,\mathbf{v}) = \mathbf{O}$

- For any tensor $\mathbf{T} = T_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$

$$\text{curl}\,\mathbf{T} = e_{ijk}T_{\alpha k,j}\;\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

- Let $\mathbf{T} = \text{grad}\,\mathbf{v}$. Clearly, in this case, $T_{\alpha\beta} = v_{\alpha,\beta}$ so that $T_{\alpha k,j} = v_{\alpha,kj}$. It therefore follows that,

$$\text{curl}\,(\text{grad}\,\mathbf{v}) = e_{ijk}v_{\alpha,kj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha = \mathbf{O}$$

On account of the symmetry of $v_{\alpha,kj}$ assuming that $\mathbf{v}$ is continuously differentiable so that the order of differentiation is immaterial, and the antisymmetry of e_{ijk}.

Show for a scalar field ϕ that $\text{curl}\,(\text{grad}\,\phi) = \mathbf{o}$

- For any tensor $\mathbf{v} = v_\alpha \boldsymbol{e}_\alpha$

$$\text{curl}\,\mathbf{v} = e_{ijk} v_{k,j}\, \boldsymbol{e}_i$$

- Let $\mathbf{v} = \text{grad}\,\phi$. Clearly, in this case, $v_k = \phi_{,k}$ so that $v_{k,j} = \phi_{,kj}$. It therefore follows that,

$$\text{curl}\,(\text{grad}\,\phi) = e_{ijk}\phi_{,kj}\,\mathbf{e}_i$$
$$= \mathbf{o}.$$

- The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the scalar field is immaterial. This will be true only if the scalar field is continuous – a proposition we have assumed in the above

Show that div $(\text{grad }\phi \times \text{grad }\theta) = 0$

$$\text{grad }\phi \times \text{grad }\theta = e_{ijk}\phi,_j\,\theta,_k\,\mathbf{e}_i$$

- The gradient of this vector is the tensor,

$$\begin{aligned}\text{grad}(\text{grad }\phi \times \text{grad }\theta) &= \left(e_{ijk}\phi,_j\,\theta,_k\right),_l\,\mathbf{e}_i \otimes \mathbf{e}_l \\ &= e_{ijk}\phi,_{jl}\,\theta,_k\,\mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk}\phi,_j\,\theta,_{kl}\,\mathbf{e}_i \otimes \mathbf{e}_l\end{aligned}$$

The trace of the above result is the divergence we are seeking:

$$\begin{aligned}\text{div }(\text{grad }\phi \times \text{grad }\theta) &= \text{tr grad}(\text{grad }\phi \times \text{grad }\theta) \\ &= e_{ijk}\phi,_{jl}\,\theta,_k\,\delta_{il} + e_{ijk}\phi,_j\,\theta,_{kl}\,\delta_{il} \\ &= e_{ijk}\phi,_{ji}\,\theta,_k + e_{ijk}\phi,_j\,\theta,_{ki} = 0\end{aligned}$$

- Each term vanishing on account of the contraction of a symmetric tensor with an antisymmetric.

Show that $\text{curl curl } \mathbf{v} = \text{grad}(\text{div } \mathbf{v}) - \text{grad}^2\mathbf{v}$

- Let $\mathbf{w} = \text{curl } \mathbf{v} \equiv e_{ijk}v_{k},_{j}\ \mathbf{e}_i$. But $\text{curl } \mathbf{w} \equiv e_{\alpha\beta\gamma}w_{\gamma},_{\beta}\ \mathbf{e}_\alpha$. Upon inspection, we find that $w_\gamma = \delta_{\gamma i}e_{ijk}v_{k},_{j} = e_{\gamma jk}v_{k},_{j}$ so that

$$\text{curl } \mathbf{w} \equiv e_{\alpha\beta\gamma}\left(\delta_{\gamma i}e_{ijk}v_{k},_{j}\right),_{\beta}\ \mathbf{e}_\alpha$$
$$= \delta_{\gamma i}e_{\alpha\beta\gamma}e_{ijk}v_{k},_{j\beta}\ \mathbf{e}_\alpha$$

- Now, it can be shown that

$$\delta_{\gamma i}e_{\alpha\beta\gamma}e_{ijk} = \delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}$$

- so that,

Show that $\text{curl curl}\,\mathbf{v} = \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$

$$\begin{aligned}\text{curl}\,\mathbf{w} &= (\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j})v_{k},_{j\beta}\,\mathbf{e}_\alpha \\ &= v_{\beta},_{j\beta}\,\mathbf{e}_j - \delta_{\beta j}v_{\alpha},_{j\beta}\,\mathbf{e}_\alpha \\ &= \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}\end{aligned}$$

- Also recall that the Laplacian (grad^2) of a scalar field ϕ is, $\text{grad}^2\phi = \delta_{ij}\phi,_{ij}$. In Cartesian coordinates, this becomes,

$$\text{grad}^2\phi = \delta_{ij}\phi,_{ij} = \phi,_{ii}$$

- as the unit (metric) tensor now degenerates to the Kronecker delta in this special case. For a vector field, $\text{grad}^2\mathbf{v} = v_{\alpha},_{jj}\,\mathbf{e}_\alpha$.
  Also note that while grad is a vector operator, the Laplacian (grad^2) is a scalar operator.

For a second-order tensor $\mathbf{T}$ define $\text{curl}\ \mathbf{T} \equiv e_{ijk}T_{\alpha k},_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$ show that for any constant vector $\mathbf{a}$, $(\text{curl}\ \mathbf{T})\ \mathbf{a} = \text{curl}\left(\mathbf{T}^\text{T}\mathbf{a}\right)$

- Express vector **a** in the invariant form with covariant components as $\mathbf{a} = a_\beta \mathbf{e}_\beta$. It follows that

$$
\begin{aligned}
(\text{curl}\ \mathbf{T})\ \mathbf{a} &= e_{ijk}T_{\alpha k},_j\ (\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{a} \\
&= e_{ijk}T_{\alpha k},_j\ a_\beta(\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{e}_\beta \\
&= e_{ijk}T_{\alpha k},_j\ a_\beta \mathbf{e}_i \delta_{\beta\alpha} \\
&= e_{ijk}(T_{\alpha k}),_j\ \mathbf{e}_i a_\alpha \\
&= e_{ijk}(T_{\alpha k} a_\alpha),_j\ \mathbf{e}_i
\end{aligned}
$$

- The last equality resulting from the fact that vector **a** is a constant vector. Clearly,

$$
(\text{curl}\ \mathbf{T})\ \mathbf{a} = \text{curl}\left(\mathbf{T}^\text{T}\mathbf{a}\right)
$$

For any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\text{curl}\,(\mathbf{u} \otimes \mathbf{v}) = [(\text{grad}\,\mathbf{u})\mathbf{v}\,\times]^{\text{T}} + (\text{curl}\,\mathbf{v}) \otimes \mathbf{u}$ where $\mathbf{v}\,\times$ is the skew tensor $e_{ijk} v_k\, \mathbf{e}_i \otimes \mathbf{e}_j$.

- Recall that the curl of a tensor $\mathbf{T}$ is defined by $\text{curl}\,\mathbf{T} \equiv e_{ijk} T_{\alpha k,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$. Clearly therefore,

$$\begin{aligned}
\text{curl}\,(\mathbf{u} \otimes \mathbf{v}) &= e_{ijk} (u_\alpha v_k)_{,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha \\
&= \left(e_{ijk} v_k \mathbf{e}_i \otimes \mathbf{e}_j\right)\left(u_{\alpha,\beta}\, \mathbf{e}_\beta \otimes \mathbf{e}_\alpha\right) + \left(e_{ijk} v_{k,j}\; \mathbf{e}_i\right) \otimes (u_\alpha \mathbf{e}_\alpha) \\
&= -(\mathbf{v}\,\times)(\text{grad}\,\mathbf{u})^{\mathbf{T}} + (\text{curl}\,\mathbf{v}) \otimes \mathbf{u} \\
&= [(\text{grad}\,\mathbf{u})\mathbf{v}\,\times]^{\mathbf{T}} + (\text{curl}\,\mathbf{v}) \otimes \mathbf{u}
\end{aligned}$$

- upon noting that the vector cross is a skew tensor.

For a second-order tensor field $\mathbf{T}$, show that $\text{div}(\text{curl}\,\mathbf{T}) = \text{curl}\left(\text{div}\,\mathbf{T}^{\text{T}}\right)$

- Define the second order tensor $\mathbf{S}$ as

$$\text{curl}\,\mathbf{T} \equiv e_{ijk}T_{\alpha k,j}\ \mathbf{e}_i \otimes \mathbf{e}_\alpha = S_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

- The gradient of $\mathbf{S}$ is $S_{i\alpha,\beta}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = e_{ijk}T_{\alpha k,j\beta}\ \mathbf{e}_i \otimes \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$
- Clearly,

$$\begin{aligned}\text{div}(\text{curl}\,\mathbf{T}) &= e_{ijk}T_{\alpha k,j\beta}\ (\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{e}_\beta \\ &= e_{ijk}T_{\alpha k,j\beta}\ \mathbf{e}_i\,\delta_{\alpha\beta} \\ &= e_{ijk}T_{\beta k,j\beta}\,\mathbf{e}_i = \text{curl}\left(\text{div}\,\mathbf{T}^{\text{T}}\right)\end{aligned}$$

Show that $((\text{curl}\,\mathbf{u})\times) = \text{grad}\,\mathbf{u} - \text{grad}^{\mathrm{T}}\mathbf{u}$

$$
\begin{aligned}
((\text{curl}\,\mathbf{u})\times) &= e_{ijk}u_{k,j}\,\mathbf{e}_i \times \\
&= e_{\alpha i\beta}e_{ijk}u_{k,j}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= \left(\delta_{\beta j}\delta_{\alpha k} - \delta_{\beta k}\delta_{\alpha j}\right)u_{k,j}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \\
&= u_{k,j}\,\mathbf{e}_k \otimes \mathbf{e}_j - u_{k,j}\,\mathbf{e}_j \otimes \mathbf{e}_k \\
&= \text{grad}\,\mathbf{u} - \text{grad}^{\mathrm{T}}\mathbf{u}
\end{aligned}
$$

Show that $\operatorname{curl}(\mathbf{u}\times\mathbf{v}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u})$

- The vector $\mathbf{w} \equiv \mathbf{u}\times\mathbf{v} = w_k\mathbf{e}_k = e_{k\alpha\beta}u_\alpha v_\beta\mathbf{e}_k$ and $\operatorname{curl}\mathbf{w} = e_{ijk}w_{k},_{j}\,\mathbf{e}_i$. Therefore,

$$\begin{aligned}\operatorname{curl}(\mathbf{u}\times\mathbf{v}) &= e_{ijk}w_{k},_{j}\,\mathbf{e}_i\\ &= e_{ijk}e_{k\alpha\beta}\left(u^\alpha v^\beta\right)_{,j}\,\mathbf{e}_i\\ &= \left(\delta_{i\alpha}\delta_{j\beta}-\delta_{j\alpha}\delta_{i\beta}\right)\left(u^\alpha v^\beta\right)_{,j}\,\mathbf{e}_i\\ &= \left(\delta_{i\alpha}\delta_{j\beta}-\delta_{j\alpha}\delta_{i\beta}\right)\left(u^\alpha{}_{,j}\,v^\beta + u^\alpha v^\beta{}_{,j}\,\right)\mathbf{e}_i\\ &= \left[u_{i,j}\,v_j + u_i v_{j,j} - \left(u_{j,j}\,v_i + u_j v_{i,j}\,\right)\right]\mathbf{e}_i\\ &= \left[\left(u_i v_j\right)_{,j} - \left(u_j v_i\right)_{,j}\,\right]\mathbf{e}_i\\ &= \operatorname{div}(\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u})\end{aligned}$$

- since $\operatorname{div}(\mathbf{u}\otimes\mathbf{v}) = \left(u_i v_j\right)_{,\alpha}\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)\mathbf{e}_\alpha = \left(u_i v_j\right)_{,j}\,\mathbf{e}_i$.

Given a scalar ϕ and a second-order tensor field $\mathbf{T}$, show that $\text{curl}\,(\phi\mathbf{T}) = \phi\,\text{curl}\,\mathbf{T} + \big((\text{grad}\,\phi)\times\big)\mathbf{T}^{\mathrm{T}}$ where $[(\text{grad}\,\phi)\times]$ is the skew tensor $e_{ijk}\phi_{,j}\,\mathbf{e}_i\otimes\mathbf{e}_k$

$$
\begin{aligned}
\text{curl}\,(\phi\mathbf{T}) &\equiv e_{ijk}(\phi T_{\alpha k})_{,j}\;\mathbf{e}_i\otimes\mathbf{e}_\alpha \\
&= e_{ijk}\big(\phi_{,j}\,T_{\alpha k} + \phi T_{\alpha k,j}\big)\,\mathbf{e}_i\otimes\mathbf{e}_\alpha \\
&= e_{ijk}\phi_{,j}\,T_{\alpha k}\,\mathbf{e}_i\otimes\mathbf{e}_\alpha + \phi e_{ijk}T_{\alpha k,j}\;\mathbf{e}_i\otimes\mathbf{e}_\alpha \\
&= \big(e_{ijk}\phi_{,j}\,\mathbf{e}_i\otimes\mathbf{e}_k\big)\big(T_{\alpha\beta}\mathbf{e}_\beta\otimes\mathbf{e}_\alpha\big) + \phi e_{ijk}T_{\alpha k,j}\;\mathbf{e}_i \\
&= \phi\,\text{curl}\,\mathbf{T} + \big((\text{grad}\,\phi)\times\big)\mathbf{T}^{\mathrm{T}}
\end{aligned}
$$

Given a tensor field $\mathbf{T}$, obtain the vector $\mathbf{w} \equiv \mathbf{T}^{\mathrm{T}}\mathbf{v}$ and show that its divergence is $\mathbf{T}\!:\!(\text{grad}\,\mathbf{v}) + \mathbf{v}\cdot \text{div}\,\mathbf{T}$

- The gradient of $\mathbf{w}$ is the tensor, $(T_{ji}v_j)_{,k}\,\mathbf{e}_i \otimes \mathbf{e}_k$. Therefore, divergence of $\boldsymbol{w}$ (the trace of the gradient) is the scalar sum, $T_{ji}v_{j,k}\,\delta_{ik} + T_{ji,k}\,v_j\delta_{ik}$. Expanding, we obtain,

$$\begin{aligned}\text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) &= T_{ji}v_{j,k}\,\delta_{ik} + T_{ji,k}\,v_j\delta_{ik}\\ &= T_{jk,k}\,v^j + T_{jk}v_{j,k}\\ &= (\text{div}\,\mathbf{T})\cdot\mathbf{v} + \text{tr}\left(\mathbf{T}^{\mathrm{T}}\text{grad}\,\mathbf{v}\right)\\ &= (\text{div}\,\mathbf{T})\cdot\mathbf{v} + \mathbf{T}\!:\!(\text{grad}\,\mathbf{v})\end{aligned}$$

- Recall that scalar product of two vectors is commutative so that

$$\text{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{T}\!:\!(\text{grad}\,\mathbf{v}) + \mathbf{v}\cdot\text{div}\,\mathbf{T}$$

Show that if ϕ is a scalar field in the Euclidean space spanned by orthogonal coordinates x_i, with unit basis vectors, $\mathbf{e}_i$, then for the position vector $\mathbf{r}$,
$\text{div grad}(\mathbf{r}\phi) = 2 \text{ grad } \phi + \mathbf{r} \text{ div grad } \phi$

- In such a coordinate system, the radius vector will be given by,

$$\mathbf{r} = x_i \mathbf{e}_i$$

- where $\mathbf{e}_i$ is the unit vector along coordinate x_i.

$$\text{grad}(\mathbf{r}\phi) = (x_i\phi)_{,j}\, \mathbf{e}_i \otimes \mathbf{e}_j$$

$$\text{grad}\big(\text{grad}(\mathbf{r}\phi)\big) = \Big((x_i\phi)_{,j}\Big)_{,k}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

$$\begin{aligned}\text{div}\big(\text{grad}(\mathbf{r}\phi)\big) = \text{tr}\Big(\text{grad}\big(\text{grad}(\mathbf{r}\phi)\big)\Big) &= \Big((x_i\phi)_{,j}\Big)_{,k}\, \mathbf{e}_i\delta_{jk} \\ &= \big(\delta_{ij}\phi + x_i\phi_{,j}\big)_{,j}\, \mathbf{e}_i \\ &= \big(\delta_{ij}\phi_{,j} + x_{i,j}\,\phi_{,j} + x_i\phi_{,jj}\big)\mathbf{e}_i \\ &= 2 \text{ grad } \phi + \mathbf{r} \text{ div grad } \phi\end{aligned}$$

Will the above expression be valid in the spherical coordinate system?

The position vector in spherical coordinates are non-linear in the coordinate variables. This form, $\mathbf{r} = x_i \mathbf{e}_i$, of the expression will not be correct. It works only for Cartesian systems.

In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that $x_{,j} = \frac{x_j}{x}$.

$x = \sqrt{x_i x_i}$ therefore,

$$
\begin{aligned}
x_{,j} &= \frac{\partial \sqrt{x_i x_i}}{\partial x_j} \\
&= \frac{\partial \sqrt{x_i x_i}}{\partial (x_i x_i)} \times \frac{\partial (x_i x_i)}{\partial x_j} \\
&= \frac{1}{2\sqrt{x_i x_i}} \left[x_i \delta_{ij} + x_i \delta_{ij} \right] \\
&= \frac{x_j}{x}.
\end{aligned}
$$

In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$

$x = \sqrt{x_i x_i}$ and $x_{,j} = \frac{x_j}{x}$, therefore,

$$x_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial \sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right)$$

$$= \frac{x\frac{\partial x_i}{\partial x_j} - x_i \frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2}$$

$$= \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$$

In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that $x_{,ii} = \frac{2}{x}$.

$x = \sqrt{x_i x_i}$ and $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$. Therefore,

$$x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3}$$
$$= \frac{3}{x} - \frac{(x)^2}{(x)^3}$$
$$= \frac{2}{x}.$$

In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ and $U_{,ii} = 0$

$x = \sqrt{x_i x_i}$ therefore,

$$U_{,j} = \frac{\partial \frac{1}{x}}{\partial x_j} = \frac{\partial \frac{1}{x}}{\partial x} \times \frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$

- Consequently,

$$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right)$$

$$= \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i \frac{\partial}{\partial x_j}(x^3)}{x^6} = \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial (x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6}$$

$$= \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

T3.54 Continued

- From the above, we have,

$$U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

- Therefore,

$$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5}$$
$$= \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$

In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that $\operatorname{div}\left(\frac{\mathbf{r}}{x}\right) = \frac{2}{x}$.

- $x = \sqrt{x_i x_i}$ therefore,

$$\operatorname{div}\left(\frac{\boldsymbol{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x} x_{j,j} + \left(\frac{1}{x}\right)_{,j}$$

$$= \frac{3}{x} + x_j \left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right) \frac{dx}{dx_j}\right)$$

$$= \frac{3}{x} + x_j \left[-\left(\frac{1}{x^2}\right) \frac{x_j}{x}\right]$$

$$= \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$

Show that curl $\mathbf{u} \times \mathbf{v} = (\text{grad}\,\mathbf{u})\mathbf{v} - (\text{div}\,\mathbf{u})\mathbf{v} - (\text{grad}\,\mathbf{v})\mathbf{u} + (\text{div}\,\mathbf{v})\mathbf{u}$

$$\mathbf{u} \times \mathbf{v} = e_{ijk} u_j v_k \mathbf{e}_i$$
$$\text{curl}\,\mathbf{w} = e_{\alpha\beta l} w_{l,\beta}\, \mathbf{e}_\alpha$$

- Clearly,

$$\begin{aligned}
\text{curl}\,\mathbf{u} \times \mathbf{v} &= e_{\alpha\beta i}\left(e_{ijk} u_j v_k\right)_{,\beta}\, \mathbf{e}_\alpha \\
&= e_{\alpha\beta i} e_{ijk} u_{j,\beta} v_k \mathbf{e}_\alpha + e_{\alpha\beta i} e_{ijk} u_j v_{k,\beta}\, \mathbf{e}_\alpha \\
&= \left(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\right) u_{j,\beta} v_k \mathbf{e}_\alpha + \left(\delta_{\alpha j}\delta_{\beta k}\right. \\
&= u_{j,k}\, v_k \mathbf{e}_j - u_{j,j}\, v_k \mathbf{e}_k + u_j v_{k,k}\, \mathbf{e}_j - u_j v_{k,j}\, \mathbf{e}_k \\
&= (\text{grad}\,\mathbf{u})\mathbf{v} - (\text{div}\,\mathbf{u})\mathbf{v} - (\text{grad}\,\mathbf{v})\mathbf{u} + (\text{div}\,\mathbf{v})\mathbf{u}
\end{aligned}$$

For a scalar function ϕ and a vector $\mathbf{v}$ show that the divergence of the vector $\mathbf{v}\phi$ is equal to, $\mathbf{v} \cdot \text{grad}\, \phi + \phi\, \text{div}\, \mathbf{v}$

$$\text{grad}\,(\mathbf{v}\phi) = (v_i\phi)_{,j}\, \mathbf{e}_i \otimes \mathbf{e}_j = \left(\phi v_{i,j} + v_i\phi_{,j}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

- Taking the trace of this equation,

$$\begin{aligned}\text{div}\,\mathbf{v}\phi &= \text{tr}\big(\text{grad}\,(\mathbf{v}\phi)\big) = \left(\phi v_{i,j} + v_i\phi_{,j}\right)\mathbf{e}_i \cdot \mathbf{e}_j \\ &= \left(\phi v_{i,j} + v_i\phi_{,j}\right)\delta_{ij} \\ &= \phi v_{i,i} + v_i\phi_{,i} \\ &= \mathbf{v} \cdot \text{grad}\,\phi + \phi\,\text{div}\,\mathbf{v}\end{aligned}$$

- Hence the result.

When $\mathbf{T}$ is symmetric, show that $\text{tr}(\text{curl}\,\mathbf{T})$ vanishes.

- **When $\mathbf{T}$ is symmetric, show that tr$(\text{curl}\,\mathbf{T})$ vanishes.**

$$\text{curl}\,\mathbf{T} = e_{ijk} T_{\beta k,j}\, \mathbf{e}_i \otimes \mathbf{e}_\beta$$
$$\text{tr}(\text{curl}\,\mathbf{T}) = e_{ijk} T_{\beta k,j}\, \mathbf{e}_i \cdot \mathbf{e}_\beta$$
$$= e_{ijk} T_{\beta k,j}\, \delta_{i\beta} = e_{ijk} T_{ik,j}$$

- which obviously vanishes on account of the symmetry and antisymmetry in i and k. In this case, For a general tensor field **T** show that,

$$\text{curl}(\text{curl}\,\mathbf{T}) = [\text{grad}^2(\text{tr}\,\mathbf{T}) - \text{div}(\text{div}\,\mathbf{T})]\mathbf{I} + \text{grad}(\text{div}\,\mathbf{T}) + \left(\text{grad}(\text{div}\,\mathbf{T})\right)^{\text{T}} - \text{grad}\big(\text{grad}\,(\text{tr}\,\mathbf{T})\big) - \text{grad}^2\mathbf{T}^{\text{T}}$$

$$\text{curl}\,\mathbf{T} = e_{\alpha st}T_{\beta t,s}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$$= S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$

$\text{curl}\,\mathbf{S} = e_{ijk}S_{\alpha k,j}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$ so

$$\text{curl}\,\mathbf{S} = \text{curl}(\text{curl}\,\mathbf{T})$$

$$= e_{ijk}e_{\alpha st}T_{kt,sj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

$$= \begin{vmatrix} \delta_{i\alpha} & \delta_{is} & \delta_{it} \\ \delta_{j\alpha} & \delta_{js} & \delta_{jt} \\ \delta_{k\alpha} & \delta_{ks} & \delta_{kt} \end{vmatrix} T_{kt,sj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

$$= \begin{bmatrix} \delta_{i\alpha}(\delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}) + \delta_{is}(\delta_{jt}\delta_{k\alpha} - \delta_{j\alpha}\delta_{kt}) \\ +\delta_{it}(\delta_{j\alpha}\delta_{ks} - \delta_{js}\delta_{k\alpha}) \end{bmatrix} T_{kt,sj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$$

$$= [\delta_{js}T_{tt,sj} - T_{sj,sj}](\mathbf{e}_\alpha \otimes \mathbf{e}_\alpha) + [T_{\alpha j,sj} - \delta_{j\alpha}T_{tt,sj}](\mathbf{e}_s \otimes \mathbf{e}_\alpha) + [\delta_{j\alpha}T_{st,sj} - \delta_{js}T_{\alpha t,sj}](\mathbf{e}_t \otimes \mathbf{e}_\alpha)$$

$$= [\text{grad}^2(\text{tr}\,\mathbf{T}) - \text{div}(\text{div}\,\boldsymbol{T})]\mathbf{I} + \left(\text{grad}(\text{div}\,\mathbf{T})\right)^{\text{T}} - \text{grad}\big(\text{grad}\,(\text{tr}\,\mathbf{T})\big) + \big(\text{grad}(\text{div}\,\mathbf{T})\big) - \text{grad}^2\mathbf{T}^{\text{T}}$$