Properties of Tensors

SSG 321 Introduction to Continuum Mechanics
Instructors: OA Fakinlede & O Adewumi
Website: www.s2pAfrica.org, www.oafak.com

Properties in General

$\mathbf{T}: \mathbb{E} \rightarrow \mathbb{E} \Rightarrow \cdots$

Having defined a tensor as a linear transformation from one vector space to another, we now look at the properties of Tensors.

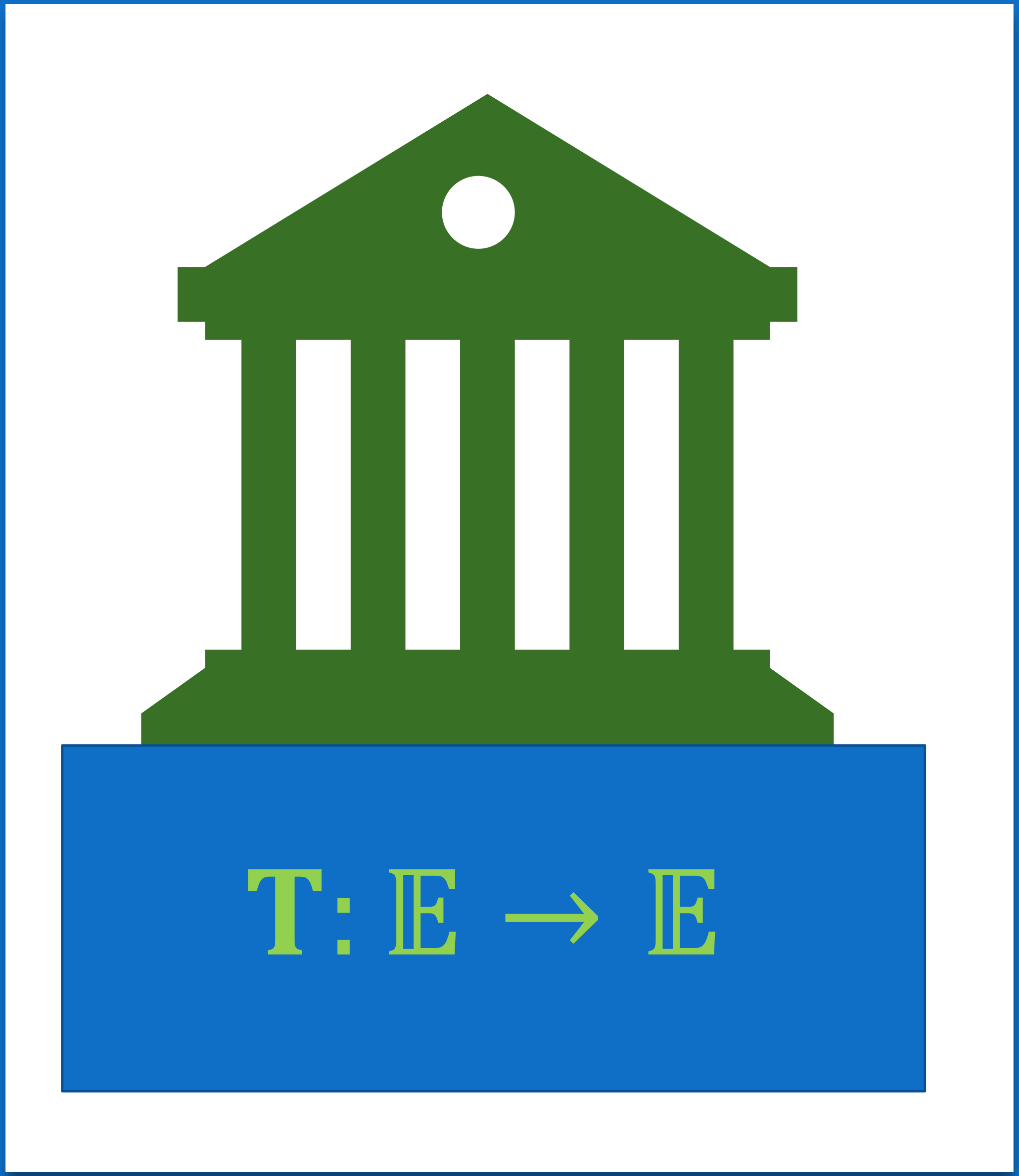
These properties are derived attributes that have the definition of tensors as their basis.

Basic definitions to establish the tensor characteristic of any object of concern.

Don’t let the Math intimidate you: They are not as hard as they look!

Foundation of Properties

- Properties are derived attributes
- They have as their bases, the definition of tensors as linear transformations from one vector space to another
- If a particular attribute gets difficult to understand, the retreat place is the definition of tensors



Equality

We defined a tensor by the linearity of its action on a vector, producing another vector.

$$\mathbf{T}: \mathbb{E} \to \mathbb{E} \Rightarrow \cdots$$

Two tensors are equal if, in transforming the same vector, they produce equal results.

$$\mathbf{T} = \mathbf{S} \Leftrightarrow \mathbf{T}\mathbf{v} = \mathbf{S}\mathbf{v} \; \forall \mathbf{v} \in \mathbb{E}$$

If we can find a counter example in the way they transform vectors; they are not equal.

The Inverse

- The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\mathbf{T} \in \mathbb{L}$ and $\mathbf{u} \in \mathbb{E}$, the mapping $\mathbf{w} \equiv \mathbf{Tu}$ produces a vector. Imagine a linear mapping $\mathbf{Y}$, if one exists, that, operating on $\mathbf{w}$, produces our original argument, $\mathbf{u}$, if we can find it
- A tensor may possess an inverse. If it does, it is said to be invertible.
- Not all tensors are invertible.

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The Inverse

$$\mathbf{Yw} = \mathbf{u}$$

- As a linear mapping, operating on a vector, clearly, $\mathbf{Y}$ is a tensor. It is called the inverse of $\mathbf{T}$ because,

$$\mathbf{Yw} = \mathbf{YTu} = \mathbf{u}$$

- So that the composition (or product) $\mathbf{YT} = \mathbf{I}$, the identity mapping. For this reason, we write,

$$\mathbf{Y} = \mathbf{T}^{-1}$$

The Inverse

- We now show that this relationship also implies that $\mathbf{TY} = \mathbf{I}$. Recall the vector defined by, $\mathbf{w} = \mathbf{Tu}$. Clearly,
$$\mathbf{TYTu} = \mathbf{TYw} = \mathbf{TIu} = \mathbf{Tu} = \mathbf{w}$$
- (First equality by the definition of $\mathbf{w}$, second by the fact that $\mathbf{YT} = \mathbf{I}$). It is clear that
$$\mathbf{TYw} = \mathbf{w}$$
- So that $\mathbf{TY} = \mathbf{YT} = \mathbf{I}$ as required.

Last Week's Question: Is the Inverse a Tensor?

If the tensor $\mathbf{T}$ is invertible, then $\exists\ \mathbf{T}^{-1}$ such that,

$$\mathbf{w} = \mathbf{T}\mathbf{u} \Rightarrow \mathbf{u} = \mathbf{T}^{-1}\mathbf{w}$$

In addition, $\mathbf{y} = \mathbf{T}\mathbf{v} \Rightarrow \mathbf{v} = \mathbf{T}^{-1}\mathbf{y}$. Furthermore, the tensor nature of $\mathbf{T}$ means that for $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) &= \alpha\mathbf{T}\mathbf{u} + \beta\mathbf{T}\mathbf{v} \\ &= \alpha\mathbf{w} + \beta\mathbf{y} \end{aligned}$$

Taking the inverse of the equation:

$$\begin{aligned} \mathbf{T}^{-1}\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) &= \mathbf{T}^{-1}(\alpha\mathbf{T}\mathbf{u} + \beta\mathbf{T}\mathbf{v}) \\ \alpha\mathbf{u} + \beta\mathbf{v} &= \mathbf{T}^{-1}(\alpha\mathbf{w} + \beta\mathbf{y}) = \alpha\mathbf{T}^{-1}\mathbf{w} + \beta\mathbf{T}^{-1}\mathbf{y} \end{aligned}$$

Showing that $\mathbf{T}^{-1}$also transforms vectors linearly and hence is a tensor.

Tensor Basis & Components

- Tensors, just like vectors, can be expressed in component form with respect to a system of coordinates created with basis vectors. Using ONB, for a typical tensor $\mathbf{T}$, we can write,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- There are nine scalar components. These can be computed using the indicial notation and the summation convention. Accordingly,

Tensor Basis & Components

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$
$$= T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 + T_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22}\mathbf{e}_2 \otimes \mathbf{e}_2$$
$$+ T_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 + T_{31}\mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32}\mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3$$

- We can find these components in terms of the tensor $\mathbf{T}$ in a way like the way we found the vector coefficient. Note that

$$\mathbf{T}\mathbf{e}_\beta = T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_\beta = T_{ij}\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_\beta) = T_{ij}\mathbf{e}_i\delta_{j\beta}$$

- so that,

$$\mathbf{e}_\alpha \cdot \mathbf{T}\mathbf{e}_\beta = T_{ij}\mathbf{e}_\alpha \cdot \mathbf{e}_i\delta_{j\beta}$$
$$= T_{ij}\delta_{i\alpha}\delta_{j\beta}$$
$$= T_{\alpha\beta}$$

With $\alpha, \beta \to i, j \Rightarrow T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$

Components of the Spherical Tensor

- Clearly,

$$\mathbf{T} = (\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j)(\mathbf{e}_i \otimes \mathbf{e}_j)$$
$$= T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- In particular, $\mathbf{e}_i \cdot \gamma\mathbf{I}\mathbf{e}_j = \gamma\mathbf{e}_i \cdot \mathbf{e}_j = \gamma\delta_{ij}$ so that the spherical tensor has the representation,

$$\gamma\mathbf{I} = \gamma(\mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j)(\mathbf{e}_i \otimes \mathbf{e}_j)$$
$$= \gamma\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$
$$= \gamma\mathbf{e}_i \otimes \mathbf{e}_i$$

- Setting $\gamma = 1$, it becomes clear that our familiar Kronecker Deltas are the scalar components of the identity tensor!
- Setting $\gamma = 0$, we find that all components of the Annihilator tensor are zero.

Identity & Annihilator Tensors

- Every component of the Annihilator Tensor is zero. The diagonal elements of the Identity Tensor are each unity with off diagonal elements vanishing:
- The Annihilator Tensor is justifiably called the Zero Tensor. In Matrix Form, these special tensors are

$$\mathbf{o} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}, \qquad \mathbf{I} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

- Note that these components have been established from the definition of the tensors. They were not assumed!

Components of the Dyad

- For $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} = v_j \mathbf{e}_j$, the dyad,

$$\mathbf{u} \otimes \mathbf{v} = u_i v_j \mathbf{e}_i \otimes \mathbf{e}_j$$

- Components of this dyad are $u_i v_j$ on the bases $\mathbf{e}_i \otimes \mathbf{e}_j$.

Products of Tensors

When you are dealing with scalars, there is only one way to define a product.

With vectors, we have defined the scalar multiplication, scalar product, vector product and tensor product. At least four different ways you can take a product.

As you move into higher order quantities, the matter becomes more complicated. Luckily, we will restrict ourselves to specific products that have practical uses. One of these is the product called “Composition”.

The Composition

- The Composition of two tensors produces a tensor of the same order. It is the product with no explicit operator. (Some textbooks use the dot; we shall decline this usage)
- Given that $\mathbf{S}$ and $\mathbf{T}$ are tensors. The sequential operation of the two tensors on a vector produces a vector. The combined effect of the two is the composition product of the two vectors.

$$\mathbf{STu} = \mathbf{S}(\mathbf{Tu}) = \mathbf{Sv} = \mathbf{w}$$

- If $\mathbf{S}, \mathbf{T} \in \mathbb{L}$, and $\mathbf{u} \in \mathbb{E}$, by the definition of the vector space, it is obvious that, $\mathbf{v}, \mathbf{w} \in \mathbb{E}$. Hence, the composition, $\mathbf{ST}$

$$\mathbf{w} = \mathbf{STu}$$

transforms vector $\mathbf{u}$ to vector $\mathbf{w}$ and is therefore a tensor.

Composition of Dyad Bases

- Apply two dyad bases sequentially to a vector:

$$\begin{aligned}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{u} &= (\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{u}) \\ &= \mathbf{e}_\alpha(\mathbf{e}_\beta \cdot \mathbf{e}_i)(\mathbf{e}_j \cdot \mathbf{u}) \\ &= (\mathbf{e}_\beta \cdot \mathbf{e}_i)(\mathbf{e}_\alpha \otimes \mathbf{e}_j)\mathbf{u}\end{aligned}$$

- Showing that composing two dyads has the same effect as obtaining a dyad from the two extreme base vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_j$ in this case, scaling the result by the dot product of the near vectors, $\mathbf{e}_\beta$ and $\mathbf{e}_i$.

Components of the Composition

- Clearly,

$$\begin{aligned}\mathbf{ST} &= \left(S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\\ &= S_{\alpha\beta}T_{ij}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = S_{\alpha\beta}T_{ij}\left(\mathbf{e}_\beta \cdot \mathbf{e}_i\right)\left(\mathbf{e}_\alpha \otimes \mathbf{e}_j\right)\\ &= S_{\alpha\beta}T_{ij}\delta_{\beta i}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_j\right)\\ &= S_{\alpha i}T_{ij}\mathbf{e}_\alpha \otimes \mathbf{e}_j = S_{ik}T_{kj}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

- The result of the product of two dyads in this section can be generalized to a larger number of dyads.

Composition of a Chain of Dyads

Given $\mathbf{a}_{i1}, \mathbf{a}_{i2} \dots \mathbf{a}_{in} \in \mathbb{E}$, the product

$$(\mathbf{a}_{i1} \otimes \mathbf{a}_{i2})(\mathbf{a}_{i3} \otimes \mathbf{a}_{i4}) \dots \left(\mathbf{a}_{i(n-1)} \otimes \mathbf{a}_{in}\right)$$

- can be shown to result in simply taking the first and the last of the vector operands and multiplying the that by the scalar products of all adjacent vectors:

$$(\mathbf{a}_{i1} \otimes \mathbf{a}_{i2})(\mathbf{a}_{i3} \otimes \mathbf{a}_{i4}) \dots \left(\mathbf{a}_{i(n-1)} \otimes \mathbf{a}_{in}\right)$$
$$= (\mathbf{a}_{i1} \otimes \mathbf{a}_{in})(\mathbf{a}_{i2} \cdot \mathbf{a}_{i3})(\mathbf{a}_{i4} \cdot \mathbf{a}_{i5}) \dots \left(\mathbf{a}_{i(n-2)} \cdot \mathbf{a}_{i(n-1)}\right).$$

- Reducing any tensor to a weighted sum of dyads is one way to simplify analyses as the dyads are much easier to deal with for this and several other reasons as we shall see.

The Transpose

- Given any two vectors, $\mathbf{u}$ and $\mathbf{v}$ and tensors $\mathbf{S}$ and $\mathbf{T}$. $\mathbf{S}$ is called the transpose of $\mathbf{T}$ if,

$$\mathbf{u} \cdot \mathbf{S}\mathbf{v} = \mathbf{v} \cdot \mathbf{T}\mathbf{u}$$

- It is customary to use the same symbols for a tensor and its transpose. Accordingly, the transpose of $\mathbf{S}$ will be written as $\mathbf{S}^{\mathrm{T}}$. Furthermore, by virtue of the definition of transpose here, if $\mathbf{S}$ is the transpose of $\mathbf{T}$, then $\mathbf{T}$ is the transpose of $\mathbf{S}$

The Transpose of a Dyad

- Given the dyad $\mathbf{a} \otimes \mathbf{b}$. For any two vectors, $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \cdot (\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{u} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{v}) = \mathbf{v} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{u}$$

- This shows that the transpose of a dyad is simply the swapping of its operands. Clearly,

$$(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = (\mathbf{b} \otimes \mathbf{a})$$

- To transpose a dyad we simply reverse its two constituent vectors as shown. Once a tensor is in component form, it is expressed in terms of dyads. Its transpose is readily obtained.

Symmetry

- A tensor indistinguishable from its transpose is said to be **symmetrical**. Tensor **S** is symmetrical if,

$$\mathbf{S} = \mathbf{S}^{\mathrm{T}}$$

- Furthermore, $\mathbf{S} = \mathbf{S}^{\mathrm{T}} \Rightarrow$

$$S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = S_{ij}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)^{\mathrm{T}}$$
$$= S_{ij}\mathbf{e}_j \otimes \mathbf{e}_i = S_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

- so that, symmetry implies $S_{ij} = S_{ji}$.

Invariants: Scalar Functions

Certain Scalar Valued Functions of a Tensor are often more the primary interest of the engineer, than the tensors themselves. Examples include:

- Mechanical Design: Von Mises or Equivalent Stress, Equivalent Strain
- Deformation Analysis: Dilatation, Area Magnification, Volume Ratio
- Vibrations & Control: Natural Frequencies, Mode Shapes, etc

These functions are scalar functions of tensors.

Most important ones are called the Principal Invariants.

Invariants

$$I_1(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr}\,\mathbf{T}$$

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr}\,\mathbf{T}^{\text{c}}, \qquad \text{and}$$

$$I_3(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det\mathbf{T}$$

- Nothing to memorize here: Note that for each arbitrarily chosen set in the triple product, each invariant operates on one vector, two vectors and three vectors respectively.

Trace is a Linear Operator on Tensors

$$I_1(\alpha\mathbf{T} + \beta\mathbf{S}) = \frac{[(\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, (\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, (\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[(\alpha\mathbf{T})\mathbf{a}, \mathbf{b}, \mathbf{c}] + [(\beta\mathbf{S})\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, (\alpha\mathbf{T})\mathbf{b}, \mathbf{c}] + [\mathbf{a}, (\beta\mathbf{S})\mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, \mathbf{b}, (\alpha\mathbf{T})\mathbf{c}] + [\mathbf{a}, \mathbf{b}, (\beta\mathbf{S})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \alpha\frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \beta\frac{[\mathbf{Sa}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Sb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Sc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \alpha I_1(\mathbf{T}) + \beta I_1(\mathbf{S})$$

Trace of a Dyad

$\otimes \to \cdot$

In Chapter One, we used this simple trick to find the trace of a dyad.

We can now show that the computation is correct:

Trace of a Dyad

$$\text{tr}\,(\mathbf{u}\otimes\mathbf{v}) \equiv I_1(\mathbf{u}\otimes\mathbf{v})$$

$$= \frac{[\{(\mathbf{u}\otimes\mathbf{v})\mathbf{e}_1\},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_2\},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_2,\{(\mathbf{u}\otimes\mathbf{v})\mathbf{e}_3\}]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]}$$

$$= \frac{1}{1}\{[v_1\mathbf{u},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,v_2\mathbf{u},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_3,v_3\mathbf{u}]\}$$

$$= \{(v_1\mathbf{u})\cdot(e_{23i}\mathbf{e}_i) + (e_{31i}\mathbf{e}_i)\cdot(v_2\mathbf{u}) + (e_{12i}\mathbf{e}_i)\cdot(v_3\mathbf{u})\}$$

$$= \{(v_1\mathbf{u})\cdot(e_{231}\mathbf{e}_1) + (e_{312}\mathbf{e}_2)\cdot(v_2\mathbf{u}) + (e_{123}\mathbf{e}_3)\cdot(v_3\mathbf{u})\} = v_i u_i$$

$$= \;\mathbf{u}\cdot\mathbf{v}$$

- Beginning from the component representation,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- Taking the trace of both sides, we have,

$$\text{tr}\,\mathbf{T} = T_{ij}\,\text{tr}\big(\mathbf{e}_i \otimes \mathbf{e}_j\big) = T_{ij}\delta_{ij} = T_{ii}$$

- as we have shown earlier that the trace of a dyad is the scalar product of its operands. We note that transposing a tensor does not alter its trace because,

$$\text{tr}\,\mathbf{T}^{\mathrm{T}} = T_{ij}\,\text{tr}\big(\mathbf{e}_j \otimes \mathbf{e}_i\big) = T_{ij}\delta_{ji} = T_{ii} = \text{tr}\,\mathbf{T}.$$

Trace of Tensor in Component Form

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Trace of the Cofactor, $I_2(\mathbf{T}) = \text{tr}\,\mathbf{T}^c$

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr}\,\mathbf{T}^c$$

- To express this in component form, we set our linearly independent set as the basis set, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Note immediately that $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1$, so that,

$$\begin{aligned}
I_2(\mathbf{T}) &= [\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3] \\
&= \left[T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1, T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_2, \mathbf{e}_3\right] \\
&+ \left[\mathbf{e}_1, T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_2, T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right] \\
&+ \left[T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_1, \mathbf{e}_2, T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right] \\
&= [T_{i1}\mathbf{e}_i, T_{\alpha 2}\mathbf{e}_\alpha, \mathbf{e}_3] + [\mathbf{e}_1, T_{\alpha 2}\mathbf{e}_\alpha, T_{i3}\mathbf{e}_i] + [T_{\alpha 1}\mathbf{e}_\alpha, \mathbf{e}_2, T_{i3}\mathbf{e}_i]
\end{aligned}$$

Trace of the Cofactor

- Continuing, we have,

$$\begin{aligned} I_2(\mathbf{T}) &= T_{i1}T_{\alpha 2}[\mathbf{e}_i, \mathbf{e}_\alpha, \mathbf{e}_3] + T_{\alpha 2}T_{i3}[\mathbf{e}_1, \mathbf{e}_\alpha, \mathbf{e}_i] + T_{\alpha 1}T_{i3}[\mathbf{e}_\alpha, \mathbf{e}_2, \mathbf{e}_i] \\ &= T_{i1}T_{\alpha 2}e_{i\alpha 3} + T_{\alpha 2}T_{i3}e_{1\alpha i} + T_{\alpha 1}T_{i3}e_{\alpha 2i} \\ &= T_{11}T_{22} - T_{21}T_{12} + T_{22}T_{33} - T_{32}T_{23} + T_{11}T_{33} - T_{31}T_{13} \\ &= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right) \end{aligned}$$

- Which is half the square of the trace minus the trace of the square of the original tensor. In several books, this is used as the definition of the Second Principal Invariant.

The Determinant, $I_3(\mathbf{T}) = \det \mathbf{T}$

- As previously observed, any three linearly independent vectors can be treated as the basis for defining the invariants. We select $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. For any tensor $\mathbf{T}$,

$$
\begin{aligned}
I_3(\mathbf{T}) &= [\mathbf{T}\mathbf{e}_1, \mathbf{T}\mathbf{e}_2, \mathbf{T}\mathbf{e}_3] \\
&= \left[T_{ij}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_1, T_{\alpha\beta}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\mathbf{e}_2, T_{rs}(\mathbf{e}_r \otimes \mathbf{e}_s)\mathbf{e}_3\right] \\
&= [T_{i1}\mathbf{e}_i, T_{\alpha 2}\mathbf{e}_\alpha, T_{r3}\mathbf{e}_r] = T_{i1}T_{\alpha 2}T_{r3}e_{i\alpha r} \\
&= T_{i1}T_{j2}T_{k3}e_{ijk} = \det \mathbf{T}
\end{aligned}
$$

Note & Poser on Linearity

- While the trace is a linear scalar function of its tensor argument, the trace of the Cofactor and the Determinant are not.
- The Inverse and the Cofactor are tensor functions of the same tensor argument.
- They are also not linear functions.
- Does this contradict the earlier assertion that the Inverse Transformation is a linear transformation and therefore a tensor?

Determinant of a Product

- For the tensors $\mathbf{A}$ and $\mathbf{B}$, we use the definition of the determinant to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$:
- Select linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$ and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent.

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

The Inner Product Tensors

- The inner product of tensors $\mathbf{S}$ and $\mathbf{T}$ is the trace
$$\mathbf{S}:\mathbf{T} \equiv \text{tr}\left(\mathbf{S}^{\text{T}}\mathbf{T}\right) = \text{tr}(\mathbf{S}\mathbf{T}^{\text{T}})$$
- For the tensor, $\mathbf{T} = T_{\alpha\beta}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)$, this definition leads to a simple formula for the components of tensors as we shall show:
- Consider the composition, $\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)$. We take the trace of this composition and obtain,
$$\text{tr}\left[\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right] = \text{tr}\left[T_{\alpha\beta}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right]$$
$$= \text{tr}\left[T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_i)\delta_{\beta j}\right]$$
$$= T_{\alpha\beta}\delta_{\alpha i}\delta_{\beta j} = T_{ij}$$

The Inner Product Tensors

- From the above definition, the scalar components of tensor $\mathbf{T}$ on the dyad bases $\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$ is given by,

$$\begin{aligned} T_{ij} &= \mathrm{tr}\left[\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right] \\ &= \mathrm{tr}\left[\mathbf{T}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)^{\mathrm{T}}\right] \\ &= \mathbf{T} : \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \end{aligned}$$

- The definition of the inner product of tensors leads to a simple way to compute the components. Just like vectors, it is by simply taking the inner product with the product base.