Properties of Tensors II

MEG 324 SSG 321 Introduction to Continuum Mechanics Instructors: OA Fakinlede & O Adewumi

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Last Week: Outstanding Issues

- It is expected that you should have read up on the notes and slides from last week. I will not ordinarily repeat them in this week's matter but there are two issues that can prevent you from coping if, you have not thoroughly mastered them:
  - Determinant of a Product
  - Trace of a Composition, and
  - Scalar Product of Two Tensors

Determinant of a Product

- For the tensors $\mathbf{A}$ and $\mathbf{B}$, we use the definition of the determinant to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$:
- Select linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$ and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent.

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

invertible

Q2.25

$$\frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]}$$

det A

det B

The Inner Product Tensors

- The inner product of tensors $\mathbf{S}$ and $\mathbf{T}$ is the trace

$$\mathbf{S}:\mathbf{T} \equiv \operatorname{tr}\left(\mathbf{S}^{\mathrm{T}}\mathbf{T}\right) = \operatorname{tr}\left(\mathbf{S}\mathbf{T}^{\mathrm{T}}\right)$$

- For the tensor, $\mathbf{T} = T_{\alpha\beta}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)$, this definition leads to a simple formula for the components of tensors as we shall show:
- Consider the composition, $\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)$. We take the trace of this composition and obtain,

$$\operatorname{tr}\left[\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right] = \operatorname{tr}\left[T_{\alpha\beta}\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right]$$

$$= \operatorname{tr}\left[T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_i)\delta_{\beta j}\right]$$

$$= T_{\alpha\beta}\delta_{\alpha i}\delta_{\beta j} = T_{ij}$$

The Inner Product Tensors

- From the above definition, the scalar components of tensor $\mathbf{T}$ on the dyad bases $(\mathbf{e}_i \otimes \mathbf{e}_j)$ is given by,

$$\begin{aligned} T_{ij} &= \text{tr}\,[\mathbf{T}(\mathbf{e}_j \otimes \mathbf{e}_i)] \\ &= \text{tr}[\mathbf{T}(\mathbf{e}_i \otimes \mathbf{e}_j)^{\mathrm{T}}] \\ &= \mathbf{T}\!:\!(\mathbf{e}_i \otimes \mathbf{e}_j) \end{aligned}$$

- The definition of the inner product of tensors leads to a simple way to compute the components. Just like vectors, it is by simply taking the inner product with the product base.

Properties Covered Here

- In continuation of our closer examination of the properties of tensors; we shall cover the properties listed here.
- The Mathematics may look cumbersome, the strategy is simple: Focus on definitions and principles, all that will be left is the repeated applications of simple rules.
- You will surprise yourself on what you will know at the end!

	Property
1.	Members of a Euclidean Vector Space
2.	Additive Decompositions (4)
3.	The Cofactor Tensor
4.	Orthogonal Tensors
5.	Vector Cross: Other side of the Coin Axial Vector

The Tensor: A Euclidean Vector Space

One consequence of the foregoing is the important fact that the set of tensors, as we have defined it, constitutes a Euclidean Vector Space.

With this approach, the definition of a tensor as a linear transformation of vectors, applies to tensors of higher orders!

Tensor Magnitude & Direction

- In the same way as vectors, the inner product of tensors induces the concept of magnitude and direction to tensors.
  - Inspired by the fact that $\mathbf{T}:\mathbf{T}$ is a scalar, we define the magnitude of a tensor

$$||\mathbf{T}|| = \sqrt{\mathbf{T}:\mathbf{T}}$$

- The angle between two tensors can be computed from,

$$\theta = \cos^{-1}\frac{\mathbf{S}:\mathbf{T}}{||\mathbf{S}||\,||\mathbf{T}||}.$$

- Unlike vectors, these values do not have the familiar geometric interpretation of directed lines and included angles. The algebra trumps the geometry.

The Tensor Set as a Vector Space

A second-order tensor fulfils all the stipulations necessary to be a Euclidean Vector Space $\mathbb{L}$:

- **Addition operation** is defined, it is commutative and associative under $\mathbb{L}$: that is, $\mathbf{T} + \mathbf{S} \in \mathbb{L}$, $\mathbf{S} + \mathbf{T} = \mathbf{T} + \mathbf{S}$, $\mathbf{T} + (\mathbf{S} + \mathbf{V}) = (\mathbf{T} + \mathbf{S}) + \mathbf{V}, \forall\ \mathbf{T}, \mathbf{S}, \mathbf{V} \in \mathbb{L}$. Furthermore, $\mathbb{L}$ is closed under addition: That is, given that $\mathbf{T}, \mathbf{S} \in \mathbb{L}$, then $\mathbf{V} = \mathbf{T} + \mathbf{S} = \mathbf{S} + \mathbf{T}, \Rightarrow \mathbf{V} \in \mathbb{L}$.

3 AXIOMS

The Tensor Set as a Vector Space

- $\mathbb{L}$ **contains a zero element** $\mathbf{O}$ such that $\mathbf{T} + \mathbf{O} = \mathbf{T}\ \forall\ \mathbf{T} \in \mathbb{L}$. For every $\mathbf{T} \in \mathbb{L},\ \ \exists -\mathbf{T}: \mathbf{T} + (-\mathbf{T}) = \mathbf{O}$.
- **Multiplication by a scalar.** For $\alpha, \beta \in \mathbb{R}$ and $\mathbf{T}, \mathbf{S} \in \mathbb{L}$, $\alpha\mathbf{T} \in \mathbb{L}$, $1\mathbf{T} = \mathbf{T}, \alpha(\beta\mathbf{T}) = (\alpha\beta)\mathbf{T}, (\alpha + \beta)\mathbf{T} = \alpha\mathbf{T} + \beta\mathbf{T},\ \alpha(\mathbf{T} + \mathbf{S}) = \alpha\mathbf{T} + \alpha\mathbf{S}$.

3 AXIOMS

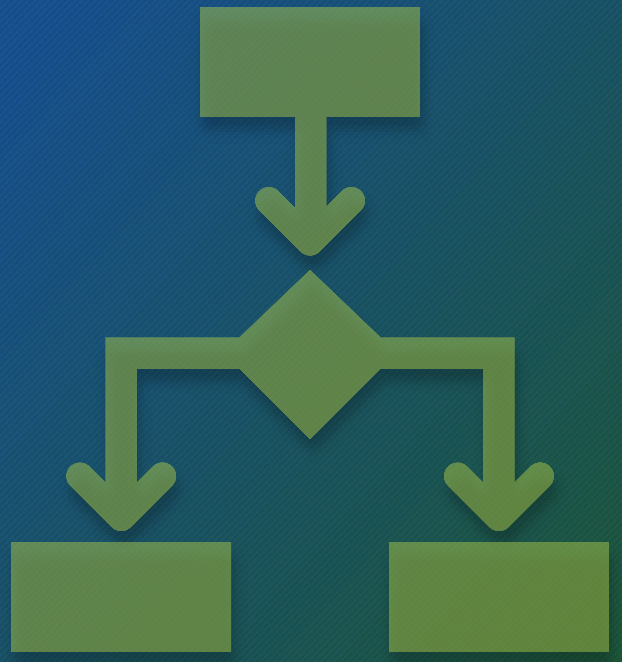
Axioms for Second-Order Tensors

- The Zero element in the Second Axiom is obviously the Annihilator Tensor.
- The remaining two axioms are easily established by recalling that dyads can be added
  - We break the tensor into its components as follows:

$$
\begin{aligned}
(\alpha\mathbf{S}+\beta\mathbf{T})\mathbf{u} &= \left(\alpha S_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)+\beta T_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)\right)u_l\mathbf{e}_l \\
&= \left(\alpha S_{ij}+\beta T_{ij}\right)u_l\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)\mathbf{e}_l \\
&= \left(\alpha S_{ij}+\beta T_{ij}\right)u_l\mathbf{e}_i\delta_{jl} = \alpha S_{ij}u_j\mathbf{e}_i \\
&= \alpha\mathbf{S}\mathbf{u}+\beta\mathbf{T}\mathbf{u}
\end{aligned}
$$

Decomposing Tensors

- We break vectors to their components in only one way. Tensors can be usefully broken down in several different ways.
- Some of these decompositions are, like the component form, additive. Others are multiplicative.
  - Additive decompositions include Skew & Symmetric Forms, Spherical and Deviatoric Forms, Spectral Representation
  - The Important Multiplicative is the Polar Decomposition.



Decomposing Tensors

Component Form: Weighted sum of Dyads adds up to the Tensor

Components are the weights. This turns the tensor into a weighted sum of up to nine dyad bases.

Three other Additive Decompositions are possible.

Symmetric & Skew Parts, and

Spherical & Deviatoric Parts

Spectral Form of eigenbases

There are also Multiplicative Decompositions of Tensors

Spherical & Deviatoric Parts

- Every tensor can be decomposed into Spherical and Deviatoric parts. The Spherical Part of a tensor is obtained by dividing its trace by three and using the result to scale an identity tensor. For a tensor $\mathbf{S}$,
  - Spherical Part, the real multiplier of the Identity Tensor, $\gamma = \frac{1}{3}\operatorname{tr}\mathbf{S}$ so that,

$$\operatorname{sph}\mathbf{S} = \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{I} = \frac{1}{3}S_{kk}\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

  - Deviatoric Part is what remains after removing the Spherical Part:

$$\operatorname{dev}\mathbf{S} = \mathbf{S} - \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{I} = \left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

Traces of Spherical & Deviatoric Parts

- For the Spherical Part, the trace,

$$\text{tr}(\text{sph}\,\mathbf{S}) = \left(\frac{1}{3}\text{tr}\,\mathbf{S}\right)\text{tr}\,\mathbf{I} = \left(\frac{1}{3}\text{tr}\,\mathbf{S}\right)3 = \text{tr}\,\mathbf{S}$$

equals the trace of the original, undecomposed tensor.

- For the Deviatoric Part, the trace,

$$\text{tr}(\text{dev}\,\mathbf{S}) = \text{tr}\,\mathbf{S} - \left(\frac{1}{3}\text{tr}\,\mathbf{S}\right)\text{tr}\,\mathbf{I} = \text{tr}\,\mathbf{S} - \text{tr}\,\mathbf{S} = 0.$$

- The deviatoric component has zero trace. A tensor with zero trace is said to be “traceless”.

Skew & Symmetric Parts

- We can also decompose a tensor $\mathbf{S}$, into Symmetrical and Anti-Symmetrical parts.
  - An Anti-Symmetric tensor, also called a Skew Tensor is defined as that which is the negative of its transpose.
- The Symmetric Part,

$$\operatorname{sym}\mathbf{S} = \frac{1}{2}\left(\mathbf{S} + \mathbf{S}^{\mathrm{T}}\right) = \frac{1}{2}\left(S_{ij} + S_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

- And the Skew Part:

$$\operatorname{skw}\mathbf{S} = \frac{1}{2}\left(\mathbf{S} - \mathbf{S}^{\mathrm{T}}\right) = \frac{1}{2}\left(S_{ij} - S_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

Skew & Symmetric Parts

- The transposes,
- $$(\operatorname{sym}\mathbf{S})^{\mathrm{T}} = \left(\frac{1}{2}\left(\mathbf{S} + \mathbf{S}^{\mathrm{T}}\right)\right)^{\mathrm{T}} = \frac{1}{2}\left(\mathbf{S}^{\mathrm{T}} + \left(\mathbf{S}^{\mathrm{T}}\right)^{\mathrm{T}}\right) = \operatorname{sym}\mathbf{S}$$
  $$(\operatorname{skw}\mathbf{S})^{\mathrm{T}} = \left(\frac{1}{2}\left(\mathbf{S} - \mathbf{S}^{\mathrm{T}}\right)\right)^{\mathrm{T}} = \frac{1}{2}\left(\mathbf{S}^{\mathrm{T}} - \left(\mathbf{S}^{\mathrm{T}}\right)^{\mathrm{T}}\right) = -\operatorname{skw}\mathbf{S}$$

Transpose of a sum equals the sum of the transposes:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{S}+\mathbf{T})^{\mathrm{T}}\mathbf{b} &= \mathbf{b}\cdot(\mathbf{S}+\mathbf{T})\mathbf{a}\\ &= \mathbf{b}\cdot\mathbf{S}\mathbf{a} + \mathbf{b}\cdot\mathbf{T}\mathbf{a}\\ &= \mathbf{a}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{b} + \mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b}\end{aligned}$$

And, further, transpose of a transpose is the original vector

Component Method

- The Symmetric Part,

$$(\text{sym}\,\mathbf{S})^{\mathrm{T}} = \frac{1}{2}\left(S_{ij} + S_{ji}\right)\mathbf{e}_j \otimes \mathbf{e}_i = \text{sym}\,\mathbf{S}$$

- And the Skew Part:

$$\begin{aligned}(\text{skw}\,\mathbf{S})^{\mathrm{T}} &= \frac{1}{2}\left(S_{ij} - S_{ji}\right)\mathbf{e}_j \otimes \mathbf{e}_i \\ &= -\frac{1}{2}\left(S_{ij} - S_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j = -\text{skw}\,\mathbf{S}\end{aligned}$$

- It is often more instructive to prove these in the direct form whenever possible.

Traces of Skew & Symmetric Parts

$$\begin{aligned}\operatorname{tr} \mathbf{S}^{\mathrm{T}} &= S_{ij} \operatorname{tr}(\mathbf{e}_j \otimes \mathbf{e}_i) \\ &= S_{ij}(\mathbf{e}_j \cdot \mathbf{e}_i) \\ &= S_{ij}(\mathbf{e}_i \cdot \mathbf{e}_j) \\ &= S_{ij} \operatorname{tr}(\mathbf{e}_i \otimes \mathbf{e}_j) = \operatorname{tr} \mathbf{S}\end{aligned}$$

$$\operatorname{tr}(\operatorname{sym} \mathbf{S}) = \frac{1}{2} \operatorname{tr}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(\operatorname{tr} \mathbf{S} + \operatorname{tr} \mathbf{S})$$

- commutative property of the scalar product makes the trace of a transpose the same as the trace of the tensor from which the transpose is obtained.
- For the same reason, a Skew tensor is traceless:

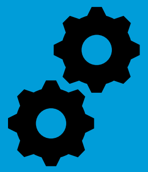
$$\operatorname{tr}(\operatorname{skw} \mathbf{S}) = \frac{1}{2} \operatorname{tr}(\mathbf{S} - \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(\operatorname{tr} \mathbf{S} - \operatorname{tr} \mathbf{S}) = 0.$$

- The Spherical Part of a tensor is a diagonal tensor, and therefore, always symmetric.

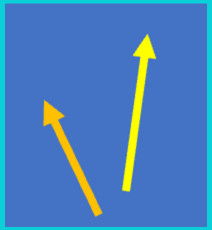
Symmetry of Deviatoric Parts

- No judgement can be made on the symmetry or skewness of a deviatoric tensor, however. Its symmetry wholly depends on the original tensor from which the deviatoric part is taken.
    - If the latter is symmetric, so will the deviatoric part. If skew, so also will the deviatoric part.
    - It is quite possible that the deviatoric tensor is neither symmetric nor skew.

Transformation of Line Elements



In Continuum Mechanics, we are concerned with changes in shape of objects that have been subjected to mechanical loads.



We shall see later that these creates important tensors such as stress, strain and deformation tensors.

We are interested in what happens to line elements in materials where such strain and deformation tensors have been created by external loads.

- It will be our business to demonstrate (Chapter 4, Kinematics) that small line elements are transformed by applying the deformation gradient tensor to the original, vectors of undeformed line elements.
- What can we say about vectors of small areas in the deformed state?
- We will show that when a tensor transforms lines, the corresponding vector areas are transformed by its Cofactor.

Cofactor Tensor & Area Vector

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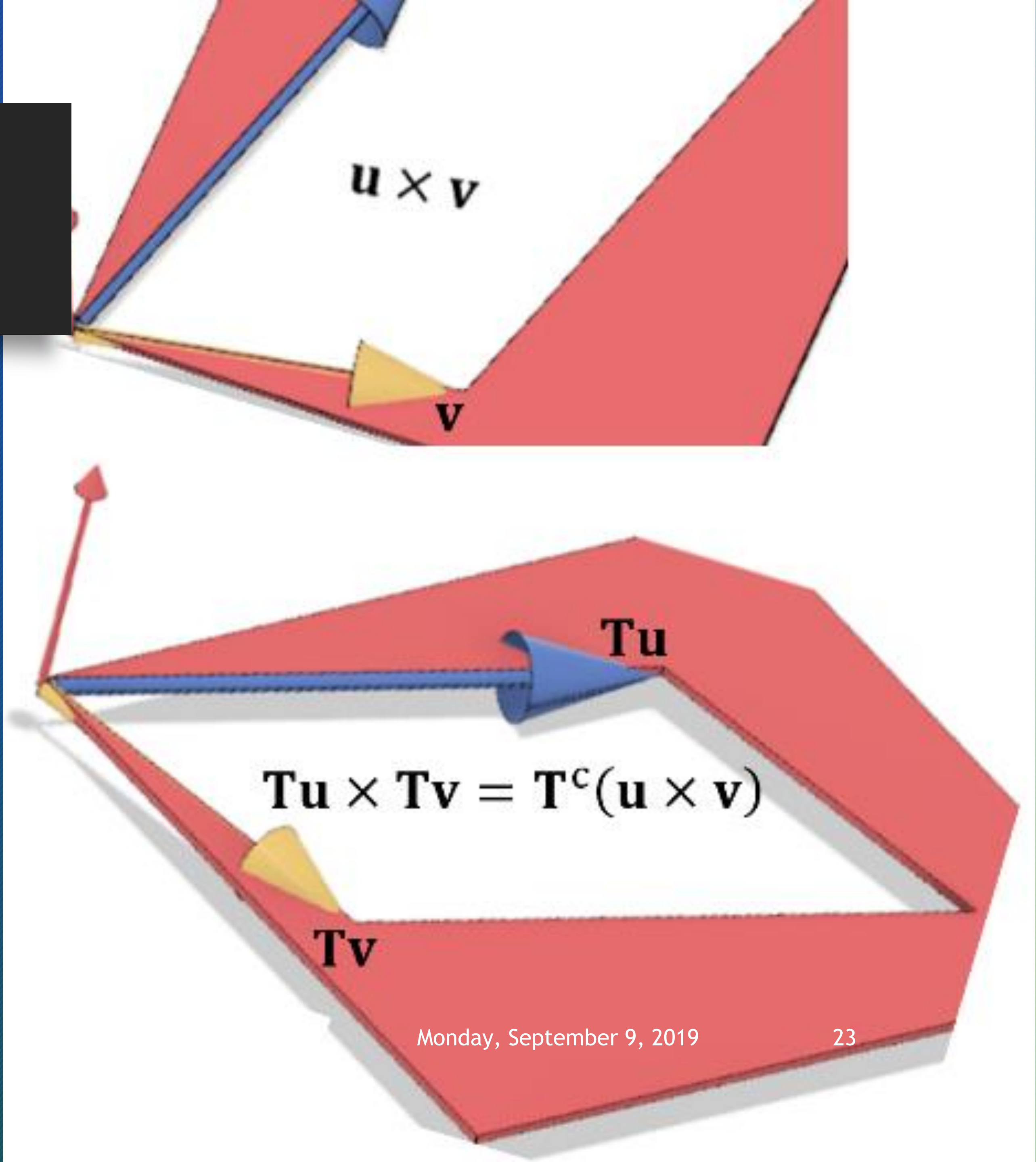
The Cofactor & Area Vector

- Vectors $\mathbf{u}$ and $\mathbf{v}$ have both been transformed by the same tensor $\mathbf{T}$ to $\mathbf{Tu}$ and $\mathbf{Tv}$.
  - How is the vector area $\mathbf{Tu} \times \mathbf{Tv}$ created by these two vectors in the transformed state related to the original vector area $\mathbf{u} \times \mathbf{v}$ of the parallelogram as shown?
- The tensor transforming the area of parallelogram sides that have been transformed by tensor $\mathbf{T}$ is called the cofactor, $\mathbf{T}^c$ of $\mathbf{T}$.
  - This is the physical meaning of the cofactor. Any other definition can be obtained from this physical definition.



The Cofactor is a Bilinear Transformation

- For example, given $\alpha, \beta \in \mathbb{R}$, and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}$, the linearity of tensor $\mathbf{T} \Rightarrow$

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{Tw} &= (\alpha\mathbf{Tu} + \beta\mathbf{Tv}) \times \mathbf{Tw} \\ &= \alpha\mathbf{Tu} \times \mathbf{Tw} + \beta\mathbf{Tv} \times \mathbf{Tw} \\ &= \alpha\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{w}) + \beta\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{w}) \\ &= \mathbf{T}^{\mathrm{c}}\big((\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w}\big)\end{aligned}$$

- The last equality coming from the linearity of cofactor tensor $\mathbf{T}^{\mathrm{c}}$, and the distributive property of the vector product over addition.

Components of the Cofactor

- For a tensor $\mathbf{T}$, let the cofactor,

$$\text{cof}\,\mathbf{T} = \mathbf{T}^{\text{c}} = T_{ij}^{c}\mathbf{e}_i \otimes \mathbf{e}_j$$

- Clearly, the component,

$$T_{ij}^{c} = \mathbf{T}^{\text{c}}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = \mathbf{e}_i \cdot \mathbf{T}^{\text{c}}\mathbf{e}_j$$

$$= \mathbf{e}_i \cdot \left[\mathbf{T}^{\text{c}}\left(\frac{1}{2}e_{jmn}\mathbf{e}_m \times \mathbf{e}_n\right)\right]$$

$$= \frac{1}{2}e_{jmn}\mathbf{e}_i \cdot [\mathbf{T}^{\text{c}}(\mathbf{e}_m \times \mathbf{e}_n)]$$

Components of the Cofactor

- In the expression, $\mathbf{e}_i \cdot (\mathbf{Te}_m \times \mathbf{Te}_n)$ we seek the i^{th} component of $\mathbf{Te}_m \times \mathbf{Te}_n$. To get this, we remember that the α and β components of each operand are $\mathbf{e}_\alpha \cdot \mathbf{Te}_m$ and $\mathbf{e}_\beta \cdot \mathbf{Te}_n$ respectively. Consequently we seek, $e_{i\alpha\beta}(\mathbf{e}_\alpha \cdot \mathbf{Te}_m)\left(\mathbf{e}_\beta \cdot \mathbf{Te}_n\right)$ so that,

$$T_{ij}^c = \frac{1}{2} e_{jmn} e_{i\alpha\beta} (\mathbf{e}_\alpha \cdot \mathbf{Te}_m)\left(\mathbf{e}_\beta \cdot \mathbf{Te}_n\right)$$
$$= \frac{1}{2} e_{i\alpha\beta} e_{jmn} T_{\alpha m} T_{\beta n}$$

- The Cofactor Tensor, in component form,

$$\mathbf{T}^c = \frac{1}{2} e_{i\alpha\beta} e_{jmn} T_{\alpha m} T_{\beta n} \left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$$

For any vector $\mathbf{u}$,
$$\mathbf{e}_i \cdot \mathbf{u} = u_i$$
The i^{th} component of $\mathbf{u}$

Components of the Cofactor

- Second principal invariant of $\mathbf{T}$ is the trace of its cofactor, $\operatorname{tr}\mathbf{T}^{\mathrm{c}}$

$$
\begin{aligned}
I_2(\mathbf{T}) &= \frac{1}{2} e_{i\alpha\beta} e_{jmn} T_{\alpha m} T_{\beta n} \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) \\
&= \frac{1}{2} \left(\delta_{\alpha m}\delta_{\beta n} - \delta_{\alpha n}\,\delta_{\beta m}\right) T_{\alpha m} T_{\beta n} \\
&= \frac{1}{2} \left(T_{mm} T_{nn} - T_{mn} T_{nm}\right) \\
&= \frac{1}{2} \left(\operatorname{tr}^2 \mathbf{T} - \operatorname{tr}\mathbf{T}^2\right)
\end{aligned}
$$

- This is exactly same expression we obtained from the definition of the second Principal Invariant.

Orthogonal Tensors

- In our study of tensor properties, we encountered some interesting tensors. The table here shows some of these tensors as we classify them into two groups as shown.
- Tensors in the first group change the magnitude of the vectors they transform; the second group are tensors that create transformed vectors with the same magnitudes as the input vector.
- The second group are just two members of a distinguished class of tensors we will need to spend a little more time to know better. They are in a class called "Orthogonal tensors".

CLASS ONE		CLASS TWO	
Projection	$\mathbf{P}_\mathbf{x} \equiv \left(\frac{1}{\|\mathbf{x}\|}\right)^2 (\mathbf{x} \otimes \mathbf{x})$	Identity	$\mathbf{I}$
Vector Cross	$(\mathbf{v}\times) \equiv -e_{ijk} v_k \mathbf{e}_i \otimes \mathbf{e}_j$	Coordinate Rotation	$\boldsymbol{\xi}_i \otimes \mathbf{e}_i$
Spherical	$\gamma \mathbf{I}$		
Annihilator	$\mathbf{O}$		
Dyad	$\mathbf{a} \otimes \mathbf{b}$		

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Spherical	$\gamma \mathbf{I}$		
Annihilator	$\mathbf{O}$		
Dyad	$\mathbf{a} \otimes \mathbf{b}$		

Orthogonal Tensors

Definition: What is an Orthogonal Tensor?

- Given a pair of vectors $\mathbf{a}$ and $\mathbf{b}$, an orthogonal tensor $\mathbf{Q}$ is said to be orthogonal if,
$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{a} \cdot \mathbf{b}$$
- Specifically, we can allow $\mathbf{a} = \mathbf{b}$, so that
$$(\mathbf{Qa}) \cdot (\mathbf{Qa}) = \mathbf{a} \cdot \mathbf{a}$$
- Or
$$\|\mathbf{Qa}\| = \|\mathbf{a}\|$$
- As we can see, one important attribute of the orthogonal tensor is that the magnitude of the input as well as that of the result remain the same: A magnitude-preserving transformation.

- Let $\mathbf{q} = \mathbf{Qa}$

$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{q} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- Recall the definition of the transpose; we have that,

$$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$

- Clearly, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}$. A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse equal to its transpose.
  - Every orthogonal tensor possesses an inverse;
  - This inverse is simply its transpose!

Definition: What is an Orthogonal Tensor?

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The Determinant of an Orthogonal Tensor

- Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) = \left(\det \mathbf{Q}^{\mathrm{T}}\right)(\det \mathbf{Q}) = (\det \mathbf{Q})^2 = 1$$

- Which clearly shows that

$$(\det \mathbf{Q}) = \pm 1$$

- When the determinant of an orthogonal tensor is strictly positive, it is called “proper orthogonal”. In particular, a rotation is a proper orthogonal tensor while a reflection is not.

Proper Orthogonal Tensors are Self-Cofactor

For any pair of vectors $\mathbf{u}, \mathbf{v}$ we show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$

- This question is the same as showing that the cofactor of $\mathbf{Q}$ is $\mathbf{Q}$ itself. That is that a rotation is self cofactor. We can write that
$$\mathbf{Q}^{\mathbf{c}}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$$
- where
$$\mathbf{T} = \mathrm{cof}(\mathbf{Q}) = \det(\mathbf{Q})\,\mathbf{Q}^{-\mathrm{T}}$$
- Now that $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = 1$, and
$$\mathbf{Q}^{-\mathrm{T}} = (\mathbf{Q}^{-1})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q}$$
- This implies that $\mathbf{T} = \mathbf{Q}$ and consequently,
$$\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$$

One Eigenvalue of a Proper Orthogonal is +1

This means that there is always a solution for the equation,

$$\mathbf{Qu} = \mathbf{u}$$

- For any invertible tensor,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

- For a proper orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = 1$. It therefore follows that,

$$\mathbf{Q}^{\mathrm{c}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

- Characteristic equation for $\mathbf{Q}$ is,

$$\det(\mathbf{Q} - \lambda\mathbf{I}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

- Or,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

- Which is obviously satisfied by $\lambda = 1$.

The Axial Vector

- The Vector Cross is a Skew Tensor.
  - This fact can be seen by transposing and showing that the transpose is its negative.
  - Like all other skew tensors, it is a traceless tensor.
- Now the question: Given a Skew tensor, can we find a vector that the tensor is a vector cross of?
  - When this happens, the vector found in this way is called the axial vector of the tensor.
  - Note that, in either case, there are only three components. This is obvious for the vector.
  - For the skew tensor, once three independent components are found, the remaining are simply the negatives of the independent components.

The Axial Vector

- From Week Six (Slide 26) the vector cross was defined as:
$$\boldsymbol{\Omega} = (\mathbf{v}\,\times) \equiv -e_{ijk}v_k\mathbf{e}_i \otimes \mathbf{e}_j = \Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$
- Given $\boldsymbol{\Omega} = \Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, can we solve the above equation for the components v_k?
  - Sure, we can! Begin from,
$$-e_{ijk}v_k = \Omega_{ij}$$

  Multiply both sides by $e_{\alpha ij}$ and we have:
$$-e_{ijk}e_{\alpha ij}v_k = -2\delta_{k\alpha}v_k = e_{\alpha ij}\Omega_{ij}$$
- The components of the axial vector are $v_i = -\frac{1}{2}e_{ijk}\Omega_{jk}$