Tutorial Two

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Q2.9 Show that $[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$

The LHS can be written as:

$$[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = e_{ijk}(\mathbf{S}^{\mathrm{c}}\mathbf{u})_j\mathbf{e}_i\otimes\mathbf{e}_k$$

where $\mathbf{S}^c = \frac{1}{2}e_{jab}e_{\beta cd}S_{ac}S_{bd}\ \mathbf{e}_j\otimes\mathbf{e}_\beta$ so that

$$\mathbf{S}^{\mathrm{c}}\mathbf{u} = \left(\frac{1}{2}e_{jab}e_{\beta cd}S_{ac}S_{bd}\mathbf{e}_j\otimes\mathbf{e}_\beta\right)(u_m\mathbf{e}_m)$$

$$= \frac{1}{2}e_{jab}e_{\beta cd}S_{ac}S_{bd}\mathbf{e}_j\delta_{\beta m}u_m$$

$$= \frac{1}{2}e_{jab}e_{\beta cd}u_\beta S_{ac}S_{bd}\mathbf{e}_j$$

Consequently,

$$[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \frac{1}{2}e_{ijk}e_{jab}e_{\beta cd}u_\beta S_{ac}S_{bd}\mathbf{e}_i\otimes\mathbf{e}_k$$

$$= \frac{1}{2}e_{\beta cd}(\delta_{ka}\delta_{ib}-\delta_{kb}\delta_{ia})u_\beta S_{ac}S_{bd}\mathbf{e}_i$$

$$= \frac{1}{2}e_{\beta cd}u_\beta(S_{kc}S_{id}-S_{ic}S_{kd})\mathbf{e}_i\otimes\mathbf{e}_k$$

$$= \frac{1}{2}e_{\beta cd}u_\beta S_{kc}S_{id}\mathbf{e}_i\otimes\mathbf{e}_k - \frac{1}{2}e_{\beta cd}u_\beta S_{ic}S_{kd}\mathbf{e}_i\otimes\mathbf{e}_k$$

$$= \frac{1}{2}e_{\beta cd}u_\beta S_{kc}S_{id}\mathbf{e}_i\otimes\mathbf{e}_k + \frac{1}{2}e_{\beta dc}u_\beta S_{ic}S_{kd}\mathbf{e}_i\otimes\mathbf{e}_k$$

$$= e_{\beta cd}u_\beta S_{kc}S_{id}\mathbf{e}_i\otimes\mathbf{e}_k$$

On the RHS

$$(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}} = (e_{\alpha\beta\gamma}u_\beta\mathbf{e}_\alpha\otimes\mathbf{e}_\gamma)(S_{ki}\mathbf{e}_i\otimes\mathbf{e}_k)$$

$$= e_{\alpha\beta\gamma}u_\beta S_{k\gamma}\mathbf{e}_\alpha\otimes\mathbf{e}_k.$$

We can therefore write,

$$\mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}} = (S_{ir}\mathbf{e}_i\otimes\mathbf{e}_r)(e_{\alpha\beta\gamma}u_\beta S_{k\gamma}\mathbf{e}_\alpha\otimes\mathbf{e}_k)$$

$$= e_{\alpha\beta\gamma}u_\beta S_{i\alpha}S_{k\gamma}\mathbf{e}_i\otimes\mathbf{e}_k$$

which on $\alpha\to d,\gamma\to c$ is the same as the LHS $\Rightarrow$

$$[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$$

as required.

Q2.20 Given an arbitrary tensor $\mathbf{T}$ a skew tensor $\mathbf{W}$ and a symmetric tensor $\mathbf{S}$. Show that

$$\mathbf{S}:\mathbf{T} = \mathbf{S}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\operatorname{sym}\mathbf{T}$$

$$\mathbf{W}:\mathbf{T} = -\mathbf{W}:\mathbf{T}^{\mathrm{T}}$$

Note that

$$\mathbf{T} = \operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T},$$

and

$$\mathbf{T}^{\mathrm{T}} = \operatorname{sym}\mathbf{T} - \operatorname{skw}\mathbf{T}$$

Also note that the inner product between a skew and a symmetric tensor vanishes. Consequently,

$$\begin{aligned}\mathbf{S}:\mathbf{T} &= \mathbf{S}:(\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T})\\ &= \mathbf{S}:\operatorname{sym}\mathbf{T} + \mathbf{S}:\operatorname{skw}\mathbf{T}\\ &= \mathbf{S}:\operatorname{sym}\mathbf{T}\\ &= \mathbf{S}:\mathbf{T}^{\mathrm{T}}\end{aligned}$$

$$\begin{aligned}\mathbf{W}:\mathbf{T} &= \mathbf{W}:(\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T})\\ &= \mathbf{W}:\operatorname{sym}\mathbf{T} + \mathbf{W}:\operatorname{skw}\mathbf{T}\\ &= \mathbf{W}:\operatorname{skw}\mathbf{T} = \mathbf{W}:\left(\operatorname{sym}\mathbf{T} - \mathbf{T}^{\mathrm{T}}\right) = -\mathbf{W}:\mathbf{T}^{\mathrm{T}}\end{aligned}$$

Q2.20b Given an arbitrary tensor $\mathbf{T}$ a skew tensor $\mathbf{W}$ and a symmetric tensor $\mathbf{S}$. Show that

$$\mathbf{S}:\operatorname{skw}\mathbf{T} = \mathbf{S}:\mathbf{W} = 0$$

To show that $\mathbf{S}:\mathbf{W} = 0$. Observe that, in component form,

$\mathbf{S} = S_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)$, $\mathbf{W} = W_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)$.

$$\mathbf{S}^{\mathrm{T}}\mathbf{W} = S_{ij}W_{\alpha\beta}(\mathbf{e}_j \otimes \mathbf{e}_i)(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)$$

$$= S_{ij}\ W_{\alpha\beta}(\mathbf{e}_j \otimes \mathbf{e}_\beta)\delta_{i\alpha} = S_{ij}W_{i\beta}\mathbf{e}_j \otimes \mathbf{e}_\beta$$

$$\operatorname{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{W}) = S_{ij}W_{\alpha\beta}\operatorname{tr}(\mathbf{e}_j \otimes \mathbf{e}_\beta)\delta_{i\alpha} = S_{ij}W_{\alpha\beta}\delta_{j\beta}\delta_{i\alpha}$$

$$= S_{ij}W_{ij} = \mathbf{S}:\mathbf{W}$$

$$= S_{ji}W_{ij} = -S_{ji}W_{ji} = -S_{ij}W_{ij} = -\mathbf{S}:\mathbf{W}$$

Which vanishes because it is equal to the negative of itself.

$$\mathbf{S}:\operatorname{skw}\mathbf{T} = 0$$

Q2.26 Use the expressions
$(\mathbf{S}+\mathbf{T})^{c} = \mathbf{S}^{c} + \mathbf{T}^{c} + \mathbf{T}^{T}\mathbf{S}^{T} + \mathbf{S}^{T}\mathbf{T}^{T} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{T} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{T} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$
and
$\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}\left(\mathbf{T}^{c}\mathbf{S}^{T}\right) + \mathrm{tr}\left(\mathbf{S}^{c}\mathbf{T}^{T}\right) + \det(\mathbf{T})$
to show that $\left(\mathbf{I} + (\boldsymbol{\omega}\,\times)\right)^{-1} = \frac{\mathbf{I}+\boldsymbol{\omega}\times+\boldsymbol{\omega}\otimes\boldsymbol{\omega}}{1+\|\boldsymbol{\omega}\|^{2}}$

For any invertible tensor $\mathbf{T}$,

$$\mathbf{T}^{-1} = \frac{\mathbf{T}^{cT}}{\det \mathbf{T}}$$

$$\det(\mathbf{I} + \boldsymbol{\omega}\,\times) = \det \mathbf{I} + \det(\boldsymbol{\omega}\,\times) + (\boldsymbol{\omega}\,\times)^{c} : \mathbf{I} + \mathbf{I}^{c} : (\boldsymbol{\omega}\,\times)$$
$$= 1 + 0 + |\omega|^{2} + 0$$

$$\left(\mathbf{I} + (\boldsymbol{\omega}\,\times)\right)^{c} = \left[(1 + \mathrm{tr}(\boldsymbol{\omega}\,\times))\mathbf{I} - (\boldsymbol{\omega}\,\times)^{T} + (\boldsymbol{\omega}\,\times)^{c}\right]$$
$$= \mathbf{I} + (\boldsymbol{\omega}\,\times) + \boldsymbol{\omega} \otimes \boldsymbol{\omega}$$

See Q2.33

so that

$$\left(\mathbf{I} + (\boldsymbol{\omega}\,\times)\right)^{-1} = \frac{\mathbf{I} + \boldsymbol{\omega}\times + \boldsymbol{\omega}\otimes\boldsymbol{\omega}}{1 + \|\boldsymbol{\omega}\|^{2}}$$

Q2.28 Use the expressions
$(\mathbf{S}+\mathbf{T})^c = \mathbf{S}^c + \mathbf{T}^c + \mathbf{T}^T\mathbf{S}^T + \mathbf{S}^T\mathbf{T}^T - \mathrm{tr}(\mathbf{T})\mathbf{S}^T - \mathrm{tr}(\mathbf{S})\mathbf{T}^T + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$
to show that for an arbitrary tensor $\mathbf{T}$,
$(\mathbf{I}+\mathbf{T})^c = \mathbf{T}^c + \mathbf{I}(1+\mathrm{tr}\,\mathbf{T}) - \mathbf{T}^T$
and that

$$(\mathbf{I}+\mathbf{u}\otimes\mathbf{v})^c = \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u}$$

Substituting the identity tensor for $\mathbf{S}$ in the given expression, we have,

$$\begin{aligned}(\mathbf{I}+\mathbf{T})^c &= \mathbf{T}^c + \mathbf{I} + (3\,\mathrm{tr}\,\mathbf{T} - \mathrm{tr}\,\mathbf{T})\mathbf{I} - 3\mathbf{T}^T - \mathbf{I}\,\mathrm{tr}\,\mathbf{T} + \mathbf{T}^T + \mathbf{T}^T \\ &= \mathbf{T}^c + \mathbf{I} + 2\,\mathrm{tr}\,\mathbf{T}\,\mathbf{I} - \mathrm{tr}\,\mathbf{T}\,\mathbf{I} - \mathbf{T}^T \\ &= \mathbf{T}^c + \mathbf{I}(1+\mathrm{tr}\,\mathbf{T}) - \mathbf{T}^T\end{aligned}$$

$$\begin{aligned}(\mathbf{I}+\mathbf{u}\otimes\mathbf{v})^c &= (\mathbf{u}\otimes\mathbf{v})^c + \mathbf{I}(1+\mathrm{tr}(\mathbf{u}\otimes\mathbf{v})) - (\mathbf{u}\otimes\mathbf{v})^T \\ &= \mathbf{0} + \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u} \\ &= \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u}\end{aligned}$$

Q2.29 Using direct notation only, show that the determinant of a vector cross is zero.

Given basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$, the third invariant of $\boldsymbol{\omega}\times$,

$$I_3(\boldsymbol{\omega}\times) = \det(\boldsymbol{\omega}\times)$$

$$= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, \boldsymbol{\omega}\times\mathbf{g}_2, \boldsymbol{\omega}\times\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, (\boldsymbol{\omega}\times)^c(\mathbf{g}_2\times\mathbf{g}_3)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, (\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\mathbf{g}_2\times\mathbf{g}_3)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

upon noting that the cofactor,

$(\boldsymbol{\omega}\times)^c = (\boldsymbol{\omega}\otimes\boldsymbol{\omega})$.

See Q2.33

And since $(\boldsymbol{\omega}\otimes\boldsymbol{\omega})$ is symmetric, the numerator above is,

$$(\boldsymbol{\omega}\times\mathbf{g}_1)\cdot(\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\mathbf{g}_2\times\mathbf{g}_3)$$

$$= (\mathbf{g}_2\times\mathbf{g}_3)\cdot(\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\boldsymbol{\omega}\times\mathbf{g}_1)$$

$$= (\mathbf{g}_2\times\mathbf{g}_3)\cdot[\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\times\mathbf{g}_1)]\boldsymbol{\omega} = 0$$

so that $I_3(\boldsymbol{\omega}\times) = \det(\boldsymbol{\omega}\times) = 0$.

Use the fact that the cofactor of any tensor can be written as

$$\mathbf{S}^{\mathrm{c}} = \left(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I}\right)^{\mathrm{T}}$$

to show that the cofactor of the sum of two tensors can be expressed in terms of the constituent tensors as,

$$(\mathbf{S} + \mathbf{T})^{\mathrm{c}}$$
$$= \mathbf{S}^{\mathrm{c}} + \mathbf{T}^{\mathrm{c}} + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} + \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$

Q2.34 Application of Cayley-Hamilton Theorem

$$(\mathbf{S}+\mathbf{T})^c = \left[(\mathbf{S}+\mathbf{T})^2 - I_1(\mathbf{S}+\mathbf{T}) + I_2\mathbf{I}\right]^{\mathrm{T}}$$
$$= \mathbf{S}^c + \mathbf{T}^c + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} + \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$

- $(\mathbf{S}+\mathbf{T})^c = \left\{(\mathbf{S}+\mathbf{T})^2 - \mathrm{tr}(\mathbf{S}+\mathbf{T})(\mathbf{S}+\mathbf{T}) + \frac{1}{2}\left[\mathrm{tr}^2(\mathbf{S}+\mathbf{T}) - \mathrm{tr}(\mathbf{S}+\mathbf{T})^2\right]\mathbf{I}\right\}^{\mathrm{T}}$

$$= \Big\{\mathbf{S}^2 + \mathbf{T}^2 + \mathbf{TS} + \mathbf{ST} - \mathrm{tr}(\mathbf{S})\mathbf{S} - \mathrm{tr}(\mathbf{T})\mathbf{T} - \mathrm{tr}(\mathbf{S})\mathbf{T} - \mathrm{tr}(\mathbf{T})\mathbf{S}$$
$$+ \frac{1}{2}\left[\mathrm{tr}^2\mathbf{S} + \mathrm{tr}^2\mathbf{T} + 2\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{S}^2) - \mathrm{tr}(\mathbf{T}^2) - \mathrm{tr}(\mathbf{TS}) - \mathrm{tr}(\mathbf{ST})\right]\mathbf{I}\Big\}^{\mathrm{T}}$$
$$= \left(\mathbf{S}^2 - \mathrm{tr}(\mathbf{S})\mathbf{S} + \frac{1}{2}\left[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S})^2\right]\mathbf{I}\right)^{\mathrm{T}} + \left(\mathbf{T}^2 - \mathrm{tr}(\mathbf{T})\mathbf{T} + \frac{1}{2}\left[\mathrm{tr}^2(\mathbf{T}) - \mathrm{tr}(\mathbf{T})^2\right]\mathbf{I}\right)^{\mathrm{T}}$$
$$+ \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$
$$= \mathbf{S}^c + \mathbf{T}^c + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathbf{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$

Remember the premise $\mathbf{S}^c = \left(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I}\right)^{\mathrm{T}}$

Q2.38. For the invertible tensor $\mathbf{T}$ and the tensors $\mathbf{F}, \mathbf{V}$ and $\mathbf{U}$, show that

$$(\mathbf{T} + \mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}\left(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U}\right)^{-1}\mathbf{VT}^{-1}$$

First consider the matrix $\begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ \mathbf{V} & \mathbf{F}^{-1} \end{pmatrix}$. Its inverse is obtained by solving the matrix equation,

$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ \mathbf{V} & \mathbf{F}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$ which yields,

$$\mathbf{A\,T} + \mathbf{BV} = \mathbf{I}$$

$-\mathbf{AU} + \mathbf{BF}^{-1} = \mathbf{O} \;\Rightarrow \mathbf{B} = \mathbf{AUF}$ so that,

$$\mathbf{A\,T} + \mathbf{AUFV} = \mathbf{A}(\mathbf{T} + \mathbf{UFV}) = \mathbf{I}$$

$$\Rightarrow \mathbf{A} = (\mathbf{T} + \mathbf{UFV})^{-1}$$

Q2.38… For the invertible tensor $\mathbf{T}$ and the tensors $\mathbf{F}, \mathbf{V}$ and $\mathbf{U}$, show that

$$(\mathbf{T}+\mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}\left(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U}\right)^{-1}\mathbf{VT}^{-1}$$

But $\mathbf{A} = \mathbf{T}^{-1} - \mathbf{BVT}^{-1}$ substituting in the second equation,

from which we can now write that $(\mathbf{T}^{-1} - \mathbf{BVT}^{-1})\mathbf{U} = \mathbf{BF}^{-1}$ so that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}$$

$$\mathbf{A} = \mathbf{T}^{-1} - \mathbf{BVT}^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$$

Finally $\mathbf{A} = (\mathbf{T}+\mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$ as required

In the special case when $\mathbf{F}$ is the identity tensor, we have,

$$(\mathbf{T}+\mathbf{UV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$$

Q2.39 Trivial

Q2.41 Given that the cofactor $\mathbf{A}^c \equiv \operatorname{cof}\mathbf{A} = \mathbf{A}^{-T}\det\mathbf{A}$ satisfies $\mathbf{Aa}\times\mathbf{Ab} = \mathbf{A}^c(\mathbf{a}\times\mathbf{b})$. Show by direct methods that transposing does not alter the determinant of a tensor

$$\det\mathbf{A} = \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \frac{\mathbf{Aa}\cdot\mathbf{Ab}\times\mathbf{Ac}}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \frac{\mathbf{Aa}\cdot\mathbf{A}^c(\mathbf{b}\times\mathbf{c})}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

$$= \frac{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{A}^{cT}\mathbf{Aa}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

$$= \frac{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{A}^{-1}\det\mathbf{A}^T\,\mathbf{Aa}}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

$$= \det\mathbf{A}^T\frac{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{A}^{-1}\mathbf{Aa}}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \det\mathbf{A}^T$$

upon noting that $\mathbf{A}^T\mathbf{Aa} = \mathbf{Ia} = \mathbf{a}$.

Q2.42 For a scalar α show that $\det \alpha \mathbf{A} = \alpha^3 \det \mathbf{A}$

Given that $\det \mathbf{A} = \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, then

$$\det \alpha \mathbf{A} = \frac{[\alpha \mathbf{Aa}, \alpha \mathbf{Ab}, \alpha \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \alpha^3 \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \alpha^3 \det \mathbf{A}$$

Q2.43 Define the inner product of tensors $\mathbf{T}$ and $\mathbf{S}$ as $\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^{\mathrm{T}})$ show that $I_1(\mathbf{T}) = \mathbf{T}:\mathbf{I}$

$$\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^{\mathrm{T}})$$

Let $\mathbf{S} = \mathbf{I}$;

$$\begin{aligned} \mathbf{T}:\mathbf{I} &= \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{I}) \\ &= \mathrm{tr}(\mathbf{T}\mathbf{I}) \\ &= I_1(\mathbf{T}) \end{aligned}$$

Q2.44. Show that every skew tensor is traceless.

$$\text{tr}\,\mathbf{W} = \mathbf{I}\colon\mathbf{W} = \mathbf{I}\colon\mathbf{W}^{\mathrm{T}} = -\mathbf{I}\colon\mathbf{W} = 0$$

The second equality because the trace operation does not change with transposing. The third equation from the fact that the transpose of a skew tensor is its opposite. The result all comes out on one line with no appeal to components.

Lastly, recall that trace is a linear operation. Hence,

$$\text{tr}\,\mathbf{W} = \text{tr}\,\mathbf{W}^{\mathrm{T}} = -\text{tr}\,\mathbf{W} = 0$$

A component-based proof is on the same page in the book

Q2.46 In component form, the third tensor invariant of a tensor $\mathbf{T}$, $I_3(\mathbf{T}) = e_{\alpha\beta\gamma}T_{1\alpha}T_{2\beta}T_{3\gamma} = \det \mathbf{T}$. Show that $e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma} = e_{\alpha\beta\gamma} \det \mathbf{T}$

We do this by first establishing the fact that the LHS is completely antisymmetric in α, β and γ. We note that the indices i, j and k are dummy and therefore,

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma} = -e_{kji}T_{i\alpha}T_{j\beta}T_{k\gamma} = -e_{kji}T_{k\gamma}T_{i\alpha}T_{j\beta} = -e_{ijk}T_{i\gamma}T_{k\alpha}T_{j\beta}$$

Showing that a simple swap of α and γ changes the sign. Thus we establish anti-symmetry in α, β and γ.

Q2.46 In component form, the third tensor invariant of a tensor $\mathbf{T}, I_3(\mathbf{T}) = e_{\alpha\beta\gamma}T_{1\alpha}T_{2\beta}T_{3\gamma} = \det \mathbf{T}$. Show that $e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma} = e_{\alpha\beta\gamma} \det \mathbf{T}$

Noting that both sides of

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma} = e_{\alpha\beta\gamma} \det \mathbf{T}$$

take the same values as the determinant of $\mathbf{T}$ when α, β and γ are equal to 1, 2 and 3 respectively. The arrangement of the indices makes this value positive or negative in the same antisymmetric way. This completes the proof

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma} = e_{\alpha\beta\gamma} \det \mathbf{T}$$