SSG 321 Introduction to Continuum Mechanics

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The Summation Convention

Week One Echoes

Vectors have Magnitude, Direction & Sense.	Can be 1. Added; 2. Scaled; 3. Two products between two vectors defined: One produces a vector result, the other produces a scalar result.
Product of vectors carry several meanings	Adjectives needed to disambiguate products of vectors. Dot product: Shorthand for the product of a projection & vector projected upon. Vector product: Area of the parallelogram formed by the vectors, outward vector to the surface as direction.

More Echoes

- Linear Independence
  - Vectors in a set that cannot be expressed in terms of the others.
  - Set containing the max number of linearly independent vectors forms a basis for the space.
  - The number of vectors in the basis set determines the dimension of the space.
- Orthonormal Basis Set (Cartesian Vectors)
  - Vectors are linearly independent, Set forms a basis
  - Main advantage: Components by simply taking a dot product.
  - Other basis are equally valid. Only linear independence matters.

Terms from Week One

- Important terms from last lecture are shown with Page numbers (P) or Q&A numbers (Q)
- Areas:
  - Rectangle → Parallelogram → Triangle →Trapezium
- Volumes:
  - Cuboid →Parallelepiped → Cone, Pyramid, Tetrahedron, etc.

Area of a Parallelogram P14	Direction P10	Projection P13	Span a Space P19
Area of a Rectangle P14	Linear Independence P16	Scalar P12	Vector P10
Basis Set of Vectors P14	Magnitude P10	Scalar Product P12	Vector Product P14
Components of a Vector P21	Normalize P21	Scalar Triple Product P22	Vector Triple Product Q21
Condition for Basis P21	Orthogonal P21	Scaling a Vector P11	Volume of a Parallelepiped P22
Dimension of Space P19	Orthonormal Set P21	Sense P10	Volume of a Tetrahedron P22,Q14

Week Two: Scope

Topic	Description	Slides
1	Echoes from Last Week	1-4
2	Sum, Index Notation & Summation Convention	6-16
3	The Kronecker Delta & Substitution Symbol	17-23
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5	Scalar, Vector and Tensor Products in Component Form	31-39

Vector: Sum of Components

- The fact that vectors can be added allows us to write any vectors as a sum of basis vectors scaled by numbers we call "Components"
$$\mathbf{f} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$$
- The above representation, for any three dimensional vector is instinctive in us. We usually assume the Cartesian Orthonormal basis vector set, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
- Simply renaming $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \to \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we proceed to write the same representation as,
$$\mathbf{f} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i$$

Sum of Components: Parsimony with Indexing

- This change in the naming of our basis as well as the new strategy of mixing numbers with alphabets in our scaling factors achieves parsimony.
- Note that the variable count reduces from six to two!
  - Initially we needed $\{\alpha, \beta, \gamma, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to describe the vector in component form.
  - Now we need only two indexed variables, $\{a_i, \mathbf{e}_i\}, i = 1, \dots, 3$
  - Imagine we were in a ten dimensional space!
    $\{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi, \zeta, \phi, \kappa, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \mathbf{q}, \mathbf{r}\} \Leftrightarrow \{a_i, \mathbf{e}_i\}, i = 1, \dots, 10$
  - If we have larger number of dimensions, it is easy to see that one scheme will exhaust its options while the other is more robust, and can go on!

Flexibility of Standard Summation

- Transformation Equations below can be written more compactly:

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \sum_{j=1}^{n} a_{1j}x_j$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \sum_{j=1}^{n} a_{2j}x_j$$

$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \sum_{j=1}^{n} a_{3j}x_j$$

Summation Compactness

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \, , i = 1, \dots, 3$$

i	LHS	RHS
1	y_1	$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = \sum_{j=1}^{n} a_{1j}x_j$
2	y_2	$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = \sum_{j=1}^{n} a_{2j}x_j$
3	y_3	$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = \sum_{j=1}^{n} a_{3j}x_j$

Move to the Einstein Summation Convention

Standard Summation Convention saves space and typing as we have seen. But the genius of the Einstein Summation Convention takes the matter one step further and has become the standard notation for Tensors and Continuum Mechanics.

The rule is simple and looks easy. Learn it as it looks. The logic is consistent.

It becomes complicated fast!

When you are confused, go back to the simple rules again: It will tell you the way out!

Points to Note:

Summation Convention:

No change to your equation.

Only helps you to express the same idea in a more compact form.

You will eventually see how a single term can replace 9 terms, or a single equation can replace twenty seven!

The rule is simple:

Make sure there is NEVER a time when any index is repeated more than ONCE.

Only allow an index to be repeated when such a repetition implies a summation over all the possible values of the index.

Apply to Slide 5

- The equation on Slide 5 becomes:
$$\mathbf{f} = a_i \mathbf{e}_i$$
- Summation over the index i is disposable as the simple fact that the same index is repeated over $i = 1, \dots, 3 \Rightarrow$ summation over i can be taken for granted.
- A repeating index such as this one is called a dummy index for the simple fact that it can be replaced by any other index that takes values over the same domain. Consequently,
$$\mathbf{f} = a_i \mathbf{e}_i = a_j \mathbf{e}_j = \cdots = a_\alpha \mathbf{e}_\alpha$$

provided it is known that $i, j, \dots, \alpha$ each take values over $1, \dots, 3$

Apply to Slide 8

- The equation on Slide 8 becomes:
$$y_i = a_{ij}x_j, i = 1, \dots, 3$$
- Again, the repetition of j means that summation sign over the j is disposable as the simple fact that the same index is repeated over $j = 1, \dots, 3 \Rightarrow$ summation over j can be taken for granted.
- Again, as before, a repeating index can be replaced by any other index that takes values over the same domain. Consequently,
$$y_i = a_{ij}x_j = a_{ik}x_k = \cdots = a_{i\alpha}x_\alpha$$

provided it is known that $j, k, \dots, \alpha$ each take values over $1, \dots, 3$. Note that i is NOT repeated in any term. Such an index is called a free index

Tell-Tale Signs

A trained eye sees almost instinctively when there is an error in an indexed object following the Summation Convention. Here are the things you look for:

A free index must occur once, once only and nothing but once in EACH term of an equation.

If a free index is missing in a term, there is an error.

A dummy index must occur twice. But it DOES NOT have to occur in another term.

Products & Terms

- We must now be very careful as to what we mean by "*a term*"
- Given that $i, j = 1, \dots, 3$, how many terms are in the following expressions?

$$b_i, \sum_{j=1}^{3} a_{ij}x_j + b_i\,, \sum_{i=1}^{3}\sum_{j=1}^{3} z_i(a_{ij}x_j + b_i)$$

- How do we represent them by the convention?

Interpreting Expressions & Equations

- Consider the equation $a_{ij}a_{jk} = b_{ik}$ representing the matrix Equations $\mathbf{AA} = \mathbf{B}$
- Note that the index j is repeated, hence there is a summation on it. But the indices i and k are free. Note that free indices occur only once in each term.
- Note also that there is a summation on the index j every time. Every expression has that summation by implication from the repetition of the index.

i	k	$a_{ij}a_{jk}$	b_{ik}
1	1	$a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31}$	b_{11}
1	2	$a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}$	b_{12}
1	3	$a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33}$	b_{13}
2	1	$a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31}$	b_{21}
2	2	$a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32}$	b_{22}
2	3	$a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33}$	b_{23}
3	1	$a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31}$	b_{31}
3	2	$a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32}$	b_{32}
3	3	$a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33}$	b_{33}

Matrix Expression

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

- It can be seen that the correct interpretation of the previous slide, by the Summation Convention Rule exactly matches the product of the matrices above.
- This must be worked out manually to get a good feel for it. It DOES NOT come naturally. Common, don't be lazy!
- Transposing the right matrix in product leads to $a_{ij}a_{kj} = b_{ik}$

Consequences of the Summation Convention

Two important constructs that come as consequences of the summation convention:

1. The Kronecker Delta
   - Substitution Symbol
   - We will see a more important interpretation after defining tensors
2. The Levi-Civita Three-Index Symbol
   - Also called the Alternating Symbol
   - Again, a more complete understanding of what it is awaits a formal definition of a tensor

The Kronecker Delta

- The Kronecker Delta is defined by the following nine Equations:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$
$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$
$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

- By the index notation, we can write these simply as a single, indexed equation,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

- No dummy index here; no summation.

The Kronecker Delta is operationally known as the “Substitution Symbol. Here is the reason why:

- Consider the arbitrary Cartesian base vectors, $\mathbf{e}_i$and $\mathbf{e}_j$ where we have not committed to the values the indices will take except for the fact that each will take a value 1 or 2 or 3.
- It is correct to write,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

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Substitution Example

Express a vector $\mathbf{v} = v_i \mathbf{e}_i$ in terms of its components.

$$\begin{aligned}\mathbf{v} \cdot \mathbf{e}_j &= v_i \mathbf{e}_i \cdot \mathbf{e}_j = v_i \delta_{ij} \\ &= v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j}\end{aligned}$$

- We now examine the value of both sides for different values of j:

$j = 1, \mathbf{v} \cdot \mathbf{e}_1 = v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{11} + v_2 \delta_{21} + v_3 \delta_{31} = v_1$;

$j = 2, \mathbf{v} \cdot \mathbf{e}_2 = v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{12} + v_2 \delta_{22} + v_3 \delta_{32} = v_2$ and

$j = 3, \mathbf{v} \cdot \mathbf{e}_3 = v_1 \delta_{13} + v_2 \delta_{23} + v_3 \delta_{33} = v_3$

We can summarize everything here abd write,

$$v_i \mathbf{e}_i \cdot \mathbf{e}_j = v_i \delta_{ij} = v_j$$

$$v_i \delta_{ij} = v_j$$

- The last expression shows the way the substitution works
  - Whenever the Kronecker delta shares an index with ANY other variable, (LHS: Kronecker Delta shares the index i)
  - Take out the shared symbol (LHS: Remove index i from v)
  - Replace the removed symbol with the remaining symbol j
  - Remove the Kronecker Delta. You obtain the result on RHS.
- Observe that following the above steps gives the expected result of the previous slide.

Substitution@Work

Product with Kronecker Delta	Shared Symbol to go	Result
$S_{\alpha\beta}\delta_{i\alpha}$	α	$S_{i\beta}$
$T_{ijk}\delta_{j\alpha}$	j	$T_{i\alpha k}$
$\delta_{ij}\delta_{\alpha j}$	j	$\delta_{i\alpha}$
$\delta_{ij}\delta_{ij}$	i or j	$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
$e_{ijk}\delta_{jk}$	j or k	$e_{ijj} = e_{ikk}$

Substitution Examples

Levi-Civita Three-Index Symbol

- It is a good idea to practice the Kronecker Delta to mastery before trying to understand the next symbol. Once you are good at the former, the latter is very easy.
- Consider a determinant made of only Kronecker Deltas:

$$e_{ijk} \equiv \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$

Where i, j or k may assume values in the usual domain of $1, \dots, 3$

Levi-Civita Three-Index Symbol

- Let us allow the values, $i = 1, j = 2$ and $k = 3$. In such a case, we have,

$$e_{123} \equiv \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Which the determinant of the identity matrix.

Levi-Civita

- Substituting the appropriate values, we can check and see that,

$$e_{123} = e_{231} = e_{312} = 1$$
$$e_{132} = e_{321} = e_{213} = -1$$

and, all the other cases returning zero. Using the fact that transposition does not alter the value of a determinant, we have an equivalent definition:

$$e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$$

Products of Alternating Symbols

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{r1}\delta_{1i} + \delta_{r2}\delta_{2i} + \delta_{r3}\delta_{3i} & \delta_{r\alpha}\delta_{\alpha j} & \delta_{r\alpha}\delta_{\alpha k} \\ \delta_{s\alpha}\delta_{\alpha i} & \delta_{s\alpha}\delta_{\alpha j} & \delta_{s\alpha}\delta_{\alpha k} \\ \delta_{t\alpha}\delta_{\alpha i} & \delta_{t\alpha}\delta_{\alpha j} & \delta_{t\alpha}\delta_{\alpha k} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Products of Alternating Symbols

- Clearly, not forgetting that repetition of an unknown index signifies a summation,

$$e_{rsk}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{11} + \delta_{22} + \delta_{33} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

- Expanding the equation, using the third row, we have:

Expanding the Product

$$
\begin{aligned}
e_{rsk}e_{ijk} &= \delta_{ki}\begin{vmatrix}\delta_{rj} & \delta_{rk}\\ \delta_{sj} & \delta_{sk}\end{vmatrix} - \delta_{kj}\begin{vmatrix}\delta_{ri} & \delta_{rk}\\ \delta_{si} & \delta_{sk}\end{vmatrix} + 3\begin{vmatrix}\delta_{ri} & \delta_{rj}\\ \delta_{si} & \delta_{sj}\end{vmatrix}\\
&= \delta_{ki}\left(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}\right) - \delta_{kj}\left(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk}\right)\\
&\qquad + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)\\
&= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)\\
&= -2\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)\\
&= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}
\end{aligned}
$$

Products of Alternating Symbols

$$\begin{aligned} e_{rjk}e_{ijk} &= \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} \\ &= 3\delta_{ri} - \delta_{ri} \\ &= 2\delta_{ri} \end{aligned}$$

Dot Product, Component Form

Recall that, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Consequently,

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_i \mathbf{e}_i) \cdot \left(b_j \mathbf{e}_j\right) \\ &= a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} \\ &= a_i b_i \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

Note on the second line how we avoided having four indices of the same type by invoking the fact that a dummy variable is mutable.

Vector Product in Component Form

- Recall that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$. The table here shows that
$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$$
- $$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= e_{ijk} a_i b_j \mathbf{e}_k \end{aligned}$$

i	j	$\mathbf{e}_i \times \mathbf{e}_j$	$e_{ijk}\mathbf{e}_k$
1	3	$1 \times 1 \sin 90\,(-\mathbf{e}_2)$	$e_{13k}\mathbf{e}_k = e_{131}\mathbf{e}_1 + e_{132}\mathbf{e}_2 + e_{133}\mathbf{e}_3 = -\mathbf{e}_2$
1	2	$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$	$e_{12k}\mathbf{e}_k = e_{121}\mathbf{e}_1 + e_{122}\mathbf{e}_2 + e_{123}\mathbf{e}_3 = \mathbf{e}_3$
2	3	$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$	$e_{23k}\mathbf{e}_k = e_{231}\mathbf{e}_1 + e_{232}\mathbf{e}_2 + e_{233}\mathbf{e}_3 = \mathbf{e}_1$
3	1	$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$	$e_{31k}\mathbf{e}_k = e_{311}\mathbf{e}_1 + e_{312}\mathbf{e}_2 + e_{313}\mathbf{e}_3 = \mathbf{e}_2$
1	1	$\mathbf{e}_1 \times \mathbf{e}_1 = 0$	$e_{11k}\mathbf{e}_k = e_{111}\mathbf{e}_1 + e_{112}\mathbf{e}_2 + e_{113}\mathbf{e}_3 = 0$
2	2	$\mathbf{e}_2 \times \mathbf{e}_2 = 0$	$e_{22k}\mathbf{e}_k = e_{221}\mathbf{e}_1 + e_{222}\mathbf{e}_2 + e_{223}\mathbf{e}_3 = 0$
2	1	$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$	$e_{21k}\mathbf{e}_k = e_{211}\mathbf{e}_1 + e_{212}\mathbf{e}_2 + e_{213}\mathbf{e}_3 = -\mathbf{e}_3$
3	2	$\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1$	$e_{32k}\mathbf{e}_k = e_{321}\mathbf{e}_1 + e_{322}\mathbf{e}_2 + e_{323}\mathbf{e}_3 = -\mathbf{e}_1$
3	3	$\mathbf{e}_3 \times \mathbf{e}_3 = 0$	$e_{33k}\mathbf{e}_k = e_{331}\mathbf{e}_1 + e_{332}\mathbf{e}_2 + e_{333}\mathbf{e}_3 = 0$

The Dyad

- One exceedingly important object that you can also produce from taking a product of two vectors is a **Tensor**. Naturally, we shall call such a product a “Tensor Product”
- The symbol is called a dyad operator, $\otimes$. It combines the product sign and a circle.
- The tensor product is therefore also called a dyad product.
- A dyad is defined by what it does when it acts on another vector:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

Components of a Dyad

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

- There are nine base dyads for expressing every tensor: $\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_1 \otimes \mathbf{e}_3, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_3, \mathbf{e}_3 \otimes \mathbf{e}_1, \mathbf{e}_3 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3$

Product	Right or wrong	Comments
$\alpha\mathbf{u}$	Correct	Scaling a vector, multiplication of a scalar and a vector; No explicit sign required
$\mathbf{u}\beta\mathbf{v}$	Error	$\mathbf{u}\beta$ is a scaled vector whose product with $\mathbf{v}$ is ambiguous. Possible additional information can make it $(\mathbf{u}\beta)\cdot\mathbf{v}$, $\mathbf{u}\times(\beta\mathbf{v})$, or $\mathbf{u}\otimes(\beta\mathbf{v})$. They have different meanings that cannot be reliable guessed unless you supply the needed information a priori.
$\beta\alpha$	Correct	Product of two scalars; No explicit sign required
$\mathbf{v}\mathbf{u}$	Error	Product of two vectors; $\mathbf{v}\cdot\mathbf{u}\neq\mathbf{v}\times\mathbf{u}\neq\mathbf{v}\otimes\mathbf{u}$ Explicit disambiguating sign required. We note here that certain authors imply this simple concatenation as the way they represent the tensor product, $\mathbf{v}\otimes\mathbf{u}$. In most current Literature on the subject, the tensor or dyad sign is the preferred way to represent this product. We retain that more popular convention here and subsequently.
$\beta(\mathbf{u}\times\mathbf{v})$	Correct	Vector product of two vectors gives a vector. Multiplying this result by a scalar does not require another sign. The order of the scaling is NOT important: $\beta(\mathbf{u}\times\mathbf{v})=\beta\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\beta\mathbf{v}=(\mathbf{u}\times\mathbf{v})\beta$ The order of the appearance of the vectors is inviolable: $\beta(\mathbf{u}\times\mathbf{v})\neq\beta\mathbf{v}\times\mathbf{u}=\mathbf{v}\times\beta\mathbf{u}\neq(\mathbf{u}\times\mathbf{v})\beta$

Product	Right or wrong	Comments
$\mathbf{u} \cdot \mathbf{v}\alpha$	Correct	The dot product of a vector with a scaled vector. No ambiguity is created with the location of α; $\mathbf{u} \cdot \mathbf{v}\alpha$, $(\mathbf{u}\alpha) \cdot \mathbf{v}$, or $\alpha\mathbf{u} \cdot \mathbf{v}$ all mean the same thing.
$\beta\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}\alpha$	Correct	Scalar triple product with vector scaling along. Result is the same as $(\beta\alpha)\mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = (\beta\alpha)\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}$
$\beta\mathbf{u} \times \mathbf{v} \times \mathbf{w}$	Error	Vector triple product with vector scaling along. Vector product is <u>not associative</u>: $$\beta\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \beta(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - \beta(\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$ $$\neq \beta(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \beta(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - \beta(\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$ Parentheses are required to show which product is intended.
$\mathbf{u} \cdot \mathbf{v} \otimes \mathbf{w}$	Correct	Only one interpretation makes sense: $(\mathbf{v} \otimes \mathbf{w})\,\mathbf{u}$.
$\mathbf{u} \times \mathbf{v} \otimes \mathbf{w}$	Correct	Treat the vector cross as a tensor, then obtain the LHS: $(\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w} = \mathbf{u} \times (\mathbf{v} \otimes \mathbf{w})$ The two different interpretations evaluate to the same value.

Vectors & Their Matrices

$$\mathbf{a} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
$$= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_i\mathbf{e}_i$$

$$\mathbf{b} = [b_1, b_2, b_3]\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$
$$= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b_j\mathbf{e}_j$$

Dyads & Matrices

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

The Trace of a Dyad

Obtain the trace of a dyad by changing the dyad operator into a dot as follows

$$\begin{aligned}\operatorname{tr}(\mathbf{a}\otimes\mathbf{b}) &= a_i b_j \operatorname{tr}\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)\\ &= a_i b_j\left(\mathbf{e}_i\cdot\mathbf{e}_j\right)\\ &= a_i b_j\,\delta_{ij}\\ &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

It is the inner product as can be seen from the matrix: The scalar product of operands.

$$\begin{bmatrix}\boldsymbol{a_1 b_1} & a_1 b_2 & a_1 b_3\\ a_2 b_1 & \boldsymbol{a_2 b_2} & a_2 b_3\\ a_3 b_1 & a_3 b_2 & \boldsymbol{a_3 b_3}\end{bmatrix}$$