Basis Vectors Euclidean Point & Vector Spaces

SSG 321 Introduction to Continuum Mechanics

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Last Week Echoes

Σ

The Summation Convention

δ_{ij}

Kronecker Delta

Explained why the epithet, Substitution Symbol.

Need to get familiarity by practice. It does not happen any other way!

e_{ijk}

Levi-Civita Symbol (Antisymmetric)

Very useful in expressing the cross product in component form.

More Echoes

- Scalar Product

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$
$$\mathbf{a} \cdot \mathbf{b} = a_\alpha b_\alpha$$

- Vector Product

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$$
$$\mathbf{a} \times \mathbf{b} = e_{ijk}a_i b_j \mathbf{e}_k$$

- Tensor Product

$$\mathbf{e}_i \otimes \mathbf{e}_j = \text{irreducible}$$
$$\mathbf{a} \times \mathbf{b} = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

We can make $\mathbf{e}_k$ subject of the formula, $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$, and obtain,

$$\mathbf{e}_k = \frac{1}{2} e_{ijk}\mathbf{e}_i \times \mathbf{e}_j$$

A bit tricky: multiply both sides by $e_{\alpha\beta\gamma}$

$$e_{\alpha\beta\gamma}\mathbf{e}_i \times \mathbf{e}_j = e_{\alpha\beta\gamma}e_{ijk}\mathbf{e}_k$$

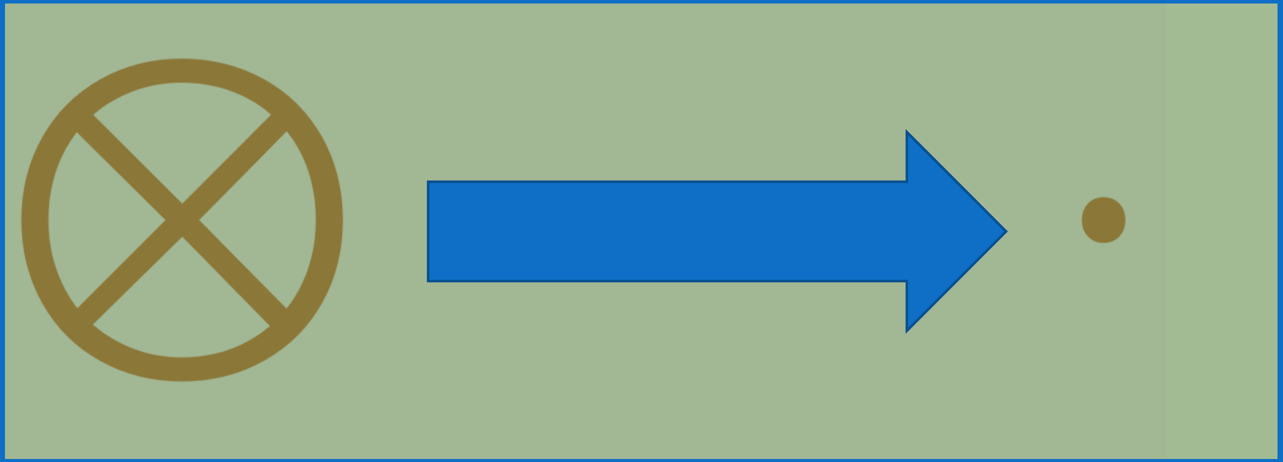
$\alpha, \beta \to i, j$ violates no rules, $\Rightarrow$

$$e_{ij\gamma}\mathbf{e}_i \times \mathbf{e}_j = e_{ij\gamma}e_{ijk}\mathbf{e}_k$$
$$= 2\delta_{\gamma k}\mathbf{e}_k = 2\mathbf{e}_\gamma$$

Or,

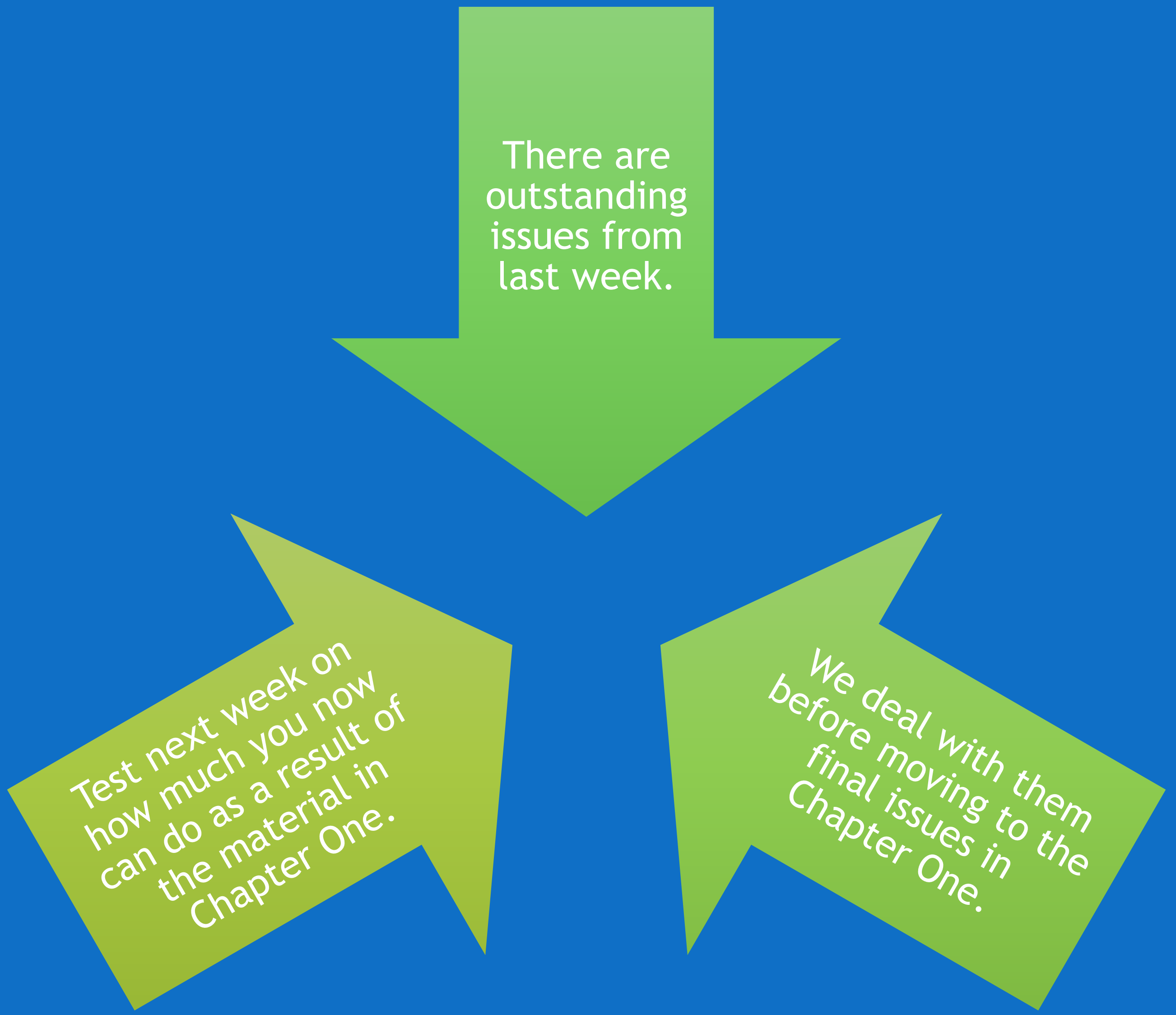
$$\mathbf{e}_k = \frac{1}{2} e_{ijk}\mathbf{e}_i \times \mathbf{e}_j$$

Last Echo

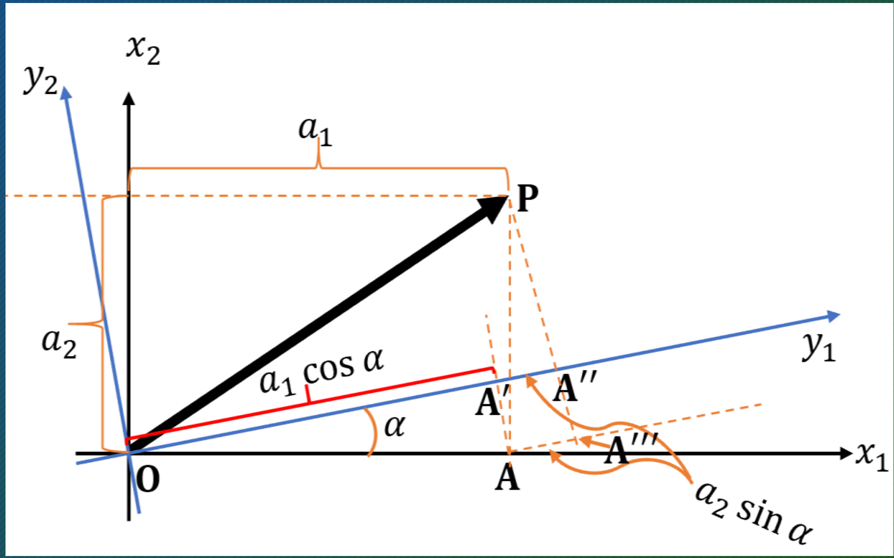


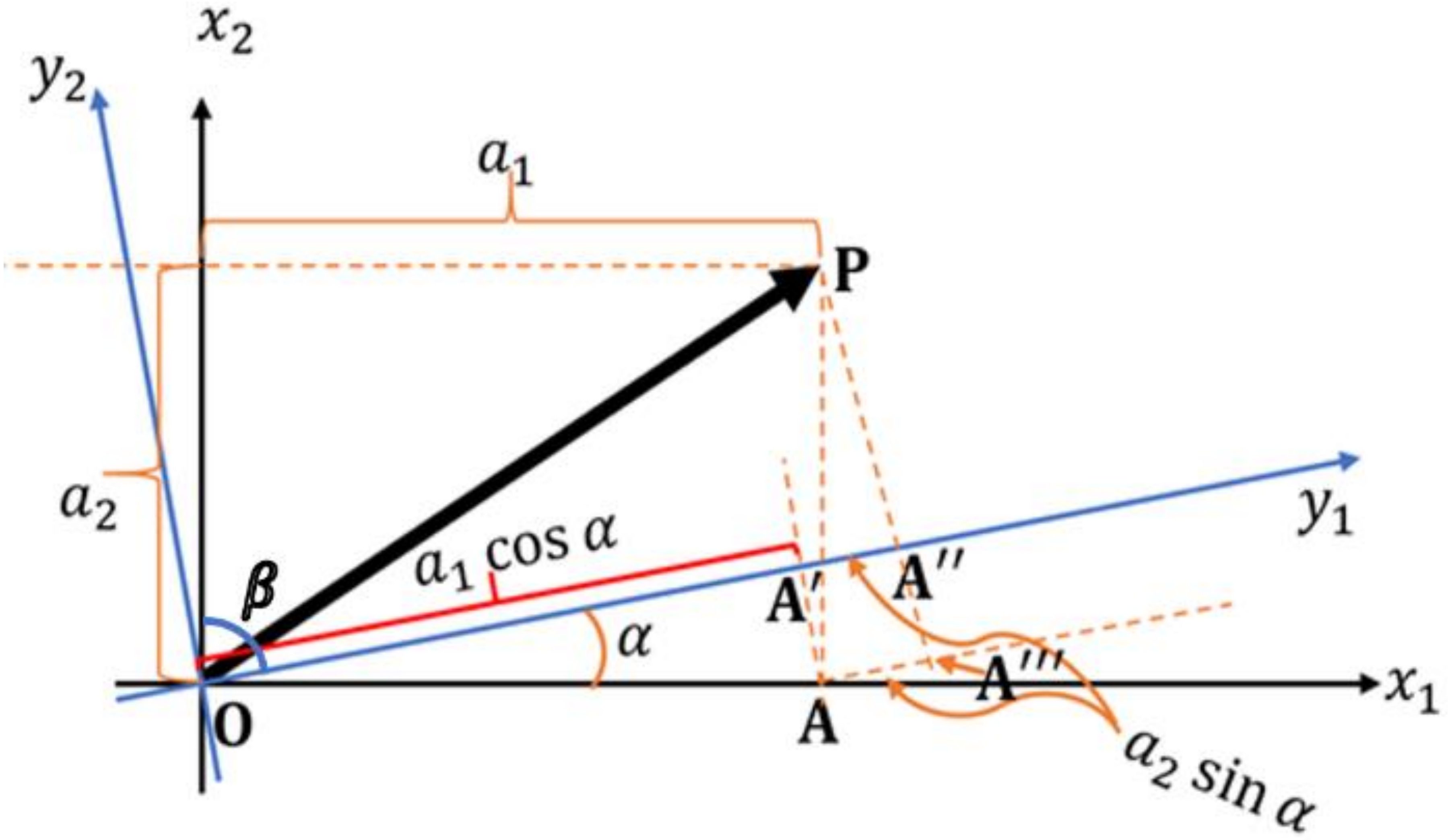
- Trace of a Dyad
    - A Linear Operator (More later)
    - Simply change the dyad operator to a dot.
    - In the matrix representation, we were able to see that the trace is the inner product which leads to the scalar product. The two are the same.

Unfinished Business



- $\mathbf{O}\ x_1\ x_2\ \to \mathbf{O}\ y_1\ y_2$, find coordinates of vector $\mathbf{OP}$ in the new system
- Let $\mathbf{OP} = \mathbf{v} = a_i\mathbf{e}_i$ where $\mathbf{e}_1$and $\mathbf{e}_2$ are unit vectors along $\mathbf{O}\ x_1\ x_2$. If the coordinates are rotated to $\mathbf{O}\ y_1\ y_2$ such that the same vector now becomes $\mathbf{v} = b_i\boldsymbol{\xi}_i$ where $\boldsymbol{\xi}_1$and $\boldsymbol{\xi}_2$ are unit vectors along the $\mathbf{O}\ y_1\ y_2$ system.
$$\mathbf{OA} = a_1; \mathbf{OB} = a_2; \mathbf{OA}'' = b_1?\ \mathbf{OB}'' = b_2?$$
- We drop perpendicular lines to the lines $\mathbf{O}\ y_1$ and $\mathbf{O}\ y_2$ meeting them at $\mathbf{A}''$and $\mathbf{B}''$respectively.
$$\mathbf{OA}' = a_1 \cos\alpha \quad \mathbf{AA}'' = a_2 \sin\alpha$$
- because $\mathbf{PA}$ is the hypotenuse of a right-angled triangle $\mathbf{APA}'''$with angle α at $\mathbf{APA}'''$ And it is easy to see that $\mathbf{AA}'\mathbf{A}''\mathbf{A}'''$is a rectangle. Its opposite sides are equal, consequently, the length
$$\mathbf{OA}'' = b_1 = a_1 \cos\alpha + a_2 \sin\alpha.$$
$$= a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)$$



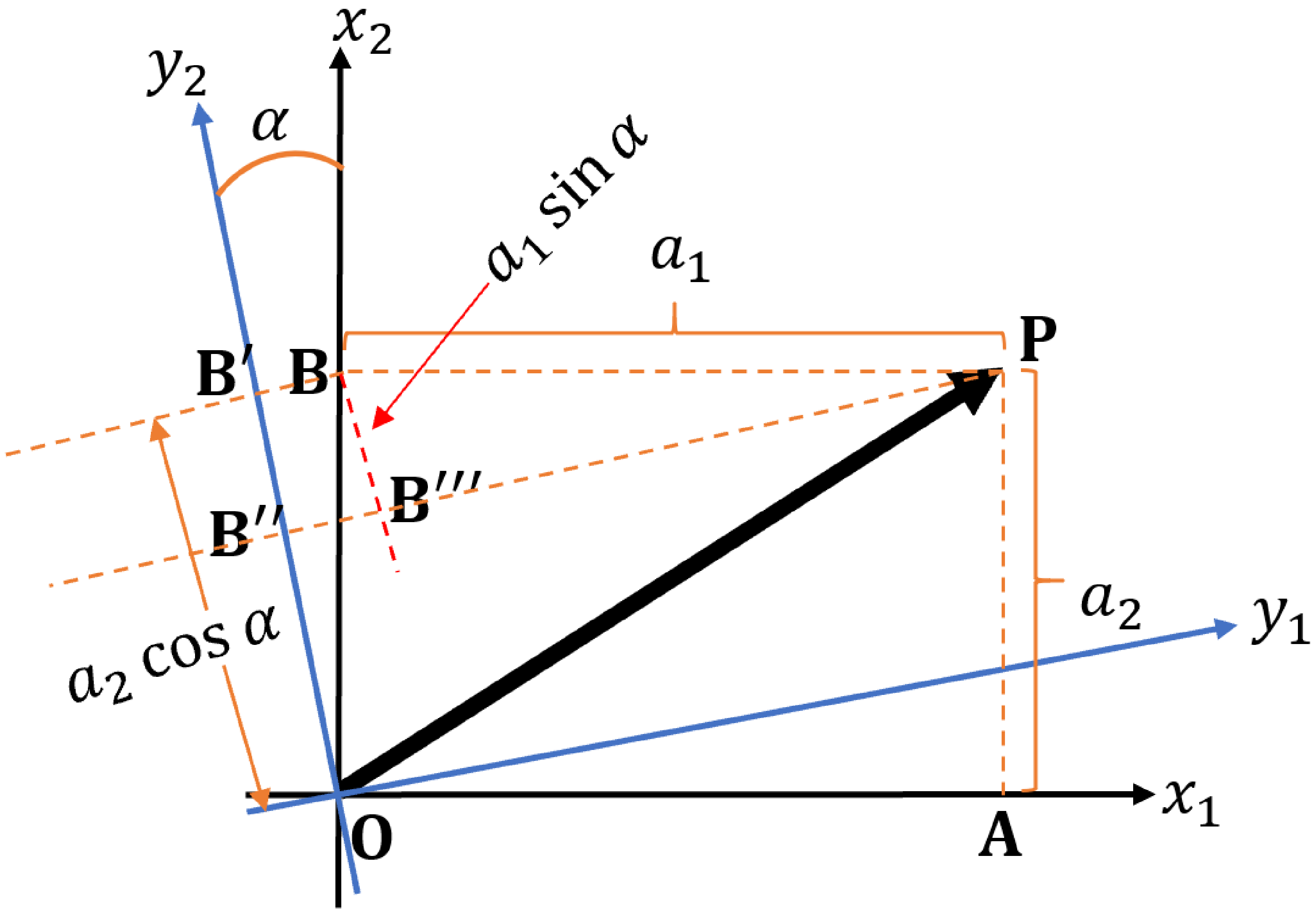

x_2
y_2
a_1
P
a_2
y_1
$a_1 \cos\alpha$
β
A′
A″
α
A‴
x_1
O
A
$a_2 \sin\alpha$

The Other Coordinate

- $\mathbf{B}''$ is the foot of the perpendicular from point $\mathbf{P}$ to the $\mathbf{O}\,y_2$-axis. $\mathbf{BB}'$is parallel to $\mathbf{PB}''$. $\mathbf{B}'''$ is the foot of the perpendicular from $\mathbf{B}$ to $\mathbf{PB}''$. By the same arguments as before, $\mathbf{BB}'\mathbf{B}''\mathbf{B}'''$ is also a rectangle. Clearly,

$$\begin{aligned}\mathbf{OB}'' &= b_2 = -a_1 \sin\alpha + a_2 \cos\alpha.\\ &= a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\end{aligned}$$

- The rotation tensor is: $\mathbf{R}^{\mathrm{T}} = \mathbf{e}_j \otimes \boldsymbol{\xi}_j$. Hence, we have



x_2
y_2
α
$a_1 \sin\alpha$
a_1
P
B′
B
B‴
B″
a_2
y_1
$a_2 \cos\alpha$
O
A
x_1

The Rotation

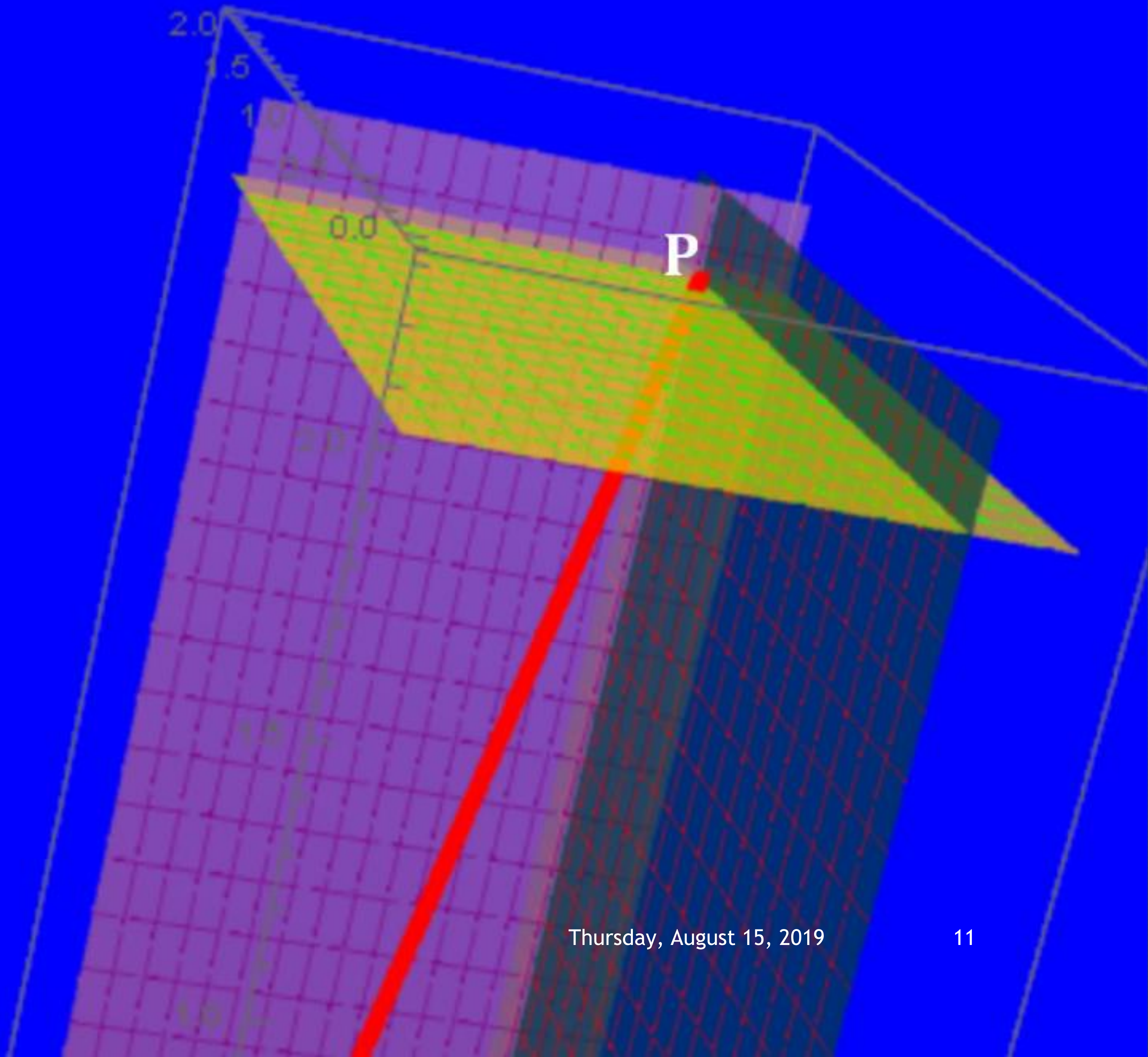
- Consider the Dyad Sum:

$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = (\mathbf{e}_j \otimes \boldsymbol{\xi}_j)a_i\mathbf{e}_i$$
$$= a_i\mathbf{e}_j(\boldsymbol{\xi}_j \cdot \mathbf{e}_i)$$
$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = \mathbf{e}_1\big(a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)\big) + \mathbf{e}_2\big(a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\big)$$

- Exactly the same expression we found geometrically!
- Look again at the dyad sum! It is simply the dyads formed by the coordinate basis vectors!
- To compute any rotation, we only need to form the dyads from the set of coordinates before and after!

The Cartesian System of Coordinates

- Three orthonormal unit vectors as basis set.
- The intersection of two planes creates a coordinate line.
- Intersection of three planes create three coordinate lines all meeting at the same point at which the three planes intersect to give the location of the point.



Mathematica Code

- The graphics you have just seen were mathematically generated. The code is something like we show here:

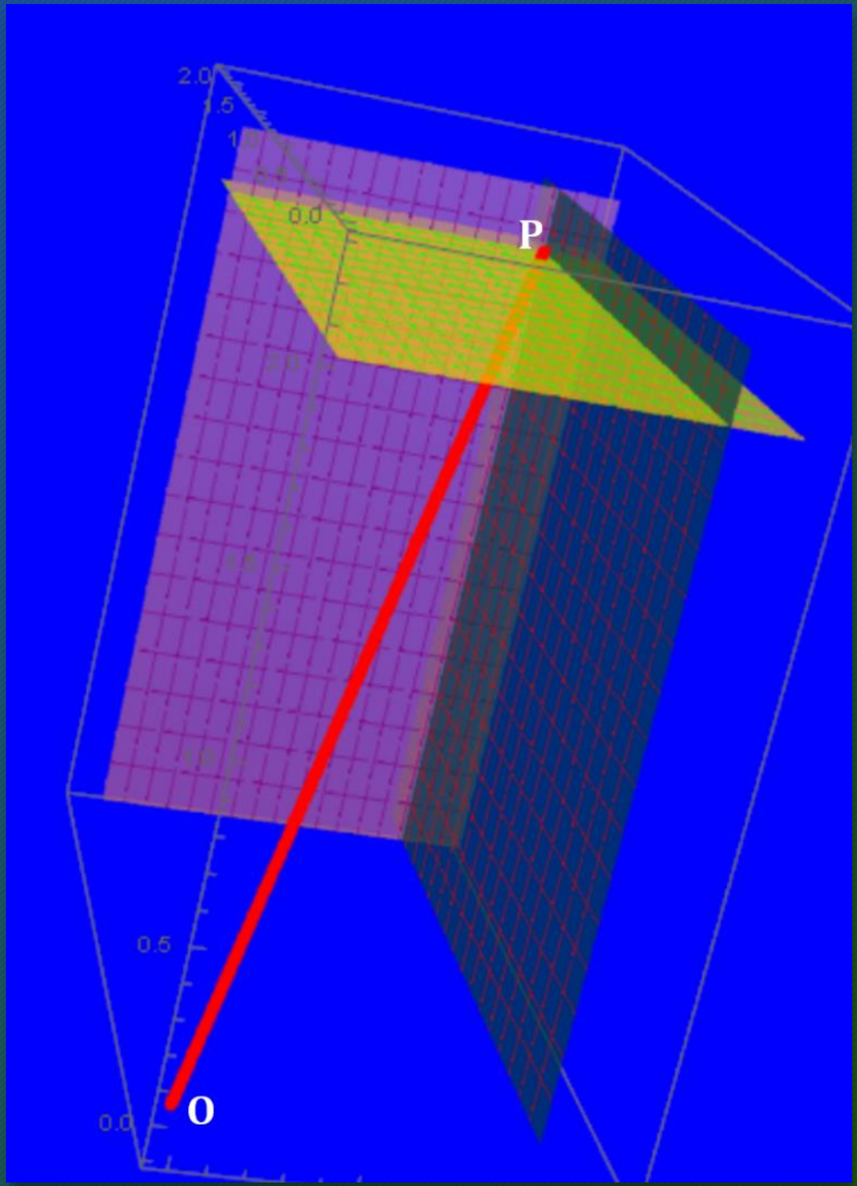
```
Cart1 = ParametricPlot3D[{1, y, z}, {y, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Red];
Cart2 = ParametricPlot3D[{x, 1, z}, {x, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Green];
Cart3 = ParametricPlot3D[{x, y, 1}, {x, 0, 1.4}, {y, 0, 1.4}, PlotStyle → Yellow];
Show[Cart1, Cart2, Cart3, PlotRange → {{0, 1.5}, {0, 1.5}, {0, 1.5}}, Ticks → None]
```

The Euclidean Point Space

- All engineering objects of interest reside. This space contains point locations that can be occupied by a location in an object at a particular time. It is often of interest to be able to do several things:
  - Locate the point in an unambiguous way,
  - Relate the point to one or more other points in its vicinity, and
  - Define quantities that take up values of interest at that point: Fields.
- Examples of Fields
  - Temperature map of this classroom (one thousand thermometers)
  - Temperature distribution, Temperature field.
  - Tensor Fields

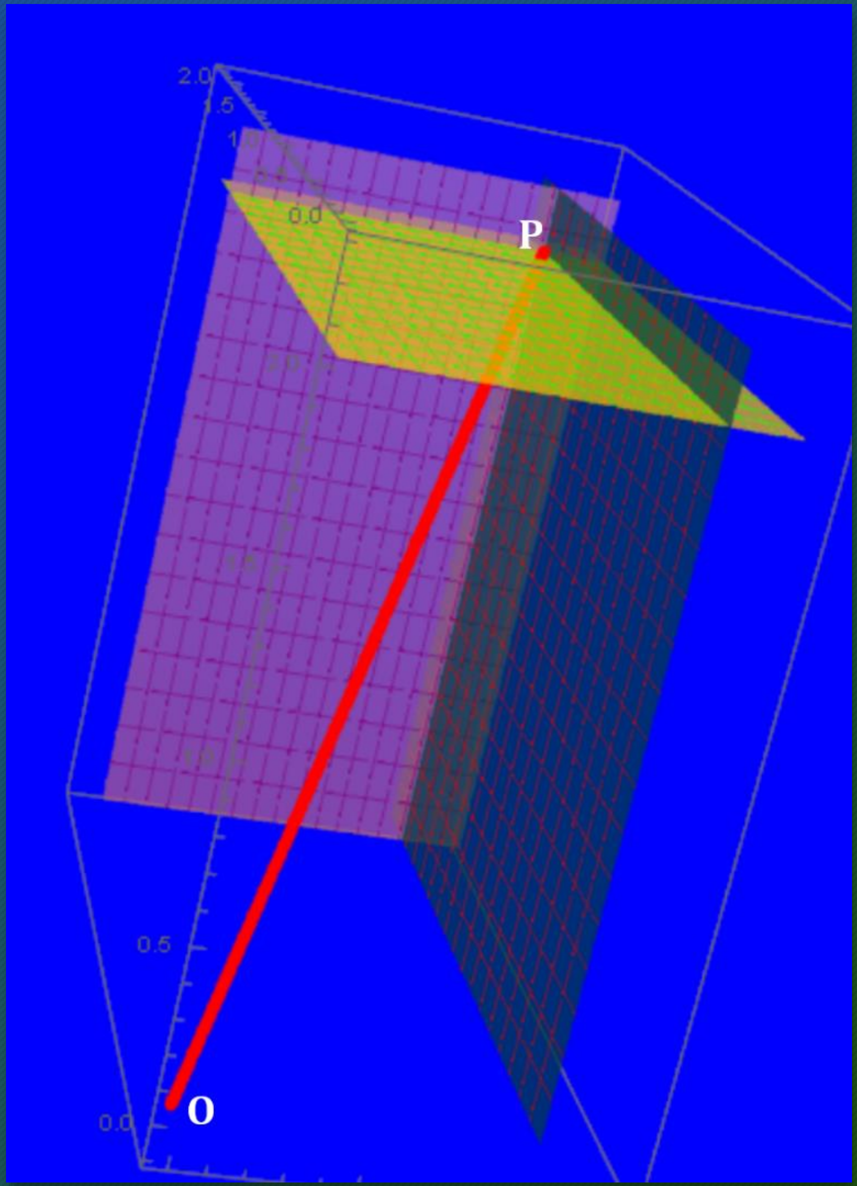
Points & Vectors

- The Euclidean Point Space that we have used our Cartesian Coordinate System to describe contains points, NOT vectors.
- It is critically important for you to be able to do graphics, to note the distinction.
- It is even more serious to note the difference in order to do mechanical analysis such as Solid Mechanics, Fluid Mechanics, Thermodynamics, etc., that our knowledge of Continuum Mechanics lead us to.



Points & Vectors

- $\mathbf{O}$ and $\mathbf{P}$ are points.
- Drawing a line between $\mathbf{O}$ and $\mathbf{P}$ creates the vector $\mathbf{OP}$ possessing all the attributes of vectors as we have previously defined:
  - Magnitude defined by the length of the line $\mathbf{OP}$,
  - Direction defined by the direction of the line $\mathbf{OP}$, and
  - Sense as from $\mathbf{O}$ to $\mathbf{P}$
- The vector we have just created is no ordinary vector. It was brought to life by joining the point $\mathbf{P}$ to the origin.



Position Vectors

- Note that we could create a vector by joining any two points.
- It is a no brainer that every point in the space can be treated as we have just treated $\mathbf{P}$: Create a vector by simply joining the point to the origin.
- The vectors created this way have a name: Position Vectors.
- We are emphasizing the fact that we have created a vector from a point by simply using a line to join the point to the origin.
- They are special vectors. Not all vectors are created this way.

Cartesian System: Special Attributes

* Each coordinate surface is a plane. The three defined at a particular point are respectively parallel to the three you can define at any other point.
* Each coordinate line: the intersection of these planes that are parallel to the axes are similarly parallel straight lines at all points in the system.
* The basis vectors – usually defined as unit vectors along the axes, are always the same at any point in the Cartesian system. It does not matter where the point P is located, the basis vectors are the same unit vectors we define as $(\mathbf{i}, \mathbf{j}$ and $\mathbf{k})$ or $(\mathbf{e}_1, \mathbf{e}_2,$ and $\mathbf{e}_3)$ along the coordinate lines at the origin.

Consequences of Attributes

- Position Vector is a linear function of the coordinates:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

- We can easily write the vector field in terms of three scalar fields that we call its components;

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

  - In order, say to find acceleration, we may need to differentiate this function spatially or temporally; We need to worry only about the components as their vector bases are all constants.
- These nice features occurs only in Cartesian System of coordinates.

Relate Position Vectors to Basis Set

- A partial differentiation of the position vector with respect to the coordinate variables yield the basis vectors for the coordinate system as shown here:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

$$\frac{\partial \mathbf{r}}{\partial x_i} = \mathbf{e}_i, i = 1,2,3.$$

- The partial derivative of the position vector to the three coordinate variables constitute a set of linearly independent vectors that can form the basis set.
- In the Cartesian System, they are the same as our usual basis set

Nonlinear Coordinate Systems

To form a coordinate system,

- Select three variables (ξ_1, ξ_2, ξ_3). When each takes a value, say, $\xi_i = \alpha_i$ where each α_i is a real number, then we have the point $(\alpha_1, \alpha_2, \alpha_3)$. We can write this point in at least two other ways: $\xi_i = \alpha_i,\ i = 1, \dots, 3$ or as $(\xi_1 = \alpha_1, \xi_2 = \alpha_2, \xi_3 = \alpha_3)$.
- For each, $\xi_i = \alpha_i$, we have defined a coordinate surface. In the case of Cartesian coordinates, given any three $\alpha_i \in \mathbb{R}, i = 1,2,3,$ we have $x_1 = \alpha_1$, defining a plane with normal along the $\mathbf{e}_1$axis, $x_2 = \alpha_2$, defining a plane with normal along the $\mathbf{e}_2$axis and $x_3 = \alpha_3$, which is a plane with normal along the $\mathbf{e}_3$ axis.
- A systematic choice leads to specific systems: Cylindrical, Spherical Polar

Cylindrical Polar Coordinate System

- We now introduce a transformation (called a polar transformation) of $\{x_1, x_2\} \rightarrow \{r, \phi\}$ such that, $x_1 = r\cos\phi$, and $x_2 = r\sin\phi$ .Note also that this transformation is invertible: $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$
- With such a transformation, we can locate any point in the 3-D space with three scalars $\{\xi_1, \xi_2, \xi_3\} \rightarrow \{r, \phi, z\}$ instead of our previous set $\{x_1, x_2, x_3\}$

Cylindrical Position Vector

- What does the position vector look like?

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$

Where we have defined,

$$\mathbf{e}_r = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$$
$$\mathbf{e}_z = \mathbf{e}_3$$

There are several methods to obtain the basis set of vectors. One instructive way is to do a partial differentiation of the position vector:

$$\frac{\partial\mathbf{r}}{\partial r} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2 = \mathbf{e}_r,\ \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\phi\,\mathbf{e}_1 + r\cos\phi\,\mathbf{e}_2 \equiv r\mathbf{e}_\phi,\ \frac{\partial\mathbf{r}}{\partial z} = \mathbf{e}_z$$

Cylindrical Basis Vector Set

By differentiating the position vector,

$$\begin{aligned}\mathbf{r} &= r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z\\ &= r\mathbf{e}_r + z\mathbf{e}_z\end{aligned}$$

with respect to the coordinate variables ξ_1, ξ_2, ξ_3 which now are r, ϕ, z, the basis vectors are shown in the table shown:

Derivative	Explicit Form	
$\frac{\partial \mathbf{r}}{\partial r}$	$\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$	$\mathbf{e}_r$
$\frac{\partial \mathbf{r}}{\partial \phi}$	$-r\sin\phi\,\mathbf{e}_1 + r\cos\phi\,\mathbf{e}_2$	$r\mathbf{e}_\phi$
$\frac{\partial \mathbf{r}}{\partial \mathbf{z}}$	$\mathbf{e}_z$	

$$\|\mathbf{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$
$$\|\mathbf{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$
$$\|\mathbf{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now let's take them in pairs:

$$\mathbf{e}_r \cdot \mathbf{e}_\phi = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$
$$\mathbf{e}_\phi \cdot \mathbf{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$
$$\mathbf{e}_z \cdot \mathbf{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$

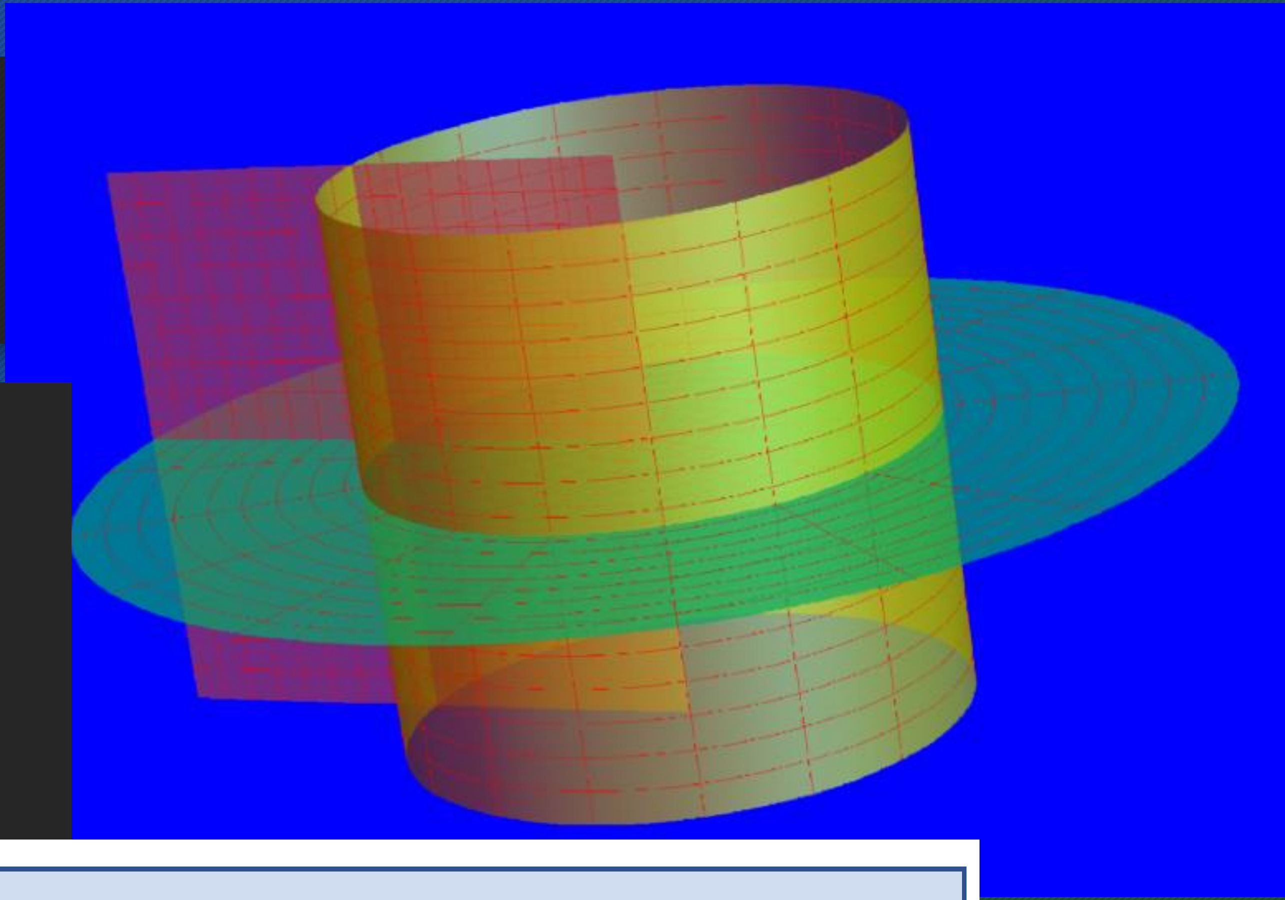
So that they are pairwise orthogonal.

Cylindrical Polar basis vectors constitute an orthonormal system

The code to plot the coordinate surfaces are given here

Type it into the Mathematica Notebook …

Coordinate Surfaces in Cylindrical System



```
c1 = ParametricPlot3D[{Sin[ϕ], Cos[ϕ], z}, {ϕ, 0, π}, {z, 1.5, 3.5}, PlotStyle → Red];
c2 = ParametricPlot3D[{r Sin[π / 3], r Cos[π / 3], z}, {r, 0, 2}, {z, 1.5, 3.5}, PlotStyle → Green];
c3 = ParametricPlot3D[{r Sin[ϕ], r Cos[ϕ], 2}, {ϕ, 0, 2 π}, {r, 0.5, 2.5}, PlotStyle → Yellow];
Show[c1, c2, c3, PlotRange -> {{0, 1.4}, {0, 1.5}, {1, 2.5}}, Ticks → None]
```

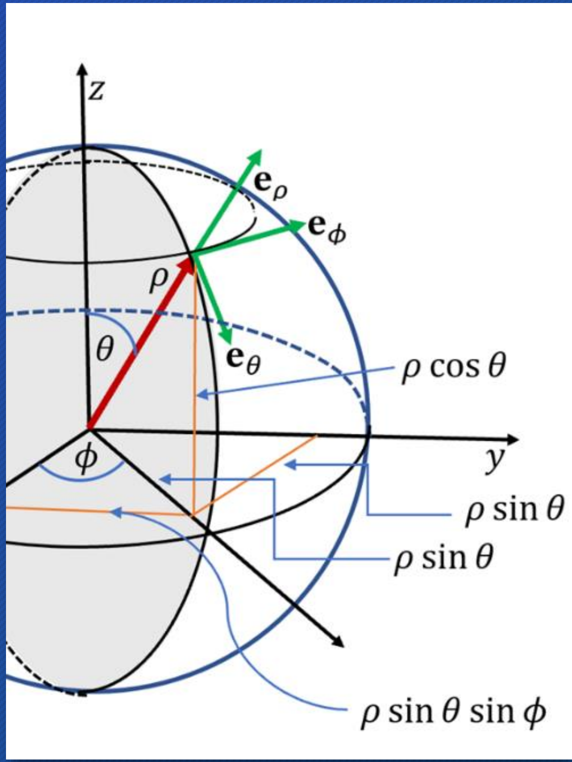
Mistakes to Avoid

- **That the Cylindrical position vector is $r\mathbf{e}_r(\phi) + \phi\mathbf{e}_\phi(\phi) + z\mathbf{e}_z$**
  - A simplistic copy of the Cartesian formula. This is wrong in at least two ways. For one thing, it is dimensionally incorrect because the unit of the middle basis component is an angle while the other components are measuring lengths. Secondly, we cannot obtain the Cartesian result from this via a coordinate transformation.
- **That the basis vectors are constants.**
  - They are NOT all constants. $\mathbf{e}_r(\phi)$ and $\mathbf{e}_\phi(\phi)$ are both functions of ϕ unlike in the Cartesian case, but $\mathbf{e}_z$ is a constant like the Cartesian case.

Spherical Coordinates

- The spherical Polar coordinate system selects its three ordered triplets with yet another strategy. This can be explained by the same transformation route we started. Continuing further with our transformation, we may again introduce two new scalars such that $\{r, z\} \to \{\rho, \theta\}$ in such a way that the position vector,
$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = \rho \sin\theta \, \mathbf{e}_r + \rho\cos\theta \, \mathbf{e}_z \equiv \rho\mathbf{e}_\rho$$
- Here, $r = \rho\sin\theta$, $z = \rho\cos\theta$. As before, we can use three scalars, $\{\rho, \theta, \phi\}$ instead of $\{r, \phi, z\}$. In comparison to the original Cartesian system we began with, we have that

Spherical Polar Coordinates



$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\\ &= \rho\sin\theta\,\mathbf{e}_r + \rho\cos\theta\,\mathbf{e}_z\\ &= \rho\sin\theta\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\cos\theta\,\mathbf{k}\\ &= \rho\sin\theta\cos\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\theta\,\mathbf{k}\\ &\equiv \rho\mathbf{e}_\rho(\theta,\phi)\end{aligned}$$

where

$$\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

a nonlinear function of the coordinate variables ρ, θ and ϕ

Spherical Coordinate Surfaces

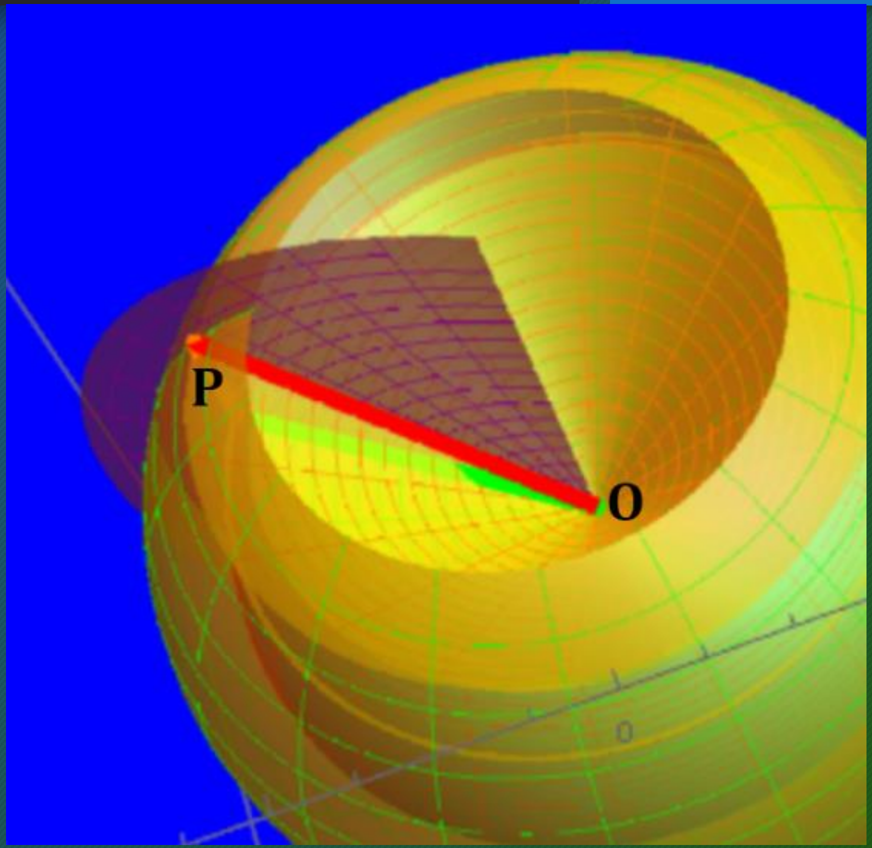
- Again, we introduce the unit vector,

$$\mathbf{e}_\theta \equiv \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$$

- and retain

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$

- as before. It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho, \phi, \theta\}$ directly by the transformation equations, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\theta$ .The orthonormal set of basis for the $\{\rho, \theta, \phi\}$ system is $\{\mathbf{e}_\rho(\theta,\phi), \mathbf{e}_\theta\,(\theta,\phi), \mathbf{e}_\phi\,(\phi)\}$



Cartesian vs Spherical Geometry

- Read Exercises 1.56-1.59 to gain an insight into representation of points and vectors, in Cartesian Coordinates.
- Observe the same points and the effects of transforming into Spherical Polar Coordinates.
- See the effect of the way unit vectors are defined and how the intersecting surfaces differ.
- See the representation of the vectors connecting points A and B are represented.

Mathematica Code

- Copy and execute the code to play with the geometry of the two coordinate systems.
- Give yourself a practice and see how much clarity you gain on what is going on.
- If there are questions, I can deal with them on the web or have another class.
- Otherwise, Test and Chapter Two from next week.

The Vector Space

- We have been using the elementary notions of vectors thus far. We present in summary form, the proper definition of a vector space. This will become necessary for subsequent work.
- Our duty here is to think about these and observe that our secondary school understanding of vectors obeys all the basic rules we shall state.
- For more advanced work, the rules here are fundamental.

A Set of Elements, Named Vectors

- Addition operation is defined and it is commutative and associative under $\mathbb{V}$: that is, $\mathbf{u}+\mathbf{v} \in \mathbb{V}$, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}, \forall\, \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}$. Furthermore, $\mathbb{V}$ is closed under addition: That is, given that $\mathbf{u},\mathbf{v} \in \mathbb{V}$, then $\mathbf{w}=\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}, \Rightarrow \mathbf{w} \in \mathbb{V}$.
- $\mathbb{V}$ contains a zero element $\mathbf{o}$ such that $\mathbf{u}+\mathbf{o}=\mathbf{u}\ \forall\ \mathbf{u} \in \mathbb{V}$. For every $\mathbf{u} \in \mathbb{V}$, $\exists -\mathbf{u}: \mathbf{u}+(-\mathbf{u})=\mathbf{o}$.
- Multiplication by a scalar. For $\alpha,\beta \in \mathbb{R}$ and $\mathbf{u},\mathbf{v} \in \mathbb{V}$, $\alpha\mathbf{u} \in \mathbb{V}$, $1\mathbf{u} = \mathbf{u}, \alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}, (\alpha+\beta)\mathbf{u} = \alpha\mathbf{u}+\beta\mathbf{u},\ \alpha(\mathbf{u}+\mathbf{v}) = \alpha\mathbf{u}+\alpha\mathbf{v}$.

Note the Following:

- By "**under** $\mathbb{V}$", we mean, so long as you are only dealing with elements of the vector space $\mathbb{V}$.
- The only multiplication needed to define a vector space is scaling.
- Our understanding of vectors thus far is admissible here.
  - Condition #1 is satisfied by our parallelogram law of vector addition.
  - The space is closed under addition because when you add two or more vectors (extending the parallelogram law to a polygon of vectors, the result you will get remains a vector, thus guaranteeing closure.
  - Commutativity as well as associativity are straightforward when we try to add more than two vectors and find that the order of addition is immaterial.

Note the following:

- For rule #2, note that a zero vector will be represented by a point; no length – resulting in a magnitude of zero. The negation of a vector is simply to retain the direction but change the sense of the arrow.
- Rule three is merely a mathematical expression of the scaling process. It should also be handled by the addition law when applied to scaled vectors.

Inner Product or Euclidean Vector Space

- An Inner-Product (also called a Euclidean Vector) Space $\mathbb{E}$ is a real vector space that defines, among its elements, the scalar product: for each pair $\mathbf{u}, \mathbf{v} \in \mathbb{E}, \exists\, l \in \mathbb{R}$ such that,
$$l = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$$
- Further, $\mathbf{u} \cdot \mathbf{u} \geq 0$, the zero-value occurring only when $\mathbf{u} = 0$. It is called "Euclidean" because the laws of Euclidean geometry hold in such a space. "Euclidean Geometry" is the totality of the geometry you have done so far, including: Adding all angle of a triangle to 180 degrees, Parallel lines never meeting, Sum of two sides of triangle always larger than the third, etc.

Inner Product or Euclidean Vector Space

- You will later get to know that there are other "geometries" where these things are not valid. These are non-Euclidean geometries.
- Note, as a simple example, these things are NOT valid on the surface of a cone or a sphere!
- The inner product, because of its operational representation as a dot between two vector operands, is also called a dot product, is the mapping

$$" \cdot ": \mathbb{V} \times \mathbb{V} \to \mathbb{R}$$

- from the product space to the real space. The notation here means nothing more than, first expressing the fact that the operational sign to denote the Inner Product is the dot, $" \cdot "$. Note: $\times$ is overloaded here.

Euclidean Point Space

- NOT a vector space;
  - Members are not vectors as we have just defined them.
  - There is a relationship between members of the Euclidean point space and vectors.
- The Euclidean Point Space is the ambient space in which **all** physical objects of interest reside.
  - To make it simple, where you are sitting, or standing, reading this, is a Euclidean Point Space. It is made up of **points** rather than **vectors**.
- What is a Point?
  - On your graph paper from high school, you are used to locating points with an ordered pair of real numbers: Cartesian coordinates of the point.
  - Extend to three dimensions. If $x = \{x_1, x_2, x_3\}, y = \{y_1, y_2, y_3\}$ and $z = \{z_1, z_2, z_3\}$ are three such points

Euclidean Point Space

- Define a particular point

$$o \equiv \{0,0,0\}$$

- the origin of coordinates in $\mathcal{E}$. The Euclidean Point Space may also be referred to non-Cartesian systems.

Euclidean Point Space: Definition

- The Euclidean Point Space, $\mathcal{E}$ is such that, for points x, y, z and an origin, if we represent the vector, $\mathbf{v}$ joining point x to point y as $\mathbf{v}(x, y) \in \mathbb{E}$, where $x, y \in \mathcal{E}$, then,

$$\mathbf{v}(x, z) = \mathbf{v}(x, y) + \mathbf{v}(y, z) \forall\, x, y, z \in \mathcal{E},$$

and

$$\mathbf{v}(x, y) = \mathbf{v}(x, z) \Leftrightarrow y = z$$

for each $x \in \mathcal{E}$

Euclidean Point Space: Definition

- A vector can be embedded into a point space.
    - One example is what we have been calling position vectors!
- Are they vectors? Are they points?
    - They are a hybrid!
- They exist in a point space. Once you define a reference point, Origin, every point now defines a vector by virtue of relating to the reference!

More Mathematica Code

- The code here may be useful in understanding the last example in the chapter
- Copy and execute. Then use the Manipulation buttons and rotations to see.

```
(*
Wriiten by: OA Fakinlede August 2019
   Graphical Demo of Cross product
*)
d1 = Cylinder[{{0, 0, 0}, {.009, .009, .009}}, 1.5];
d2 = Cylinder[{{0, 0, 0}, {.01, .01, .01}}, 2.5];
d3 = Cylinder[{{0, 0, 0}, {.008, .008, .008}}, 2.0];
Arrow1 = Arrow[Tube[{{0, 0, 0}, {1, 1, 1}}, 0.1]];
Manipulate[
 Manipulate[
  Graphics3D[{{Green, Rotate[d1, α π / 20, {1, 1, -1}, {.005, .005, .005}],
     Rotate[Arrow1, α π / 20, {1, 1, -1}, {.005, .005, .005}]},
    {Blue, Rotate[Arrow1, β π / 20, {1, -.5, -.5}, {.005, .005, .005}],
     Rotate[d3, β π / 20, {1, -.5, -.5}, {.005, .005, .005}]}, {Red, Arrow1, d2}},
   Boxed -> False], {α, 0, 20}], {β, 0, 20}]
```