ONE

Vectors: A Review of Elementary Principles & an Extension to Abstract Concepts

"*God made the integers, all else is the work of man*" – Leopold Kronecker

MetaData

The prose, video, slides and the Q&A in this chapter are directed at scoring the following points:

1. A set of **linearly independent** vectors is a set where one member cannot be expressed as a linear combination of the others.
2. When you have the maximum number of such vectors in a set, all other vectors in that space can be expressed as linear combinations of the members of this set. The set of orthonormal vectors we are used to in the Cartesian form is only one kind of such a set. Occasions will arise that will make other linearly independent vectors useful to know.

3. When a set is complete – having the maximum number of vectors, it is said to form **a basis** of the vector space that it **spans**. These words are codes to express the fact that they can be used to represent all other vectors in the space. All that will be needed is the set of scalar weights (or scaling factors) on basis vectors will represent each vector.
4. These scalars are called **components** of the specific vectors represented. Once they are found, with the basis in mind, we use them instead of the vectors they represent because analyses are easier done with the components.
5. #4 above can lead to confusing the vector with its matrix representation. The components of a vector are meaningless unless we specify the basis vectors underlying the representation. This is where the vector, and as we shall see later, the tensor objects, significantly differ from the matrices they look like.
6. The number of vectors constituting a basis spanning the space is the **dimension** of that space.
7. Volume of a cone, pyramid or tetrahedron is one third the base area times the height. For vectors that are linearly independent, the tetrahedron formed by their vertices MUST not vanish. This gives a geometrical meaning to linear independence. For an orthonormal system, this tetrahedron has the value of 1/6.
8. We gain valuable compactness by the **Summation convention**. Mastering it early is a great advantage for later work.
9. Make sure that the worked examples are attempted first before looking at the solutions. You learn faster by that method.

Notation

In this chapter, we shall be dealing with vectors and scalars. We adopt the following notation from elementary set theory:

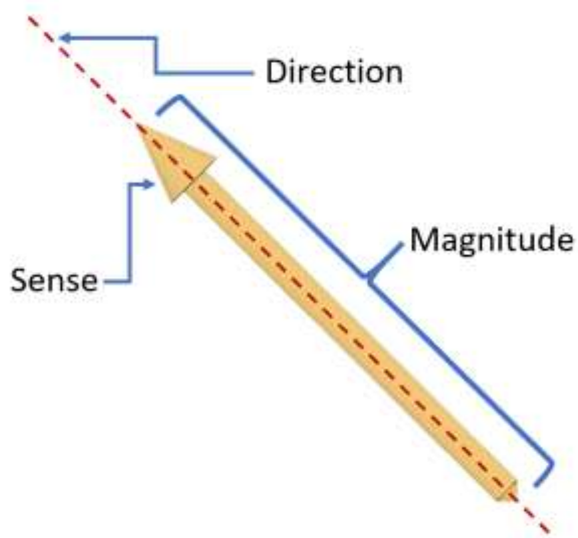
Table 1.1 Notation

Notation	Meaning
$\alpha.\beta \in \mathbb{R}$	α and β belong to the space of real numbers. Or simply, α and β are real numbers.
$\mathbf{v}, \mathbf{w} \in \mathbb{E}$	$\mathbf{v}$ and $\mathbf{w}$ belong to (are members of) the Euclidean vector space $\mathcal{E}$ This is a set of vectors that allow the definition of the dot product. In three dimensions, it also allows the definition of the cross product.
δ_{ij}, δ_j^i	Kronecker Delta, Mixed Kronecker Delta; Coefficients of the Identity tensor

e_{ijk} $\epsilon^{ijk}, \epsilon_{ijk}$	Alternating, Levi-Civita Symbol. Also the coefficients of the Alternating tensor covariant and contravariant alternating tensor components
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots$	ONB Basis Vectors
$\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3,$ $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$	Covariant Base vectors Contravariant Base Vectors
g_{ij}, g^{ij}	Covariant and contravariant metric tensors, Non-Cartesian identity tensor components
$\mathbb{R}$	Real space; Set of real numbers
$\mathbb{V}$ $\mathbb{V} \times \mathbb{V}$	Real Vector Space Product Vector Space
$\mathbb{E}$	Euclidean Vector Space
$\in$	Belongs to
$\mathcal{E}$	Euclidean Point Space
$\forall$	for all
$\otimes$	Binary operator for Dyad or Tensor Product
$\exists$	There exists

Introduction

A vector, roughly speaking, is an abstract representation of quantities that have magnitude, direction and sense. Vectors remind us of forces, velocities, moments, angular velocities, displacements and several quantities that have, in common, the fact that magnitude is not sufficient to quantify them; we must add direction and sense, for full characterization. A plane area, for example, can be thought of as a vector quantity if we add the outwardly drawn unit vector to its full description. It is widely applicable and props up in virtually everything we do. A more abstract – hence more widely applicable definition will be given later. It is very well and good to be clear on the meaning of the elementary notions at the outset. More accurate definitions will still include this as a special case as we shall see.



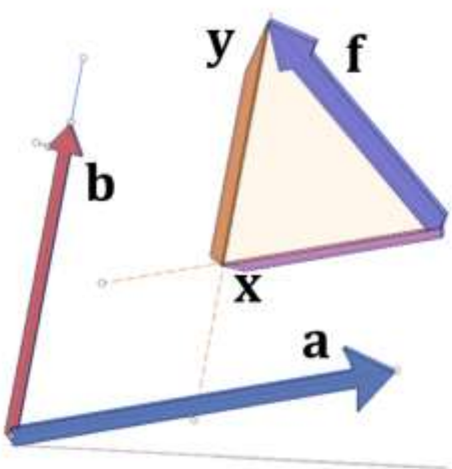
In figure ___, the length of the line gives us the magnitude of the vector; the direction of the line gives us the direction of the vector while the arrow head indicates the sense of the vector. Furthermore, we assume that two vectors are equal if they have the same magnitude and are directed the same way.

Defined in this way, a vector may represent a force, an acceleration, a moment of an angular velocity. While these quantities are divers and represent vastly different things, in so far as each requires a magnitude, as well as a direction and a sense for full representation, the concept of a vector can be used to represent each; and we gain valuable analytical ability for doing so.

In the first instance, we further assume that this vector is contained in a plane. Suppose we introduce two new vectors $\mathbf{a}$ and $\mathbf{b}$. The only thing we require is that these two should not be collinear; their directions are different. We do not require these two new vectors to have unit magnitude; neither do we require them to be orthogonal, but they are not collinear.

Linear Independence, Basis Vectors

We will argue that these two vectors can be used to express any other vector on the plane in the sense that we only need two scaled versions of them to add up to any other vector. If we succeed in showing that, we then say that the two vectors span the space given by the plane. This idea of spanning comes from the fact that we can always select two scaling factors for the two vectors. With these, we can represent any vector as the weighted sum of the two vectors using the two scaling factors (or scalars)



At the tip of the vector $\mathbf{f}$, we draw a line parallel to $\mathbf{b}$. At the tail of the same vector, we draw another line parallel to $\mathbf{a}$. It is easy to see that the vectors $\mathbf{x}$ and $\mathbf{y}$, chosen along these lines are parallel to $\mathbf{a}$ and $\mathbf{b}$ respectively. Consequently, we may write that $\mathbf{x} = \alpha\mathbf{a}$**;** and $\mathbf{y} = \beta\mathbf{b}$ where α and β are the scaling factors (real numbers that can be positive or negative). From the forgoing, we see clearly that any vector $\mathbf{f}$ on this plane can be expressed as

$$\mathbf{f} = \alpha\mathbf{a} + \beta\mathbf{b}$$

$\alpha.\beta \in \mathbb{R}$. Where the above shorthand simply means that the scaling factors belong to the class of real numbers.

The proviso that the two vectors MUST not be collinear is paramount. If they were collinear, it would not be possible to guarantee that every vector in this plane can be so represented. We hereby conclude by this geometrical arrangement that in a 2-D plane, the maximum number of vectors that can be used in this way is two because a third vector can be expressed in terms of the other non-collinear two.

Another way of expressing the fact that these two vectors can be used, with appropriate scalars, in a weighted addition, to represent any other vector, is to say that the set $\{\mathbf{a}, \mathbf{b}\}$ forms a basis for the plane in question.

Notice that it is ***a basis***. There could be other pairs that can equally form a basis for this plane. One such famous pair is the coordinate unit vectors $\{\mathbf{i}, \mathbf{j}\}$ that have unit magnitude and are directed (orthogonal to each other) along the x and y −axes in a Cartesian system of coordinates when the plane in question is the $x - y$ plane. There are several ways you can obtain the vectors to form the basis in any plane. One thing they must have in common is that it MUST not be possible to express one as a scaled version of another. When that condition is satisfied, we say that the vectors are *Linearly Independent*. A set containing the maximum number of linearly independent vectors is what you need to form a basis in any situation.

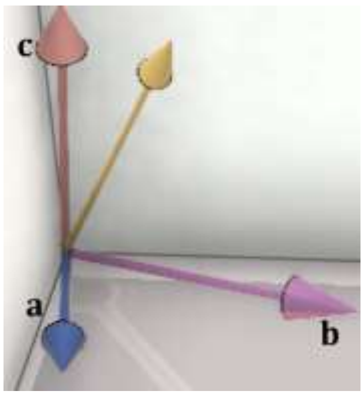
A further observation about the basis vectors. It is possible to complete the parallelogram with the other two sides parallel to the basis vectors. One geometric way to check if the vectors truly form a basis (equivalently, are linearly independent), is that the parallelogram formed must have a non-zero area. Given that θ is the angle between the two vectors, the area of this parallelogram is given by the base times the perpendicular height,

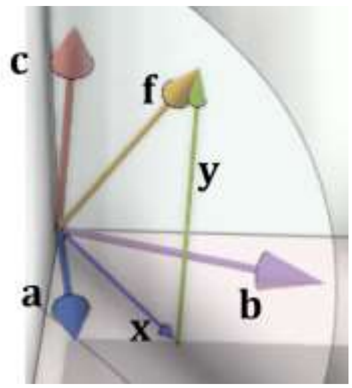
$$A = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = \|\mathbf{a} \times \mathbf{b}\|$$

Hence, we can say that any two vectors such that $\mathbf{a} \times \mathbf{b} \neq \mathbf{o}$ can be used as basis in a 2-D plane.

Linear Independence, 3-Dimensional Space

In three-dimensional space, we must require, in addition to the fact that our three vectors be non collinear, they must not all be contained in the same plane. If this condition is not satisfied, they will not be able to represent the vectors that are not contained in the plane. We hereby introduce the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ that are not collinear and not coplanar as shown in figure 1.3 below.





The first two basis vectors $\{\mathbf{a}, \mathbf{b}\}$are drawn on the x-y plane. The third vector,$\mathbf{c}$ is shown in pink and drawn near the z-axis. A typical vector in this 3-D space can be constructed as shown in the directed line in yellow. In order to represent this vector in terms of the three basis vectors, construct the plane containing vectors , $\mathbf{c}$ and $\mathbf{f}$. Drop a line from the tip of $\mathbf{f}$ to the $x - y$ plane containing $\{\mathbf{a}, \mathbf{b}\}$parallel to vector $\mathbf{f}$. Call the vector image of $\mathbf{f}$ on the $x - y$ plane $\mathbf{x}$. The vector on this oblique plane, parallel to $\mathbf{c}$ is called $\mathbf{y}$. The fact that $\mathbf{x}$ on the same plane as means we can, as we just did in the 2-D case represent it by the two basis vectors in that plane. Therefore, we can easily find $\alpha.\beta \in \mathbb{R}$ such that,

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b}$$

We recall that $\mathbf{y}$ is parallel to $\mathbf{c}$, hence,$\exists\gamma \in \mathbb{R}$ such that, $\mathbf{y} = \gamma\mathbf{c}$. Consequently, any 3-D vector $\mathbf{f}$ can be expressed as,

$$\mathbf{f} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

The set, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ forms a basis for the 3-D space. A tetrahedron formed by joining the tips of this set of basis vectors has a base that is half the size of the parallelogram base.

Dimensionality of Space

One Dimension

A **collection** or a bag full of vectors, which when each is uniquely identifiable is **all** we mean by a **set of vectors**. Somehow, some influential people feel we should call a set of vectors, a **vector space**. Consider the collection, or vector space in the picture below.

Imagine it goes on both sides such that we have a large number of elements in the set. If we take one vector in the list, call it $\mathbf{a}$**;** any other vector $\mathbf{b}$ can be represented as a scalar multiple of $\mathbf{a}$. Alternatively. Given any other vector $\mathbf{b}$ in the vector space, the equation

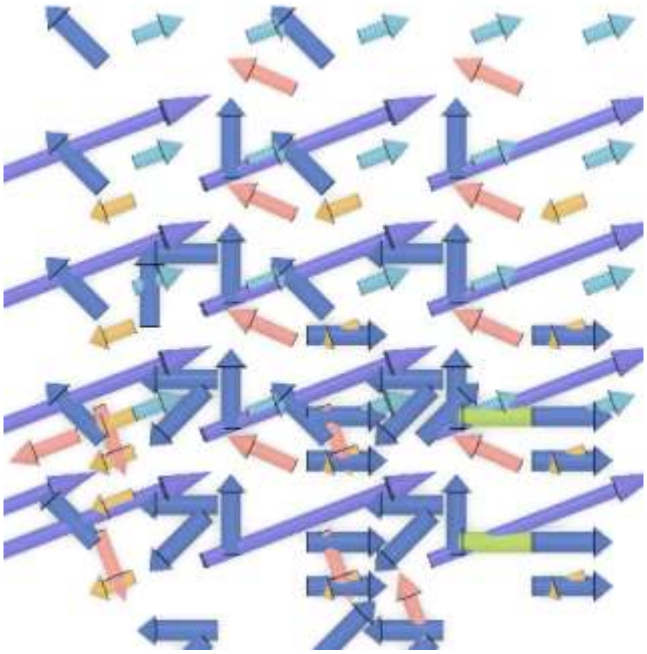
$$\alpha\mathbf{a} + \beta\mathbf{b} = 0$$

Assuming $\alpha \neq 0$, and $\boldsymbol{\beta} \neq 0$, this equation can be simplified to $\mathbf{b} = -\frac{\alpha}{\beta}\mathbf{a}$**.** In which case, once we have identified vector $\mathbf{a}$**,** all we need to represent any other vector in the space is the scalar $-\frac{\alpha}{\beta}$, which can take fractional, decimal, positive or negative values, as a multiple of $\mathbf{a}$. A single vector $\mathbf{a}$spans this space. It forms a basis of this space; and the dimension of this space is one. It is just a basis because we could have chosen any other vector to perform this function. That will be another basis, leading to different set of choices for the scalar $-\frac{\alpha}{\beta}$ to define define each element in the new basis. The fact that any basis we correctly select contains only one vector what makes this a **one-dimensional vector space**.

Two-Dimensional Space

Consider another bag, collection or set of vectors as shown below. Here, all the vectors are contained in a single flat plane. We showed earlier that any two non-collinear vectors, say $\mathbf{a}$ and $\mathbf{b}$ among these can be chosen in such a way that the other vectors in the vector space can be expressed in terms of scalar multiples of the two. We also showed further that once these are chosen, any other vector $\mathbf{x}$ can be expressed as a sum of scaled versions

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b}$$



Of this two so that, $\mathbf{a}$ that have been so chosen have formed a basis of the vector space. Furthermore, given any vector c in the space, the equation,

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0$$

can be solved for $\mathbf{c}$ provided γ is not zero. This means that the maximum number of linearly independent vectors in this space is two. This makes the plane a **two-dimensional vector space**. In such a space, the maximum number of linearly independent vectors you can have is two. There is no uniqueness about the choice, as another two non collinear vectors may as well have been chosen. .

Three Dimensions

The arguments above can be carried to three dimensions. A geometric interpretation can be given. With a more accurate mathematical definition of vectors, we can even go to higher dimensions. Once we are past three dimensions, however, a geometric interpretation will no longer be possible, but the concept can remain useful for analytical purposes.

The maximum number of linearly independent vectors in a three-dimensional space is three. These will, in addition to not been collinear, they MUST NOT all be coplanar. That means that once you have four or more vectors, one will be expressible in terms of the other three.

Components in Different Bases

Up till this point, you may have taken for granted, the fact that you could express any vectors in terms of the basis vector set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which are orthogonal unit vectors along the three coordinate axes in a Cartesian system of coordinates. Two properties of these vectors are that they are mutually orthogonal and that they have unit magnitudes are quite useful. They not only allow you to express any given vector in terms of these basis vectors, they also, by these attractive properties of normality (unit magnitude) and orthogonality (being at right angles to one another) make the computation of the coordinates along the basis vectors very simple.

Despite this, it is important to note that, we DO NOT have to require these properties in order to conclude that a set of vectors can form a basis. What we have proved here is that, in three dimensions, a set of linearly independent vectors (orthogonal or not, normalized or not) can form a basis set. Any other vector in the space, as we have shown above, can be expressed in terms of their components along these vectors. The method of computing their components along these axes may be more difficult; the fact remains they can be found.

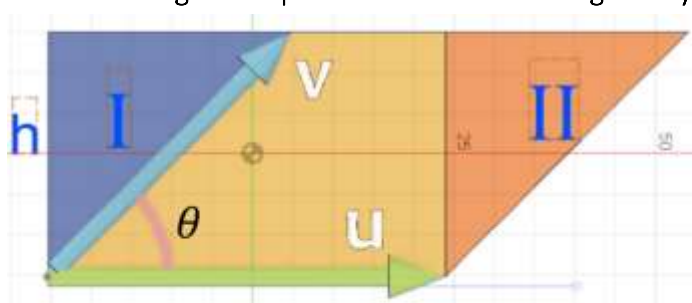
It turns out that occasions will arise when we will no longer require our basis vectors to be orthonormal. However, the linear independence requirement will always be made because it is only linearly independent vectors that can form a basis for any space. Orthonormal sets form basis; not all basis vector sets are orthonormal; Orthonormal sets are linearly independent; not all linearly independent sets are orthonormal.

Area of a Parallelogram

Consider the rectangle whose base is vector $\mathbf{u}$ with height h as shown. Its area is obviously

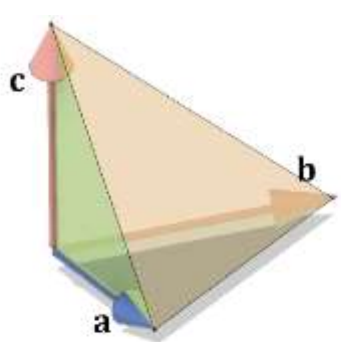
$$A_r = base \times height = \|\mathbf{u}\| h.$$

Triangle II completes the parallelogram so that its slanting side is parallel to vector $\mathbf{v}$. Congruency of I and II is assured as they are both right angled triangles. Removing I gives the parallelogram, keeping it and removing II gives the rectangle. The rectangle is therefore of the same area as the parallelogram. But $h = \|\mathbf{v}\| \sin\theta$. Area of the parallelogram is therefore,



$$A_p = A_r = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$$

Volume of a Tetrahedron



The area of the triangular base of the tetrahedron formed by three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is half the parallelogram formed by the same vectors. Hence this base is $\frac{1}{2}\mathbf{a} \times \mathbf{b}$ with the vector area directed at the normal to this plane. If we take the dot product of this with vector $\mathbf{c}$, we have obtained the base times height. However, for a tetrahedron, or any volume obtained by a flat area lofted linearly to a single point (See Conramid, Problem ___) is one third of this as we shall show. Consequently, a tetrahedron formed by the three vectors has the volume

$$V=\frac{1}{3}\left(\frac{1}{2}\mathbf{a}\times\mathbf{b}\right)\cdot\mathbf{c}=\frac{1}{6}|\mathbf{a}\times\mathbf{b}\cdot\mathbf{c}|=\frac{1}{6}|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|$$

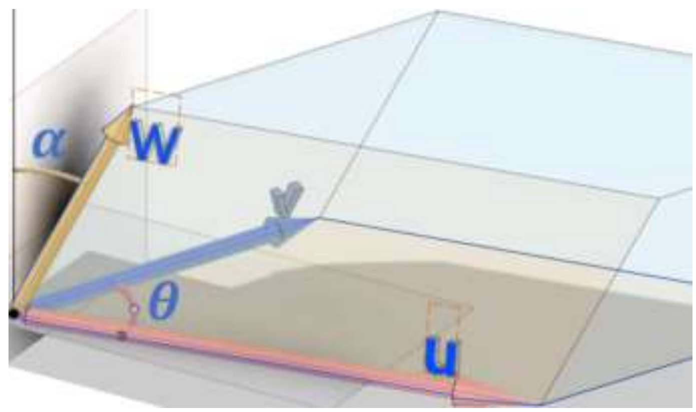
As before, linear independence requires that the volume of this tetrahedron be nonzero. That means that no two of them can be colinear, and the three cannot be coplanar.

Volume of a Parallelepiped

A parallelepiped with sides bound by vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{u}$ subtending an angle θ on the horizontal plane while $\mathbf{w}$ is inclined at angle α to the vertical axis. The base area

$$A=\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta=\|\mathbf{u}\times\mathbf{v}\|$$

Vertical height, h, of the object is $\|\mathbf{w}\|\cos\alpha$. Volume therefore is



$$V=Ah=\|\mathbf{u}\times\mathbf{v}\|\|\mathbf{w}\|\cos\alpha=|\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}|.$$

Orthonormal Basis Vectors

It is often (not always) convenient to use the Cartesian System of coordinates. We can choose a convenient set of linearly independent vectors that are unit vectors and mutually orthogonal to one another. Instead of the calling this set $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ it is found more convenient to refer to them as $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. In this case, $\mathbf{e}_1\times\mathbf{e}_2\cdot\mathbf{e}_3=1$.

A typical vector $\mathbf{f}$ can be written in terms of the basis vectors as,

$$\mathbf{f}=\alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3$$

The scalars a_1,a_2,a_3in this case are easily found by taking the dot product of the equation with $\mathbf{e}_1$,

$$\mathbf{f}\cdot\mathbf{e}_1=a_1\mathbf{e}_1\cdot\mathbf{e}_1+a_2\mathbf{e}_2\cdot\mathbf{e}_1+a_3\mathbf{e}_3\cdot\mathbf{e}_1=a_1.$$

And we can similarly take products with $\mathbf{e}_2$ and $\mathbf{e}_3$ respectively and obtain that, $a_2=\mathbf{f}\cdot\mathbf{e}_2$, and $a_3=\mathbf{f}\cdot\mathbf{e}_3$.

The Einstein Summation Convention

We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we later deal with higher ranked objects (for example, tensors), such notational conveniences become even more important. We shall often deal with coordinate transformations.

When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

Consider the following set of transformation equations:

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j$$

$$y_2 = \sum_{j=1}^{n} a_{2j}x_j$$

$$y_3 = \sum_{j=1}^{n} a_{3j}x_j$$

In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j;\ y_2 = a_{2j}x_j;\ y_3 = a_{2j}x_j$$

Finally, we observe that y_1, y_2, and y_3 can be represented as we have been doing by y_i, $i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \quad i = 1,2,3$$

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij}x_j$$

to represent, more compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

i	k	$a_{ij}a_{jk}$	b_{ik}
1	1	$a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31}$	b_{11}
1	2	$a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}$	b_{12}
1	3	$a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33}$	b_{13}
2	1	$a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31}$	b_{21}
2	2	$a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32}$	b_{22}
2	3	$a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33}$	b_{23}
3	1	$a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31}$	b_{31}
3	2	$a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32}$	b_{32}
3	3	$a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33}$	b_{33}

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}$$

In this same way, we could have also proved that,

$$a_{ij}a_{kj} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

Points to note:

1. An index must not be repeated more than once in any term. A repeated index is called a dummy index.
2. Dummy indices are mutable. Changing them to another unused index in the object does not change value. For example, $a_k a_{kj} = a_\alpha a_{\alpha j} = a_m a_{mj} = a_1 a_{1j} + a_2 a_{2j} + a_3 a_{3j}$
3. Because of #2, use a pair of new dummy variables to avoid situations that could have caused more repeats than allowed.

Also do not forget that the Einstein summation convention is a matter of convenience to avoid the need to write too many summation symbols. The meaning of the expressions and equations are not affected by the correct use of this convention. A great deal of reduction in unnecessary terms can be achieved nevertheless.

Orthonormal vector components again

In a previous section, we introduced the orthonormal basis vectors, $\mathbf{e}_i,\ i = 1,2,3$ With respect to this basis, we can express vectors $\mathbf{v}, \mathbf{w}$ in terms of the basis as, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i$, $\mathbf{w} = w_i\mathbf{e}_i$. The summation sign is no longer needed because of the summation convention. Each v_i is called the component of $\mathbf{v}$, while w_i is called the component of $\mathbf{w}$

The Kronecker Delta. δ_{ij}

The Kronecker delta is a symbol with two indices. The value attained depends on the values of the indices. In our case, each can assume values ranging from 1 to 3. The value of the symbol itself depends, not so much on the indices directly, but on their equality or non-equality. When the indices are equal, the Kronecker Delta takes the value of one; otherwise, its value is zero. Here are examples:

$$\delta_{11} = 1,\ \delta_{12} = 0,\ \delta_{13} = 0$$

$$\delta_{21} = 0,\ \delta_{22} = 1,\ \delta_{23} = 0$$

$$\delta_{31} = 0,\ \delta_{32} = 0,\ \delta_{33} = 1$$

These nine equations can be summarized in the simple form:

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

The Kronecker Delta, for reasons that will later become obvious, is called the **substitution symbol.** We will later also see that they are the components of the **Identity Tensor** when referred to Cartesian coordinates.

Consider the scalar product of two Cartesian base vectors, $\mathbf{e}_i$ and $\mathbf{e}_j$.

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

From which it is clear that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

For any $\mathbf{v} \in \mathbb{V}$,

$$\mathbf{v} = v_i \mathbf{e}_i$$

is the vector expressed in component form using the summation convention. Taking the inner product of the above equation with the basis vector $\mathbf{e}_j$, we have

$$\mathbf{v} \cdot \mathbf{e}_j = v_i \mathbf{e}_i \cdot \mathbf{e}_j = v_i \delta_{ij}$$
$$= v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j}$$

When

$$j = 1, v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{11} + v_2 \delta_{21} + v_3 \delta_{31} = v_1;$$
$$j = 2, v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{12} + v_2 \delta_{22} + v_3 \delta_{32} = v_2 \text{ and}$$
$$j = 3, v_1 \delta_{13} + v_2 \delta_{23} + v_3 \delta_{33} = v_3$$

In all cases, therefore,

$$\mathbf{v} \cdot \mathbf{e}_j = v_j$$

which contains the expressions for v_1, v_2, and v_3 as we allow $j = 1,2,3$ in the above equation.

Substitution Symbol

The epithet of "substitution symbol, as applied to the Kronecker Delta is the result of the above result: $v_i \delta_{ij} = v_j$! It is a general rule: When you have the product of the Kronecker Delta and another object with which it shares an index, the result of that product is to remove the Kronecker Delta and allow a substitution of the symbol that was not shared as in this expression. Look at the following examples:

Product with Kronecker Delta	Result
$S_{\alpha\beta}\delta_{i\alpha}$	$S_{i\beta}$
$T_{ijk}\delta_{j\alpha}$	$T_{i\alpha k}$

$\boldsymbol{\delta_{ij}\delta_{\alpha j}}$	$\delta_{i\alpha}$
$\boldsymbol{\delta_{ij}\delta_{ij}}$	$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
$\boldsymbol{e_{ijk}\delta_{jk}}$	$e_{ijj} = e_{ikk}$

The Alternating Levi-Civita Symbol.

Consider the determinant,

$e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$, where the indices i, j and k, varying *column to column*, can take the values 1,2 or 3. Clearly, the values $i = 1, j = 2$ and $k = 3$ gives the determinant,

$$e_{ijk} = e_{123} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

the determinant of the **Identity Tensor**. A simple check reveals the fact that

$$e_{123} = e_{231} = e_{312} = 1$$

$$e_{132} = e_{321} = e_{213} = -1$$

and the value of this quantity is zero in every other case as can be checked by a simple determinant expansion. Those cases include situations when one or more of the indices is equal to another.

We can arrive at the same relationship if, going *row-wise*, we define

$$e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}.$$

Again, just as the previous case,

$$e_{123} = e_{231} = e_{312} = 1$$

$$e_{132} = e_{321} = e_{213} = -1$$

with all the other cases returning zero. In either of these cases, the symbol, e_{ijk} or e_{rst} as we have defined it, is called the **Levi-Civita** or **Alternating Symbol**. An even permutation of its symbols retains sign while any odd permutation negates the sign. This behavior can be predicted from the knowledge of determinants. A row or column swap negates sign while two row or columns swaps becomes a double negation of sign and gives positive. Consequently, even permutations result in sign preservation while odd permutations negative. It is said to be perfectly anti-symmetric.

Continuing with the determinant interpretation, equality of the indices denotes a determinant with repeated rows or columns. Clearly, we have zero value for such a determinant.

Products of Alternating tensors

Consider the product, $e_{rst}e_{ijk}$ of the alternating symbols – the determinants we just defined. We will proceed to show that,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

The definition of e_{ijk} and of δ_{ij} immediately shows that,

$$e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}, \text{ and } e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$$

The product,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{r1}\delta_{1i} + \delta_{r2}\delta_{2i} + \delta_{r3}\delta_{3i} & \delta_{r\alpha}\delta_{\alpha j} & \delta_{r\alpha}\delta_{\alpha k} \\ \delta_{s\alpha}\delta_{\alpha i} & \delta_{s\alpha}\delta_{\alpha j} & \delta_{s\alpha}\delta_{\alpha k} \\ \delta_{t\alpha}\delta_{\alpha i} & \delta_{t\alpha}\delta_{\alpha j} & \delta_{t\alpha}\delta_{\alpha k} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

We now consider a situation when on of the indices of the alternating symbols in a product are the same. To do this, we begin from the above result:

Given that

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix} \text{ Show that } e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{rj}\delta_{si}$$

Clearly, not forgetting that repetition of an unknow index signifies a summation,

$$e_{rsk}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{11} + \delta_{22} + \delta_{33} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

Expanding the equation, using the third row, we have:

$$\begin{aligned} e_{rsk}e_{ijk} &= \delta_{ki} \begin{vmatrix} \delta_{rj} & \delta_{rk} \\ \delta_{sj} & \delta_{sk} \end{vmatrix} - \delta_{kj} \begin{vmatrix} \delta_{ri} & \delta_{rk} \\ \delta_{si} & \delta_{sk} \end{vmatrix} + 3 \begin{vmatrix} \delta_{ri} & \delta_{rj} \\ \delta_{si} & \delta_{sj} \end{vmatrix} \\ &= \delta_{ki}\left(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}\right) - \delta_{kj}(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk}) + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= -2\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj} \end{aligned}$$

It is instructive to observe the two terms in the last expression. Notice that there is a change in partners in the pairs. This observation, if we remember, means that once we can form one term, the other is a simply pairing exchange.

We now proceed to look at the example where two of the indices of the alternating symbols in the product are the same. Beginning from our most recent result, Given that

$$e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$$

Show that $e_{rjk}e_{ijk} = 2\delta_{ri}$.

In the equation, $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$ set $s = j$, we have,

$$e_{rjk}e_{ijk} = \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} = 3\delta_{ri} - \delta_{ri} = 2\delta_{ri}.$$

Component Form of Products of Vectors

Invoking the Einstein summation convention and using the Cartesian system of coordinates, we can write the component form of vectors $\mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = b_j\mathbf{e}_j$. We can go ahead to write the scalar and vector products in their component forms:

Scalar, Dot Product

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_i\mathbf{e}_i) \cdot \left(b_j\mathbf{e}_j\right) \\ &= a_ib_j\mathbf{e}_i \cdot \mathbf{e}_j = a_ib_j\delta_{ij} \\ &= a_ib_i \\ &= a_1b_1 + a_2b_2 + a_3b_3. \end{aligned}$$

Which is the meaning of the compact form, a_ib_i. (**Note**: It is correct that $b_i\mathbf{e}_i = b_j\mathbf{e}_j$. Any dummy index would be ok. However, using the first would have led to $a_ib_i\mathbf{e}_i \cdot \mathbf{e}_i$ which would not only violate the summation convention ruled that no index be repeated more than once in any term. It would also have led to wrong results).

Vector, Cross Product.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= e_{ijk} a_i b_j \mathbf{e}_k\end{aligned}$$

The last step requires us to show that the cross product of the base vectors, $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$. This important result comes from a compendium of repeated application of the definition of the cross product as shown in the table below:

i	j	$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$
1	3	$e_{13k}\mathbf{e}_k = e_{131}\mathbf{e}_1 + e_{132}\mathbf{e}_2 + e_{133}\mathbf{e}_3 = -\mathbf{e}_2$
1	2	$e_{12k}\mathbf{e}_k = e_{121}\mathbf{e}_1 + e_{122}\mathbf{e}_2 + e_{123}\mathbf{e}_3 = \mathbf{e}_3$
2	3	$e_{23k}\mathbf{e}_k = e_{231}\mathbf{e}_1 + e_{232}\mathbf{e}_2 + e_{233}\mathbf{e}_3 = \mathbf{e}_1$
3	1	$e_{31k}\mathbf{e}_k = e_{311}\mathbf{e}_1 + e_{312}\mathbf{e}_2 + e_{313}\mathbf{e}_3 = \mathbf{e}_2$
1	1	$e_{11k}\mathbf{e}_k = e_{111}\mathbf{e}_1 + e_{112}\mathbf{e}_2 + e_{113}\mathbf{e}_3 = 0$
2	2	$e_{22k}\mathbf{e}_k = e_{221}\mathbf{e}_1 + e_{222}\mathbf{e}_2 + e_{223}\mathbf{e}_3 = 0$
2	1	$e_{21k}\mathbf{e}_k = e_{211}\mathbf{e}_1 + e_{212}\mathbf{e}_2 + e_{213}\mathbf{e}_3 = -\mathbf{e}_1$
3	2	$e_{32k}\mathbf{e}_k = e_{321}\mathbf{e}_1 + e_{322}\mathbf{e}_2 + e_{323}\mathbf{e}_3 = -\mathbf{e}_1$
3	3	$e_{33k}\mathbf{e}_k = e_{331}\mathbf{e}_1 + e_{332}\mathbf{e}_2 + e_{333}\mathbf{e}_3 = 0$

Note that we only need to specify the i and j values as there is indexing into all the values of k because it is a dummy index in the above expression.

Expansion of the vector product is straightforward:

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= e_{ijk} a_i b_j \mathbf{e}_k \\ &= e_{123}a_1 b_2 \mathbf{e}_3 + e_{132}a_1 b_3 \mathbf{e}_2 + e_{231}a_2 b_3 \mathbf{e}_1 + e_{213}a_2 b_1 \mathbf{e}_3 + e_{312}a_3 b_1 \mathbf{e}_2 + e_{321}a_3 b_2 \mathbf{e}_1\end{aligned}$$

$$= a_1 b_2 \mathbf{e}_3 - a_1 b_3 \mathbf{e}_2 + a_2 b_3 \mathbf{e}_1 - a_2 b_1 \mathbf{e}_3 + a_3 b_1 \mathbf{e}_2 - a_3 b_2 \mathbf{e}_1$$

By avoiding repeated indices, we gain speed in ignoring zero elements in the expression.

You will see that only the six non-vanishing values of e_{ijk} appear in the expression here. We gain valuable time and avoid unnecessary evaluation by following a simple strategy:

1. Once the first index, $i = 1$, only two non-zero cases exist: $j = 2, k = 3$ and $j = 3, k = 2$
2. When $i = 2$, again, only two non-zero cases exist: $j = 3, k = 1$ and $j = 1, k = 3$
3. Lastly, When $i = 3$, again, only two non-zero cases exist: $j = 1, k = 2$ and $j = 2, k = 1$.

Using this approach, it becomes unnecessary to write 27 terms when 21 of them vanish. Instead, we can pick out only the six non-vanishing terms.

The Dyad

We are used to producing scalars or vectors by taking a product of two vectors. One exceedingly important object that you can also produce from taking such a binary product is a **Tensor**. Naturally, we shall call such a product a "Tensor Product".

Its symbol, $\otimes$, is not a dot or a cross. It is a symbol that may look strange. That symbol combines the product sign and a circle. It is called a dyad operator. Therefore, as before, a tensor product also has a nickname, "the Dyad", or a "Dyad Product".

The **dyad** is **defined** by the result of its action on a vector. Consider the dyad $\mathbf{a} \otimes \mathbf{b}$**.** Its action on a vector $\mathbf{c}$ is defined as follows:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

That is, it produces a vector in the direction of its first argument scaled by a factor of the scalar product of its second argument with the vector it acts upon.

The most elementary tensor you can get is the dyad product of two base vectors: $\mathbf{e}_i \otimes \mathbf{e}_j$

The product of two vectors can be expressed in terms of this dyad base:

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes \left(b_j \mathbf{e}_j\right) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

The summation convention still applies so that it is easy to see that the above expression contains nine components.

Observe immediately that, in 3D, just as you express a vector in terms of three basis vectors, there are nine base dyads for expressing every tensor: $\mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{e}_1 \otimes \mathbf{e}_2$, $\mathbf{e}_1 \otimes \mathbf{e}_3$, $\mathbf{e}_2 \otimes \mathbf{e}_1$, $\mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{e}_2 \otimes \mathbf{e}_3$, $\mathbf{e}_3 \otimes \mathbf{e}_1$, $\mathbf{e}_3 \otimes \mathbf{e}_2$, $\mathbf{e}_3 \otimes \mathbf{e}_3$
To find the components of a tensor is to find nine scalar coefficients to these base dyads.

Binary, Ternary Operations

We will introduce tensors more formally in the next chapter. For our purpose here, remember that with two vectors, we have defined three different products that may result. These are: scalar or dot product; vector or cross product; tensor or dyad product. This means that, unlike scalars, you DO NOT simply "multiply" two vectors. To say that creates an ambiguity because we have at least these three possible results: a scalar, a vector or a tensor. The particular product we have in mind MUST be specified. While the statement, multiply two scalars is sensible, the same statement, applied to two vectors is ambiguous. When we are dealing with vector multiplication, we must disambiguate by being specific on which vector multiplication or product we have in mind. We do this in prose, we also do it in the equation in which vector products are involved. The disambiguation method is the sign, dot, cross or the dyad circle on a product sign that signifies a tensor product. It is therefore an error, to simply concatenate two vectors to signify a product, as you would be permitted to do when dealing with two scalar variables or numbers. Given that α and β are scalars, and that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, the following table provides examples of products explaining why some may not be correct statements:

Product	Right or wrong	Comments
$\alpha\mathbf{u}$	Correct	Scaling a vector, multiplication of a scalar and a vector; No explicit sign required
$\mathbf{u}\beta\mathbf{v}$	Error	$\mathbf{u}\beta$ is a scaled vector whose product with $\mathbf{v}$ is ambiguous. Possibilities include $(\mathbf{u}\beta)\cdot\mathbf{v}$, $\mathbf{u}\times(\beta\mathbf{v})$, or $\mathbf{u}\otimes(\beta\mathbf{v})$
$\beta\alpha$	Correct	Product of two scalars; No explicit sign required

$\mathbf{vu}$	Error	Product of two vectors; $\mathbf{v}\cdot\mathbf{u}\neq\mathbf{v}\times\mathbf{u}\neq\mathbf{v}\otimes\mathbf{u}$ Explicit disambiguating sign required
$\beta(\mathbf{u}\times\mathbf{v})$	Correct	Vector product of two vectors gives a vector. Multiplying this result by a scalar does not require another sign
$\mathbf{u}\cdot\mathbf{v}\alpha$	Correct	The dot product of a vector with a scaled vector. No ambiguity is created with the location of α; $\mathbf{u}\cdot\mathbf{v}\alpha$, $(\mathbf{u}\alpha)\cdot\mathbf{v}$, or $\alpha\mathbf{u}\cdot\mathbf{v}$ all mean the same thing.
$\beta\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}\alpha$	Correct	Scalar triple product with vector scaling along. Result is the same as $(\beta\alpha)\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}=(\beta\alpha)\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$
$\beta\mathbf{u}\times\mathbf{v}\times\mathbf{w}$	Error	Vector triple product with vector scaling along. Vector product is <u>not associative</u>: $\beta\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-\beta(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ $\neq\ \beta(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ $=\ \beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-\beta(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ Parentheses are required to show which product is intended.
$\mathbf{u}\cdot\mathbf{v}\otimes\mathbf{w}$	Correct	Only one interpretation makes sense: $(\mathbf{v}\otimes\mathbf{w})\,\mathbf{u}$**.**
$\mathbf{u}\times\mathbf{v}\otimes\mathbf{w}$	Correct	Treat the vector cross as a tensor, then obtain the LHS: $(\mathbf{u}\times\mathbf{v})\otimes\mathbf{w}=\ \mathbf{u}\times(\mathbf{v}\otimes\mathbf{w})$ The two different interpretations evaluate to the same value.

More on the Tensor Product

Given vectors $\mathbf{a}=a_i\mathbf{e}_i$ and $\mathbf{b}=b_j\mathbf{e}_j$, we may use matrix notation, in two different ways, and write,

$$\mathbf{a}=[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3=a_i\mathbf{e}_i$$

$$\mathbf{b}=[b_1,b_2,b_3]\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3=b_j\mathbf{e}_j$$

The dyad $\mathbf{a} \otimes \mathbf{b} = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$ which can be given in its full component form as,

$$\begin{aligned}\mathbf{a} \otimes \mathbf{b} &= a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j \\ &= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes [b_1, b_2, b_3] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\ &= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\ &= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}\end{aligned}$$

The matrices of scalars can cross the dyad sign because only one product is defined for scalars. For vectors, the case is different. Three different products are defined between two vectors. We must always be consistent with the product involved. The matrix for the dyad $\mathbf{a} \otimes \mathbf{b}$ is

$$\begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

The dyad itself is,

$$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

or,

$$= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

The matrix representation of the vector is $\mathbf{a}$ is $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ or $\mathbf{b}$, $[b_1, b_2, b_3]$**.** The vectors, in component form are expressed as, $\mathbf{a} = a_i \mathbf{e}_i = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ or $\mathbf{b} = b_j \mathbf{e}_j = [b_1, b_2, b_3] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$. The matrix elements will change if we change the basis vectors to which the vector or dyad is referred. Again, as you can see, the matrix representations, in all cases, are not the same as the tensor or the vector.

Trace of a Dyad

A very important **Linear Operation** on a dyad is the trace operation. It turns a dyad into a scalar quantity. It is achieved by simply changing the dyad operator into a dot as follows:

$$\begin{aligned}\operatorname{tr}(\mathbf{a}\otimes\mathbf{b}) &= a_i b_j \operatorname{tr}\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)\\ &= a_i b_j\left(\mathbf{e}_i\cdot\mathbf{e}_j\right)\\ &= a_i b_j\,\delta_{ij}\\ &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3.\end{aligned}$$

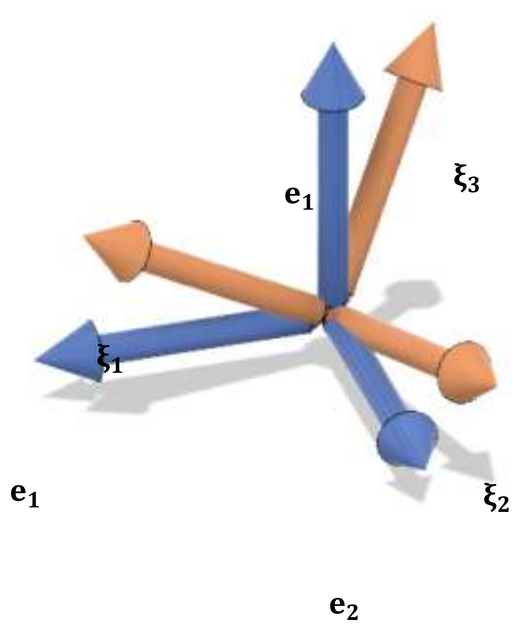
A simple observation will show that this is the sum of the diagonal elements

$$\begin{bmatrix} \boldsymbol{a_1 b_1} & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & \boldsymbol{a_2 b_2} & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & \boldsymbol{a_3 b_3} \end{bmatrix}$$

of the dyad matrix representation as shown above. There is more to say about linearity, linear operators and linear functions in the next chapter.

Coordinate Transformation

Consider a set of Cartesian coordinate orthonormal vectors, $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$ shown in blue in figure 1.4. These vectors are position vectors at $\{1,0,0\},\{0,1,0\}$ and $\{0,0,1\}$ respectively. Consider another orthonormal system, shown in pink, whose unit vectors are oriented as shown in the figure. Let these unit vectors be $\{\boldsymbol{\xi_1},\boldsymbol{\xi_2},\boldsymbol{\xi_3}\}$. The set $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$, since they are orthonormal, are also linearly independent. Consequently, each member of the set, $\{\boldsymbol{\xi_1},\boldsymbol{\xi_2},\boldsymbol{\xi_3}\}$ can be expressed in terms of the basis vectors in $\{\mathbf{e_1},\mathbf{e_2},\mathbf{e_3}\}$. (We note that the opposite is also possible: we could express the original vectors in terms of the rotated system). Taking these vectors one by one, we may write,



$$\boldsymbol{\xi}_\mathbf{1} = \alpha_1\mathbf{e_1} + \beta_1\mathbf{e_2} + \gamma_1\mathbf{e_3}$$
$$\boldsymbol{\xi}_2 = \alpha_2\mathbf{e_1} + \beta_2\mathbf{e_2} + \gamma_2\mathbf{e_3}$$
$$\boldsymbol{\xi}_3 = \alpha_3\mathbf{e_1} + \beta_3\mathbf{e_2} + \gamma_3\mathbf{e_3}$$

The coefficients can be found by taking the dot products as usual. Note that we can gain more compactness and use only one symbol for all the nine coefficients if we adopt this simple arrangement: Let $\alpha_i \equiv a_{i1}, \beta_i \equiv a_{i2}$, and $\gamma_i \equiv a_{i3}$. The three equations can therefore be written more compactly as,

$$\boldsymbol{\xi}_{\boldsymbol{i}} = a_{ij}\mathbf{e}_{\boldsymbol{j}}$$

We can find each of the nine coefficients by taking the scalar product of this equation with $\mathbf{e}_{\boldsymbol{\alpha}}$:

$$\boldsymbol{\xi}_{\boldsymbol{i}} \cdot \ \mathbf{e}_{\boldsymbol{\alpha}} = a_{ij}\mathbf{e}_{\boldsymbol{j}} \cdot \mathbf{e}_{\boldsymbol{\alpha}} = a_{ij}\delta_{j\alpha} = a_{i\alpha}$$

Or, $a_{ij} = \boldsymbol{\xi}_{\boldsymbol{i}} \cdot \ \mathbf{e}_j$.These linear equations can always be inverted and we may have the converse:

$$\mathbf{e}_{\boldsymbol{j}} = b_{jk}\boldsymbol{\xi}_k$$

$\mathbf{B} = [b_{ij}]$ is obviously the inverse of the coefficient matrix $\mathbf{A} = [a_{ij}]$. This inverse relationship can be obtained easily using the indicial notation. Starting with $\boldsymbol{\xi}_{\boldsymbol{i}} = a_{ij}\mathbf{e}_{\boldsymbol{j}}$, we could substitute for e_j and write,

$$\boldsymbol{\xi}_{\boldsymbol{i}} = a_{ij}\mathbf{e}_{\boldsymbol{j}} = a_{ij}b_{jk}\boldsymbol{\xi}_k$$

Taking scalar products again, we have,

$$\begin{aligned}
\boldsymbol{\xi}_{\boldsymbol{i}} \cdot \boldsymbol{\xi}_\alpha = \delta_{i\alpha} &= a_{ij}b_{jk}\boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_\alpha \\
&= a_{ij}b_{j\alpha} \\
&= (a_{ij}\mathbf{e}_{\boldsymbol{j}}) \cdot (a_{\alpha\beta}\mathbf{e}_{\boldsymbol{\beta}}) \\
&= a_{ij}a_{\alpha\beta}\delta_{j\beta} \\
&= a_{ij}a_{\alpha j}
\end{aligned}$$

These equations in matrix form can be written as,

$$\mathbf{I} = \mathbf{AB} = \mathbf{AA}^{\mathrm{T}}$$

Showing that the inverse transformation matrix is the transpose of the original transformation. The inverse transformation can now be re-written, using this result:

$$\mathbf{e}_j = b_{jk}\boldsymbol{\xi}_k = a_{kj}\boldsymbol{\xi}_k$$

So that, in a transformation of from one orthonormal system to another, if

$$\boldsymbol{\xi}_i = a_{ij}\mathbf{e}_j,$$

then

$$\mathbf{e}_i = a_{ji}\boldsymbol{\xi}_j$$

because the inverse of the transformation is simply its transpose.

The Euclidean Point Space

The 3D Euclidean Point Space we live in is where all engineering objects of interest to us reside. This space contains point locations that can be occupied by a location in an object at a particular time. It is often of interest to be able to do several things:

1. Locate the point in an unambiguous way,
2. Relate the point to one or more other points in its vicinity, and
3. Define quantities that take up values of interest at that point.

* Temperature map of this classroom (one thousand thermometers)
* Temperature distribution, Temperature field.
* Tensor Fields

Cartesian & Other Coordinate Systems

Our coordinate systems so far have very interesting features: They are based on spatially constant unit vectors orthogonal to each other. These are called Rectangular Cartesian or Orthonormal Base (ONB) Systems. We have seen that we are only required to have, for basis vector sets to span a space, that they are linearly independent.

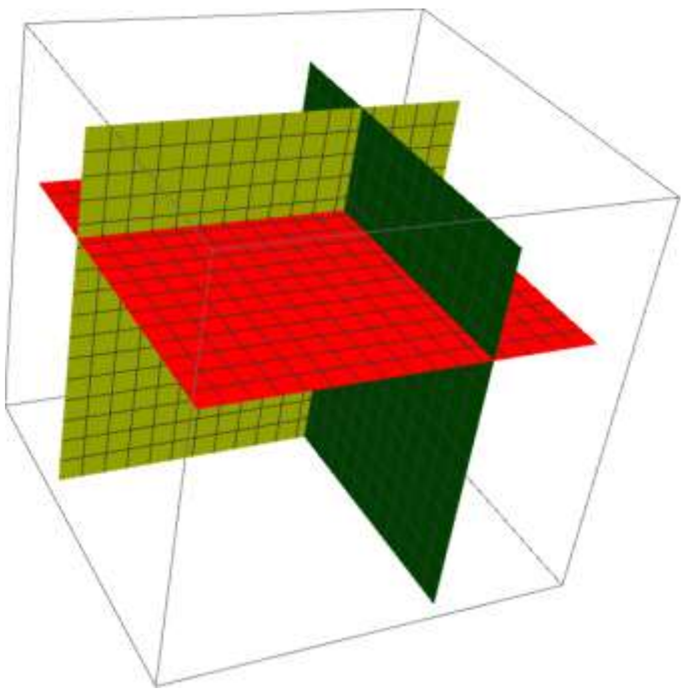
ONBs are more than linearly independent; their orthonormal attributes make the computation of coordinates for any vector referred to them, very easy to obtain. There are other advantages:

* We can refer the room to a set of Cartesian coordinates (x, y, z).
* In this system, each location is represented by three ordered numbers. The first represents the x coordinate, the second the y coordinate, and the third, the z coordinate respectively.
* The basis vector set is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. These are along the constant coordinate lines which are straight line intersections of the coordinate planes as shown below.
* Following $Mathematica^{®}$ code implements this idea

```
Cart1=ParametricPlot3D[{1,y,z},{y,0,1.4},{z,0,1.4},PlotStyle->Red];
Cart2=ParametricPlot3D[{x,1,z},{x,0,1.4},{z,0,1.4},PlotStyle->Green];
Cart3=ParametricPlot3D[{x,y,1},{x,0,1.4},{y,0,1.4},PlotStyle->Yellow];
Show[Cart1,Cart2,Cart3,PlotRange->{{0,1.5},{0,1.5},{0,1.5}}, Ticks->None]
```

In locating point $\mathbf{P}(x_1, y_1, z_1)$ above, we constructed three coordinate planes:

* A dark colored plane perpendicular to the x −axis,
* A yellow plane perpendicular to the y −axis, and
* A red plane perpendicular to the z −axis.



Position Vector

* Furthermore, we can define a vector for the point location $\boldsymbol{P}(x_1, y_1, z_1)$ .Such a vector is defined by joining the point $\boldsymbol{P}$ to the origin to form the vector **OP** represented by the line shown.
* The vector whose magnitude is defined by the length of **OP**, and whose direction is indicated by the direction of OP, a ***Position Vector***.
* We defined a vector (a member of the Euclidean Vector Space, that is now embedded in the Euclidean point space of our daily experience.
    * The latter contains just points, the former is a collection of objects that obey certain rules that make us label them "vectors".
    * This particular one is not just a vector, it is a position vector because it is the point $\boldsymbol{P}(x_1, y_1, z_1)$ that gave birth to it. At any other point we define by three numbers, we can also get a position vector in this simple way.

Notice several things that are attractive in the Cartesian system we have described.

* Each coordinate surface is a plane. The three defined at a particular point are respectively parallel to the three you can define at any other point.

* Each coordinate lines: the intersection of these planes that are parallel to the axes are similarly parallel straight lines at all points in the system.
* The basis vectors – usually defined as unit vectors along the axes, are always the same at any point in the Cartesian system. It does not matter where the point P is located, the basis vectors are the same unit vectors we define as $(\mathbf{i}, \mathbf{j}$ and $\mathbf{k})$ or $(\mathbf{e}_1, \mathbf{e}_2,$ and $\mathbf{e}_3)$ along the coordinate lines at the origin.

These properties combine to make the Cartesian coordinate system very simple and easy to use. It is no wonder that it is the first coordinate system you get introduced to – for most people, as early as secondary school!

The first important advantage of the Cartesian system is the simplicity of the expression for a position vector. The position vector OP can be written simply as,

$$\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

Or, more conveniently as,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

Where we have replaced (x_1, y_1, z_1) by (x_1, x_2, x_3) so we may benefit from the compactness of the Einstein's summation convention. This expression is linear in the coordinate variables. There are two other hidden reasons why this coordinate system is so simple and easy to use. It may not be obvious that the simple expression of the position vector we have here is possible only in the Cartesian system.

In other coordinate systems, the position vector is usually a much more complicated function of the coordinate variables and the basis vectors. In general, if we do not assume that we are using the Cartesian system,

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2, \alpha_3, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$$

where α_i, $i = 1,2,3$ are the coordinate variables and $\mathbf{g}_i$, $i = 1,2,3$ are the basis vectors. The simple linear form we have for the Cartesian case, as we shall see is a rare exception and a special case. The functional for of the position vectors can be complicated.

A second reason that the Cartesian system is so easy, useful and pervasive is the related fact of the constancy of the basis unit vectors. To illustrate this, imagine we continue with our thought experiment to get a temperature map for the room, then we have a scalar field $T(x_1, x_2, x_3)$. If we have a vector function defined at each point, then we get a vector field $\boldsymbol{v}(x_1, x_2, x_3)$ .We can easily write the vector field in terms of three scalar fields that we call its components; hence, we may write,

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

Where $v_i(x_1, x_2, x_3),\ i = 1,2,3$ are the components of the velocity vector. The fact that the basis vectors $\boldsymbol{e}_i, i = 1,2,3$ neither varies temporally nor spatially means that differential and integral calculus with the Cartesian system take a particularly easy form. Differentiating the above equations, whether with respect to time or to space, we simply focus on the functions, $v_i(x_1, x_2, x_3)$ and ignore the constants $\mathbf{e}_i, i = 1,2,3$!

A third reason for the simplicity of the Cartesian system is in the fact that the three numbers representing the coordinates are of the same dimensionality.

The numbers, x_1, x_2,and x_3 (coefficients of the basis vectors) for the coordinates of $\mathbf{P}$ are all lengths. They are all the same dimension. There is nothing compelling you to use lengths for your coordinate variables in a coordinate system.

Observation: A partial differentiation of the position vector with respect to the coordinate variables yield the basis vectors for the coordinate system as shown here:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

$$\frac{\partial \mathbf{r}}{\partial x_i} = \mathbf{e}_i, i = 1,2,3.$$

This applies to the other coordinate systems as well.

In fact, the two next most popular systems – the Spherical and Cylindrical systems use a combination of lengths and angles! If you are not careful, and you use these coordinate systems just the way you do the Cartesian, your first error might be that you are adding quantities of different dimensions and units in the same expression and will be guaranteed to obtain wrong results.

Cylindrical Polar Coordinates

In the cylindrical system, we select the three numbers that we shall use to represent a typical point P using a different strategy. We select two lengths and an angle. Since we already are quite used to the Cartesian system, let us first note that the third coordinate in the Cylindrical Polar

System is shared with the Cartesian. Even if we represent it with a different symbol, note that the z-coordinate as well as the $\mathbf{k}$, $\mathbf{e}_3$ or $\mathbf{e}_z$ essentially remain the same in both Cartesian and the Cylindrical Polar system.

Begin with our familiar Cartesian system of coordinates. We can represent the position of a point (position vector) with three coordinates x_1, x_2, x_3 $(\in R)$ such that,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

That is, the choice of any three scalars can be used to locate a point. We now introduce a transformation (called a polar transformation) of $\{x_1, x_2\} \to \{r,\ \phi\}$ such that, $x_1 = r\cos\phi$, and $x_2 = r\sin\phi$ .Note also that this transformation is invertible: $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$

With such a transformation, we can locate any point in the 3-D space with three scalars $\{r,\ \phi, z\}$ instead of our previous set $\{x_1, x_2, x_3\}$ .Our position vector is now,

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$

where we define $\boldsymbol{e}_r \equiv \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2$,$\boldsymbol{e}_z$ is no different from $\mathbf{e}_3$ or $\boldsymbol{k}$. In order to complete our triad of basis vectors, we need a third vector, $\boldsymbol{e}_\phi$ .In selecting $\boldsymbol{e}_\phi$,we want it to be such that $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ can form an orthonormal (pairwise orthogonal and individually normalized) basis. Let

$$\mathbf{e}_\phi = \xi\mathbf{e}_1 + \eta\mathbf{e}_2$$

To satisfy our conditions, $\mathbf{e}_\phi \cdot \mathbf{e}_r = 0$,$\mathbf{e}_\phi \cdot \mathbf{e}_z = 0$ (automatically satisfied by not choosing a different third coordinate) and $\sqrt{\xi^2 + \eta^2} = 1$.

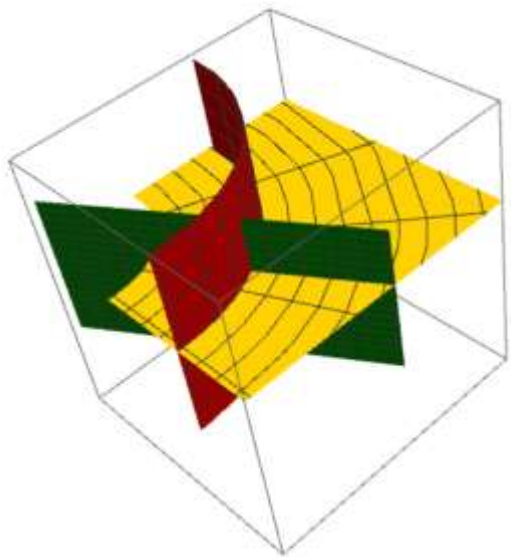
It is easy to see that $\mathbf{e}_\phi \equiv -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$ satisfies these requirements. $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ forms an orthonormal (that is, each member has unit magnitude and they are pairwise orthogonal) triad just like $\mathbf{e}_i, i = 1,2,3$.

The transformation we have just described can be given a geometric interpretation. In either case, it is the definition of the **Cylindrical Polar coordinate system**.

Unlike our Cartesian system, we note that $\{\mathbf{e}_r(\phi), \mathbf{e}_\phi(\phi), \mathbf{e}_z\}$ as the first two of these are not constants but spatial variables dependent on angular orientation. $\mathbf{e}_z$ remains a constant vector as in the Cartesian case.

Geometric Interpretation

The coordinate system just described requires us, as before, to select three ordered numbers to uniquely represent a point in the Euclidean point space. The first is a length, r, the second, an angle ϕ, and the third, a length, z. These are the coordinate variables.



```
c1=ParametricPlot3D[{ Sin[ϕ], Cos[ϕ],z},{ ϕ,0,Pi},{z,1.5,3.5},PlotStyle->Red];
c2=ParametricPlot3D[{r Sin[Pi/3],r Cos[Pi/3],z},{r,0,2},{z,1.5,2.3},PlotStyle->Green];
c3=ParametricPlot3D[{r Sin[θ],r Cos[ϕ],2},{ ϕ,0,2 Pi},{r,0.5,2.5},PlotStyle->Yellow];
Show[c1,c2,c3, PlotRange->{{0,1.4},{0,1.5},{1,2.5}},Ticks->None]
```

Recall that in the Cartesian case, the coordinate planes have equations, $x_1 = const$, $x_2 = const$, and $x_3 = const$ giving us three planes that intersect at the point defined by those three values of the constants used.

In a similar way, the coordinate planes in the Cylindrical Polar are: $r = const$ describing a cylinder with the z-axis as its axis, $\phi = const$ describing a plane through the axis and another plane, $z = const$ describing a plane that is perpendicular to the cylinder axis. This is as shown in the figure below:

We can obtain the basis vectors by differentiation of the position vector:

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z$$

$$= r\mathbf{e}_r + z\mathbf{e}_z$$

$$\frac{\partial\mathbf{r}}{\partial r} = \mathbf{e}_r;\ \ \frac{\partial\mathbf{r}}{\partial\phi} = r\mathbf{e}_\phi;\ \ \frac{\partial\mathbf{r}}{\partial z} = \mathbf{e}_z$$

The basis vectors obtain by differentiation also compels dimensional consistency but they are no longer orthonormal even though they remain mutually orthogonal.

Mistakes to avoid

Two easy mistakes that can be made are:

1. **That the Cylindrical position vector is** $r\mathbf{e}_r(\phi) + \phi\boldsymbol{e}_\phi + z\boldsymbol{e}_z$ which is a simplistic copy of the Cartesian formula. This is wrong in at least two ways. For one thing, it is dimensionally incorrect because the unit of the middle basis component is an angle while the other

components are measuring lengths. Secondly, we cannot obtain the Cartesian result from this via a coordinate transformation.

2. **That the basis vectors are constants**. They are NOT all constants. $\mathbf{e}_r(\phi)$ and $\boldsymbol{e}_\phi(\phi)$ are both functions of ϕ unlike in the Cartesian case, but $\boldsymbol{e}_z$ is a constant like the Cartesian case.

Spherical coordinates

The spherical Polar coordinate system selects its three ordered triplets with yet another strategy. This can be explained by the same transformation route we started. Continuing further with our transformation, we may again introduce two new scalars such that $\{r,\ z\} \to \{\rho, \theta\}$ in such a way that the position vector,

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = \rho \sin\theta\, \mathbf{e}_r + \rho\cos\theta\, \mathbf{e}_z \equiv \rho\mathbf{e}_\rho$$

Here, $r = \rho\sin\theta,\ z = \rho\cos\theta$. As before, we can use three scalars, $\{\rho,\ \theta,\ \phi\}$ instead of $\{r,\ \phi, z\}$. In comparison to the original Cartesian system we began with, we have that,

$$\begin{aligned}\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} &= \rho\sin\theta\,\mathbf{e}_r + \rho\cos\theta\,\mathbf{e}_z \\ &= \rho\sin\theta\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\cos\theta\,\mathbf{k} \\ &= \rho\sin\theta\cos\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\theta\,\mathbf{k} \\ &\equiv \rho\mathbf{e}_\rho\end{aligned}$$

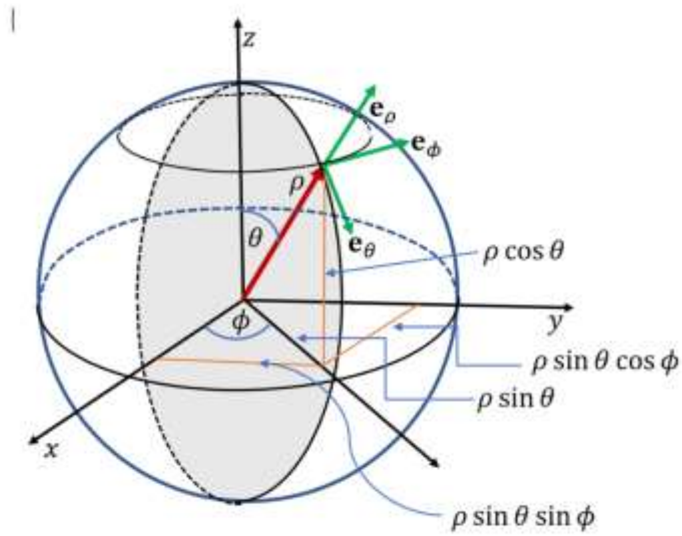
it is clear that the unit vector

$$\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

.Again, we introduce the unit vector, $\mathbf{e}_\theta \equiv \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$ and retain $\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$ as before**.** It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho,\ \phi, \theta\}$ directly by the transformation equations, $x = \rho\sin\phi\cos\theta,\ y =$

$\rho \sin\phi \sin\theta$ and $z = \rho\cos\theta$ .The orthonormal set of basis for the $\{\rho,\ \theta,\ \phi\}$ system is $\{\mathbf{e}_\rho(\theta,\ \phi), \mathbf{e}_\theta(\theta,\ \phi), \mathbf{e}_\phi(\phi)\}$

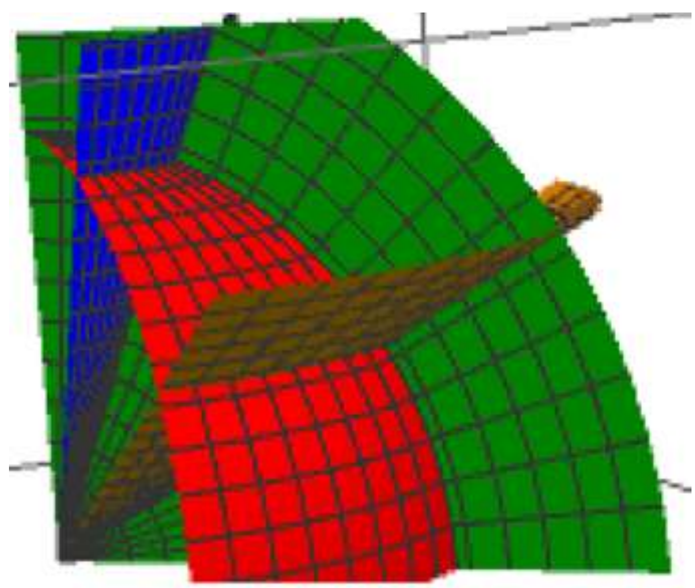
$$\mathbf{r}(\rho, \theta,\ \phi) \equiv \rho\mathbf{e}_\rho(\theta,\ \phi)$$



Showing that the position vector depends on the three coordinate variables representing the radial distance, ρ, from the origin on the azimuthal (great circle, longitudinal) plane inclined at an angle θ to the meridian plane $(x - z)$, with a polar angle ϕ as shown below: the orthonormal basis vectors are shown at the point of interest. The projection of the radial distance to the "equatorial" plane is also shown

Coordinate Surfaces

Here we have two points on the same sphere (equal radii) and the same θ but with two values of θ. The coordinate surfaces are spheres for $\rho = const$; planes through the vertical axis for $\phi = const$ and cones through the origin for $\theta = const$. As before, the coordinate surfaces are orthogonal as well as the tangents to the coordinate lines that are at the intersections of the coordinate planes. Just the same way we obtained the basis vectors by differentiation in the cylindrical system, we can obtain the same for the spherical:



For spherical polar,

$$\mathbf{r} = \rho\mathbf{e}_\rho(\theta,\ \phi)$$

$$\frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{e}_\rho;\ \frac{\partial\mathbf{r}}{\partial\theta} = \rho\mathbf{e}_\theta;\ \frac{\partial\mathbf{r}}{\partial\phi} = \rho\sin\theta\mathbf{e}_\phi$$

The vectors $\mathbf{e}_\rho, \mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors. The multipliers in each case are the magnitudes of the basis vectors obtained from differentiation.

Other Coordinate Systems

There are many other ways of selecting three ordered scalars to create a coordinate system. The ones we have seen so far are all orthogonal coordinate systems because the coordinate planes meet at all points at right angles. Other orthogonal coordinate systems that have engineering significance include:

1. Parabolic and Parabolic Cylindric
2. Elliptic Cylinder, Elliptic, Bipolar,
3. Confocal,
4. Prolate and Oblate spheroidal, Toroidal

The strategy of definition is similar in each case. A few:

Parabolic Cylinder Coordinate System

Parabolic Cylinder Coordinates are (ξ, η, z). Here the first two are square roots of length while the third scalar is length. Transformation equations are: $x_1 = \xi\eta, x_2 = \frac{1}{2}(\xi^2 - \eta^2)$ and $x_3 = z$. Substituting these in the Cartesian position vector,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \xi\eta\mathbf{e}_1 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_2 + z\mathbf{e}_3$$

Again, by differentiating this with respect to the coordinate variables, ξ, η, z, we obtain the following basis vectors for the Parabolic Cylinder System:

$$\eta\mathbf{e}_1 + \xi\mathbf{e}_2; \xi\mathbf{e}_1 + \eta\mathbf{e}_2; z\mathbf{e}_3$$

Coordinate System	Position Vector	Basis Vectors

Cartesian, $x_1 x_2 . x_3$	$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Cylindrical Polar r, ϕ, z	$r\mathbf{e}_r(\phi) + z\mathbf{e}_z$	$\mathbf{e}_r, r\mathbf{e}_\phi, \mathbf{e}_z$
Spherical Polar ρ, θ, ϕ	$\rho\mathbf{e}_\rho(\theta, \phi)$	$\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi$
Parabolic Cylindric ξ, η, z	$\eta\mathbf{e}_1 + \xi\mathbf{e}_2; \xi\mathbf{e}_1 + \eta\mathbf{e}_2; z\mathbf{e}_3$	$\eta\mathbf{e}_1 + \xi\mathbf{e}_2; \xi\mathbf{e}_1 + \eta\mathbf{e}_2; z\mathbf{e}_3$
1		
1		

Vector Spaces

Not so sure about all that Kronecker included in “all else” in his famous quote. Initially, it appears that Vectors are the work of man. It is no wonder that, unlike the celestial integers, their meanings are often varied and shifty! Wait a minute, you may find the Integers themselves, constitute a vector space!

We are now in a position to provide a more exact definition of what a vector really is. What you should observe in the following is that the definition is satisfied by our elementary notions about vectors. However, a vector is a more abstract object than we have been looking at. The abstraction is useful because it allows the analytical treatment of quantities that do not appear to be similar or related to the notions brought from elementary considerations.

We begin by assuming we have a bag containing real numbers. We call this collection, the set $\mathbb{R}$. That is the foundation of our vector space. It is possible to build the vector space upon a different foundation, such as complex numbers, or rational numbers. For this reason, our definition has indicator, “real”, in it.

Definition. A **real vector space** $\mathbb{V}$ is a set of elements (called vectors) such that,

1. **Addition operation** is **defined** and it is **commutative** and **associative** under $\mathbb{V}$: that is, $\mathbf{u}+\mathbf{v}\in\ \mathbb{V}$, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$, $\forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}$. Furthermore, $\mathbb{V}$ is **closed** under addition: That is, given that $\mathbf{u},\mathbf{v}\in\ \mathbb{V}$, then $\mathbf{w}=\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $\Rightarrow\mathbf{w}\ \in\ \mathbb{V}$.
2. $\mathbb{V}$ **contains** a **zero element** $\mathbf{o}$ such that $\mathbf{u}+\mathbf{o}=\mathbf{u}\ \forall\ \mathbf{u}\in\ \mathbb{V}$. For every $\mathbf{u}\in\ \mathbb{V}$**,** $\exists-\mathbf{u}$: $\mathbf{u}+(-\mathbf{u})=\mathbf{o}$**.**
3. **Multiplication by a scalar**. For $\alpha,\beta\in\mathbb{R}$ and $\mathbf{u},\mathbf{v}\in\ \mathbb{V}$, $\alpha\mathbf{u}\in\ \mathbb{V}$, $1\mathbf{u}=\mathbf{u}$, $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}$, $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$, $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$**.**

End of definition

Note the following:

1. The only multiplication needed to define a vector product is scaling. Not scalar, vector nor tensor products among tensors are needed to define a vector space. Consequently, there are several structures that would qualify as a vector space.
2. Our understanding of vectors thus far is admissible here. Condition #1 is satisfied by our parallelogram law of vector addition. The space is closed under addition because when you as two or more vectors (extending the parallelogram law to a polygon of vectors, the result you will get remains a vector, thus guaranteeing closure. Commutativity as well as associativity are straightforward when we try to add more than two vectors and find that the order of addition is immaterial.
3. For rule #2, note that a zero vector will be represented by a point; no length – resulting in a magnitude of zero. The negation of a vector is simply to retain the direction but change the sense of the arrow.
4. Rule three is merely a mathematical expression of the scaling process. It should also be handled by the addition law when applied to scaled vectors.

The Inner Product or Euclidean Vector Space

* An **Inner-Product** (also called a **Euclidean Vector**) **Space** $\mathbb{E}$ is a real vector space that defines the scalar product: for each pair $\mathbf{u},\mathbf{v}\in\mathbf{E},\exists\ l\in\mathbb{R}$ such that, $l=\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$. Further, $\mathbf{u}\cdot\mathbf{u}\geq 0$, the zero value occurring only when $\mathbf{u}=0$. It is called “Euclidean” because the laws of Euclidean geometry hold in such a space.
* The inner product also called a dot product, is the mapping

$$" \cdot ": \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$$

from the product space to the real space. The notation here means nothing more than, first expressing the fact that the operational sign to denote the Inner Product is the dot, " · ". The "product" is not the same meaning of multiplication of any type we are used to, but simply expressing the fact that we took one element of a vector space, and went back again to take another element of a vector space in order to perform the operation. And the right pointing arrow in the expression shows that the result of the operation is a member of the Real set: a complicated way of saying that it is a real number! If we had needed three element from the vector space, then we would have had, $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$ for the scalar triple product. These operations will be written as,

$$" \cdot\times ": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$$

We could also have written,

$$"[\ ,\ ,]": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$$

Because we may in fact prefer this notation as it emphasizes that only the ordering of the vectors is important, NOT the locations of the dot and the cross.

For the scalar tripple product – showing that the symbolic representation of the operation which produces a scalar result but requires a dot and a cross, while,

$$" \times ": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$$

will represent the vector triple product as it requires three vectors to produce a single vector.

Our dyads require two vectors to produce a tensor. We can write,

$$" \otimes ": \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{L}$$

If we represent the linear transformation that we call tensors by the symbol $\mathbb{L}$.

The inclusion of a definition for the Scalar product induces the concept of length. To make it easy, note that we have used spaces that have no concept of length – hence, it is not always necessary to include the concept into every structure we intend to develop. As a quick example, the thermodynamic plot of pressure to volume remains very useful even though the concept of distance between two arbitrary points is meaningless. In case of the vector space, for our use, the extension to the inclusion of the inner product as well as its induction of the length idea is, though not essential, is very useful indeed.

Magnitude The norm, length or magnitude of $\mathbf{u}$**,** denoted $\|\boldsymbol{u}\|$ is defined as the positive square root of $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$**.** When $\|\mathbf{u}\| = 1$, $\mathbf{u}$ is said to be a unit vector. When $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal.

Direction Furthermore, for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the angle between them is defined as,

$$\cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

The scalar **distance** $d \in \mathcal{R}$ between two position vectors $\mathbf{u}$ and $\mathbf{v}$

$$d = \|\mathbf{u} - \mathbf{v}\|.$$

Notice that we do not include the definition of the vector product in the definition of any vector space. The fact is that, once the concept of magnitude exists, we can define several other things on that basis. The vector product is just one of the many consequences of the scalar product. The latter being the more fundamental concept.

The Euclidean Point Space

It is a good thing to get a firm grasp of the **Euclidean Point Space**. It is NOT a vector space because its members are not vectors as we have defined them. There is a relationship, as we shall see. However, it is the ambient space in which **all** objects of interest physically reside. To make it simple, where you are sitting, or standing, reading this, is a Euclidean Point Space. Shall we then have its definition:

What is a Point? On your graph paper from High School, you are used to locating points with an ordered pair of real numbers. These are the Cartesian coordinates of the point. We are also used to the extension of this concept to three dimensions. If $x = \{x_1, x_2, x_3\}, y = \{y_1, y_2, y_3\}$ and $z = \{z_1, z_2, z_3\}$ are three such points, we can define the vectors joining them to a given point

$$\mathrm{o} \equiv \{0,0,0\}$$

the origin of coordinates in $\mathcal{E}$. The Euclidean Point Space may be referred to non-Cartesian systems. The three numbers may no longer represent distances. They must be in correct order. The dimensionality of a space determines the number of elements contained in the description of a point in $\mathcal{E}$

Definition: The Euclidean Point Space, $\mathcal{E}$ is such that, for points x, y, z and an origin, if we represent the vector, $\mathbf{v}$ joining point x to point y as $\mathbf{v}(x, y) \in \mathbb{E}$, where $x, y \in \mathcal{E}$, then, ***diagram needed here***

1. $\mathbf{v}(x,z) = \mathbf{v}(x,y) + \mathbf{v}(y,z) \; \forall \; x,y,z \in \mathcal{E}$, and
2. $\mathbf{v}(x,y) = (x,z) \Leftrightarrow y = z$ for each $x \in \mathcal{E}$

End of Definition

Consequences:

From Rule 1, we can see that, $\mathbf{v}(x,x) = \mathbf{v}(x,y) + \mathbf{v}(y,x) = \mathbf{o}$ as the vector joining a point to itself must necessarily be the zero vector. This neutral additive concept of a zero vector in the Euclidean POINT space leads to an additive inverse as the last equation immediately implies that,

$$\mathbf{v}(x,y) = -\mathbf{v}(y,x) \; \forall \; x,y \in \mathcal{E}$$

The Position Vector

The question of the true nature of what is called a "position vector" can now be addressed. Remember, all points are resident in the Euclidean Point Space. A position vector joins a point to the origin of coordinates. It is a vector defined by the location of two points in $\mathcal{E}$. Consequently, we have,

$$\mathbf{v}(x) \equiv \mathbf{v}(x,0) = \mathbf{x}(0) = \mathbf{x} - \mathbf{o}$$

Where $\mathbf{x}(0), \mathbf{x}(y) \in \mathbb{E}$ and we define $\mathbf{x}(y) \equiv \mathbf{x} - \mathbf{y} = \mathbf{x} - \mathbf{o} - (\mathbf{y} - \mathbf{o})$.

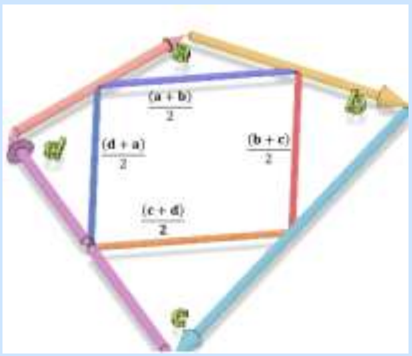
The vector $\mathbf{x} = \mathbf{x} - \mathbf{o}$ joining the point $x \in \mathcal{E}$ to the origin is called a **Position Vector**. The vector itself resides in the vector space, (in the sense that it takes its characteristics among vectors) the points defining it dwell in the Euclidean point space. Mathematically, this is called an embedding of a Vector Space (defining the Position Vectors) in the Euclidean Point Space (defining the points that create them).

The distance between two position vectors becomes sensible: It is the magnitude of the vector $\mathbf{v}(x,y)$ joining point x to point y in the Euclidean Point Space.

$$d(x-y) = \|\mathbf{v}(x,y)\| = \|\mathbf{x}(y)\| = \|\mathbf{x} - \mathbf{y}\|$$

Examples

1.0 Given that vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ form a closed circuit, show that the vectors joining their midpoints form a parallelogram



In the picture, $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$. Clearly, $\frac{(\mathbf{a}+\mathbf{d})}{2}=-\frac{(\mathbf{b}+\mathbf{c})}{2}$, also, $\frac{(\mathbf{a}+\mathbf{b})}{2}=-\frac{(\mathbf{c}+\mathbf{d})}{2}$

Opposite sides of the lines joining the midpoints are parallel. This is a parallelogram.

1.1 Given that $\mathbf{a}$ and $\mathbf{b}$ are vectors, use indicial notation to show that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$

$$\mathbf{a}\times\mathbf{b}=e_{ijk}a_jb_k\mathbf{e}_i=-e_{ikj}b_ka_j\mathbf{e}_i=-\mathbf{b}\times\mathbf{a}$$

1.2 Given that $\mathbf{a}$ and $\mathbf{b}$ are vectors, use indicial notation to show that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\mathbf{0}$

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=e_{ijk}a_jb_k\mathbf{e}_k\\(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}&=\left(e_{ijk}a_jb_k\mathbf{e}_k\right)(a_\alpha\mathbf{e}_\alpha)\\&=e_{ijk}a_jb_ka_\alpha\mathbf{e}_i\cdot\mathbf{e}_\alpha\\&=e_{ijk}a_jb_ka_\alpha\delta_{i\alpha}\\&=e_{ijk}a_jb_ka_i\\&=-e_{jik}a_jb_ka_i\\&=-e_{ijk}a_jb_ka_i=0\end{aligned}$$

The expression in symmetrical in i and j, it is also anti-symmetrical in the same two indices at the same time. The same situation occurs on the RHS.

1.3 Show that (a) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\|\mathbf{a}\|^2-\|\mathbf{b}\|^2$, and that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-2\mathbf{a}\times\mathbf{b}$

a	(a) Opening the parentheses, $$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b} = \|\mathbf{a}\|^2-\|\mathbf{b}\|^2$$ (b) Similarly, $$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = \mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b} = -\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b} = -2\mathbf{a}\times\mathbf{b}$$

1.4	Given that $\forall\, \mathbf{v}\in\mathbb{E}, \mathbf{a}\cdot\mathbf{v}=\mathbf{b}\cdot\mathbf{v}$, Show that $\mathbf{a}=\mathbf{b}$
	We are given that $\forall\, \mathbf{v}\in\mathbb{E}, \mathbf{a}\cdot\mathbf{v}=\mathbf{b}\cdot\mathbf{v}$ this implies, $$\mathbf{a}\cdot\mathbf{v}-\mathbf{b}\cdot\mathbf{v}=(\mathbf{a}-\mathbf{b})\cdot\mathbf{v}=0$$ Define the vector $\mathbf{c}\equiv\mathbf{a}-\mathbf{b}$. The equation becomes, $$\mathbf{c}\cdot\mathbf{v}=\|\mathbf{c}\|\|\mathbf{v}\|\cos\theta=0.$$ Because v can be any vector, it does not have to be perpendicular to $\boldsymbol{c}$ and we can rule out the trivial case of its being the zero vector. This leaves us with the only choice that $\|\mathbf{c}\|=0$. And, the only vector that has zero magnitude is the zero vector. So that, $\mathbf{c}\equiv\mathbf{a}-\mathbf{b}=\mathbf{o}$, or $\mathbf{a}=\mathbf{b}$.
1.5	Given that for any vector $\mathbf{v}$, $\mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v}$, Show that $\mathbf{a}=\mathbf{b}$
	We are given that $\forall\, \mathbf{v}\in\mathbb{E}$, $\mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v}$, Now take a dot product with $\mathbf{a}$, we have that, $$\mathbf{a}\cdot\mathbf{b}\times\mathbf{v}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}=0=\mathbf{o}\cdot\mathbf{v}$$ for all $\mathbf{v}$ proving that $\mathbf{a}\times\mathbf{b}=\mathbf{o}$. This shows that $\mathbf{a}$ and $\mathbf{b}$ are collinear. We can therefore write that $\mathbf{b}=\alpha\mathbf{a}$ Hence, $\mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v}=\alpha\mathbf{a}\times\mathbf{v}$ where α is a scalar. So that $$(\mathbf{a}\times\mathbf{v})(1-\alpha)=\boldsymbol{o}\Rightarrow 1=\alpha$$ showing that $\mathbf{a}=\mathbf{b}$ as was required.
1.5a	Identify all the equations contained in the expression $e_{ijk}T_{jk}=0$

For each free index, there is an equation:

$$i = 1 \Rightarrow e_{1jk}T_{jk} = e_{123}T_{23} + e_{132}T_{32} = T_{23} - T_{32} = 0 \Rightarrow T_{23} = T_{32}$$

$$i = 2 \Rightarrow e_{2jk}T_{jk} = e_{213}T_{13} + e_{231}T_{31} = T_{13} - T_{31} = 0 \Rightarrow T_{13} = T_{31}$$

$$i = 3 \Rightarrow e_{3jk}T_{jk} = e_{312}T_{12} + e_{321}T_{21} = T_{12} - T_{21} = 0 \Rightarrow T_{12} = T_{21}$$

Notice that this same expression could have been written in the full invariant form:

$$e_{ijk}T_{jk}\mathbf{e}_i = \mathbf{o}$$

In this form, there is no free index, all indices are dummy. Notice that the RHS is a vector zero. Strictly speaking, this can be fully expanded to

$$\begin{aligned} e_{ijk}T_{jk}\mathbf{e}_i &= 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 = \mathbf{o} \\ &= e_{1jk}T_{jk}\mathbf{e}_1 + e_{2jk}T_{jk}\mathbf{e}_2 + e_{3jk}T_{jk}\mathbf{e}_3 \end{aligned}$$

which are three equations:

$$e_{1jk}T_{jk} = 0, e_{2jk}T_{jk} = 0, e_{3jk}T_{jk} = 0.$$

And this is the meaning of the single equation,

$$e_{ijk}T_{jk} = 0$$

where the free index facilitates the production of the three equations. It follows that either form of writing gives us the same set of equations if correctly interpreted. That is one reason why we should be careful to note whether the zero we are dealing with is a scalar zero, a vector zero or a tensor zero. It matters!

1.5a

1.5a

Given that $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Write the equation $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I}$ in indicial notation.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} & a_{31}a_{21} + a_{32}a_{22} + a_{33}a_{23} & a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33} \end{pmatrix}$$

Observe that in each cell, every term maintains the cell's row and column numbers in their first term.

$$a_{(row\ no)1}a_{(col\ no)1} + a_{(row\ no)2}a_{(col\ no)2} + a_{(row\ no)3}a_{(col\ no)3}$$

Is true of EVERY cell in the above array! Look at it closely! It is also clear that we are summing over the second number in each term. Is it not clear that we can gain a significant amount of space if we simply writing, for the i^{th} and j^{th} column,

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3}$$

And is it not obvious that this can be written, using the summation convention as,

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = a_{i\alpha}a_{j\alpha} = a_{ik}a_{ik}$$

On the right hand side, the identity matrix is,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$$

Observe that anywhere the row and column numbers are the same, the value is 1. When they are not, the value is zero. So that the typical element is δ_{ij}. Hence, we can write the equation as,

$$a_{ik}a_{ik} = \delta_{ij}$$

1.5a Show that a transformation of every vector to the vector $4\mathbf{e}_2 + \mathbf{e}_1$ cannot be a tensor.

Let us first transform an arbitrary vector $\mathbf{u}$;

$$\mathbf{Tu} = 4\mathbf{e}_2 + \mathbf{e}_1$$

For any scalar α, let us also transform $\alpha\mathbf{u}$, since $\alpha\mathbf{u}$ is a vector, this transformation, since it transforms every vector the same way, transforms to

$$\mathbf{T}(\alpha\mathbf{u}) = 4\mathbf{e}_2 + \mathbf{e}_1 = \mathbf{T} \neq \alpha\mathbf{Tu}$$

Hence, it is not a linear transformation. A tensor is a linear transformation of a vector to another vector. The transformation is NOT a tensor.

1.5a Can a transformation of every vector to the zero vector be a tensor? Why?

Let us transform an arbitrary vector $\mathbf{u}$;

$$\mathbf{Tu} = \mathbf{o}$$

For any scalar α, let us also transform $\alpha\mathbf{u}$, since $\alpha\mathbf{u}$ is a vector, this transformation, since it transforms every vector the same way, transforms to

$$\mathbf{T}(\alpha\mathbf{u}) = \mathbf{o} = \alpha\mathbf{Tu}$$

Hence this transformation is a tensor: It transforms linearly, and from tensor to tensor. It is the Annihilator Tensor.

1.5a	Explain the terms, **Vector Cross**, Dual Vector, Deviatoric Tensor, Spherical Tensor.
	Vector Cross is a tensor that operates on a vector, yielding the same vector result that would have been obtained were there to have been a cross product on that vector. For any skew tensor, a Dual vector is a vector that is a vector cross of the tensor. A Deviatoric tensor is what remains after subtracting the spherical part of the tensor from the tensor A Spherical Tensor is a tensor that has the value zero in each non-diagonal element. The diagonal elements are of equal value. It follows that the tensor can be written in the form, $\alpha\mathbf{I}$ where α is a scalar, and $\mathbf{I}$ is the identity tensor.
1.5a	Given that the vector cross formula is $(\mathbf{u}\,\times) = e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$, Find the vector cross of $\mathbf{u} = 4\mathbf{e}_2 + \mathbf{e}_1 - 3\mathbf{e}_3$. Is it a deviatoric tensor? Why?
	Let $\boldsymbol{\Omega} = (\mathbf{u}\,\times) = e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$ $i = 1, k = 2 \Rightarrow \Omega_{12} = e_{132}u_3 = -u_3 = 3$ $i = 1, k = 3 \Rightarrow \Omega_{13} = e_{123}u_2 = u_2 = 1$ $i = 2, k = 3 \Rightarrow \Omega_{23} = e_{213}u_1 = -u_1 = -4$ $[\Omega_{ij}] = \begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ -1 & 4 & 0 \end{pmatrix}$ The trace of a skew tensor is zero. It has no spherical part. Hence, the skew tensor is deviatoric.
1.5a	
1.5a	
1.5a	

1.5a	
1.5a	
1.6	Given $[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ and $[a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ evaluate (a) S_{ii}, (b) $S_{ji}S_{ji}$, (c) $S_{jk}S_{kj}$, (d) $a_m a_m$, (e) $S_{mn}a_m a_n$, (f) $S_{nm}a_m a_n$
a	Because the subscript index is repeated, summation is implied for the full range of acceptable values – that is, 1,2 and 3; therefore, $S_{ii} = S_{11} + S_{22} + S_{33} = 1 + 1 + 3 = 5$.
b	In this case, two different indices are repeated. There is summation on both of them. To get it right, we must apply such one by one. We do it, starting with the first index, i, and later, after that is fully completed, we take the second index j, as follows: $S_{ij}S_{ij} = S_{1j}S_{1j} + S_{2j}S_{2j} + S_{3j}S_{3j} = S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{2j}S_{2j} + S_{3j}S_{3j}$ $= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{3j}S_{3j}$ $= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33}$ $= 1 \times 1 + 0 \times 0 + 2 \times 2 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 0 + 3 \times 3 = 28$

c	Proceeding as in the earlier two examples, we write, $$S_{jk}S_{kj} = S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3}$$ $$= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{2k}S_{k2} + S_{3k}S_{k3}$$ $$= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{3k}S_{k3}$$ $$= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{3k}S_{k3}$$ $$= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{31}S_{13} + S_{32}S_{23} + S_{33}S_{33}$$ $$= 1 \times 1 + 0 \times 0 + 2 \times 3 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 2 + 3 \times 3$$ $$= 23$$
d	Here, again, one index is repeated. A summation over all the allowable values of that index is implied. Accordingly, $a_m a_m = a_1 a_1 + a_2 a_2 + a_3 a_3 = 1 \times 1 + 2 \times 2 + 3 \times 3 = 14$
e	This is another example of a double summation. We proceed as we have done previously: $$S_{mn}a_m a_n = S_{1n}a_1 a_n + S_{2n}a_2 a_n + S_{3n}a_3 a_n$$ $$= S_{11}a_1 a_1 + S_{12}a_1 a_2 + S_{13}a_1 a_3 + S_{2n}a_2 a_n + S_{3n}a_3 a_n$$ $$= S_{11}a_1 a_1 + S_{12}a_1 a_2 + S_{13}a_1 a_3 + S_{21}a_2 a_1 + S_{22}a_2 a_2 + S_{23}a_2 a_3 + S_{31}a_3 a_1 + S_{32}a_3 a_2 + S_{33}a_3 a_3$$ $$= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 2 \times 1 \times 3 + 0 \times 2 \times 1 + 1 \times 2 \times 2 + 2 \times 2 \times 3 + 3 \times 3 \times 1 + 0 \times 3 \times 2 + 3 \times 3 \times 3$$ $$= 59$$
f	As in the above example, everything unchanged except that location of m and n indices in the first term are now revered. It is good to work this out fully manually and draw lessons from the result. This can have far reaching effects on your understanding of other materials later.

$$= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 3 \times 1 \times 3 + 0 \times 2 \times 1 + 1 \times 2 \times 2 + 0 \times 2 \times 3 + 2 \times 3 \times 1 + 2 \times 3 \times 2 + 3 \times 3 \times 3$$
$$= 59$$

The fact that the last two examples gave the same answer is NOT a coincidence. This example may look like some easy problem that is merely tedious. However, it strikes at the very heart of understanding the skills involved in the summation convention. These skills are not elementary nor are they trivial. We pause a little moment to look again at the problems 2.1f and 2.1g. By the time it fully sinks in, you will see that it should not be necessary for you to do the tedious arithmetic to see that they MUST give the same answer. Here is the proof:

$$S_{ij}a_ia_j = S_{ij}a_ja_i$$

Is true for the simple fact that multiplying a_i and a_j will always give us the same answer no matter in what order the operands are given: multiplication is Commutative. The fact that i and j in the above equations are repeated means that they are dummy variables. They can therefore be exchanged for any other set of dummy variables provided we are consistent. Accordingly, replace i by m and j by n on the left-hand side, and replace i by n and j by m on the right-hand side, we obtain,

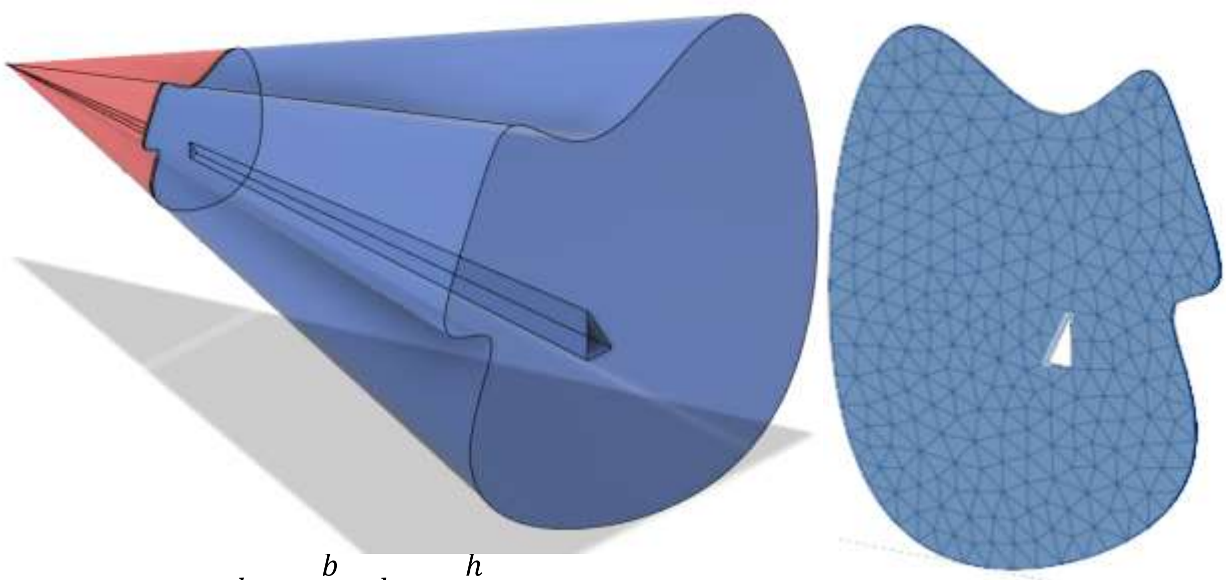
$$S_{mn}a_ma_n = S_{nm}a_ma_n$$

Never forget that we are only able to arbitrarily replace variables on a side without doing exactly the same at each object because we are here dealing with variables that have repeated themselves and are dummy variables.

Volume of a ConRamid.

Definition: A ConRamid is an object with a flat base of any shape that tapers, linearly, to a point maintaining the same shape in any horizontal section. A cone, pyramid or tetrahedron are all special cases of a conramid.

Volume. We will show that the volume of a conramid is one third the height times the area of the base. In doing this, we shall first demonstrate that as the lengths vary linearly with distance from the tip, the elemental areas vary as the square of this distance. Let us assume that the base area is A, and the perpendicular distance between the base and the vertex is H. We consider an element at a distance x from the vertex. To make things easy we have selected a right angled triangle at the centerline – through the perpendicular. The breadth, b_{tx} and height, h_{tx} of this triangle, compared to the image (height h, breadth b) at the base is



$$b_{tx} = \frac{b}{H}x, h_{tx} = \frac{h}{H}x$$

. The area of this triangle is therefore,

$$A_{x_t} = \frac{1}{2}\left(\frac{b}{H}x\right)\left(\frac{h}{H}x\right) = \frac{bh}{2H^2}x^2 = \frac{1}{H^2}A_t x^2$$

We can easily mesh the entire disk in a set of triangles as shown. In this case, the total area of the disk at point x will be the sum of all the triangular areas:

The volume of the typical disk is,

$$\frac{1}{H^2}(A_1 + A_2 + \cdots + A_n)x^2 dx = \frac{1}{H^2}Ax^2 dx$$

where H is the height of the triangle at the base. The volume of the conramid is therefore,

$$V = \frac{A}{H^2}\int_0^H x^2 dx = \frac{1}{3}AH$$

Which is one third the area of the base times the height. This applies to a cone, a pyramid or a tetrahedron as we have assumed previously.

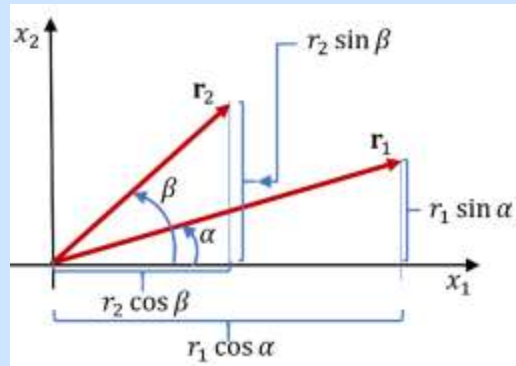
1.7	Evaluate $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$
	$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \big((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\big) \cdot \mathbf{d} \\ &= \big((\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c}\,)\mathbf{a}\big) \cdot \mathbf{d} \\ &= (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c}\,)(\mathbf{a} \cdot \mathbf{d})\end{aligned}$$

1.8	Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})^2$
	$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) &= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}]\mathbf{a} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\mathbf{c} \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\big((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}\big) \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}\,) \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})^2\end{aligned}$$

1.9	Given that position vectors $\mathbf{r}_1 and\ \mathbf{r}_2$ on the $x_1 - x_2$ plane are inclined at angles α, and β respectively to the x_1 axis. Find expressions for the component forms

of these vectors and (a) use the dot product to show that, $\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.(b) Use the cross product to show that $\sin(\beta - \alpha) = \cos\alpha - \sin\alpha\cos\beta$

(a) Writing $r_1 \equiv \|\mathbf{r}_1\|$, and $r_2 \equiv \|\mathbf{r}_2\|$ In the sketch below, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit coordinate vectors for x_1 and x_2 respectively. Clearly,



$$\mathbf{r}_1 = r_1\cos\alpha\,\mathbf{e}_1 + r_1\sin\alpha\,\mathbf{e}_2$$
$$\mathbf{r}_2 = r_2\cos\beta\,\mathbf{e}_1 + r_2\sin\beta\,\mathbf{e}_2$$
$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \|\mathbf{r}_1\|\|\mathbf{r}_2\|\cos(\beta-\alpha)$$
$$= (r_1\cos\alpha\,\mathbf{e}_1 + r_1\sin\alpha\,\mathbf{e}_2)$$
$$\cdot\,(r_2\cos\beta\,\mathbf{e}_1 + r_2\sin\beta\,\mathbf{e}_2)$$
$$= r_1\cos\alpha\, r_2\cos\beta + r_1\sin\alpha\, r_2\sin\beta$$

so that,

$$\cos(\beta-\alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

(b)

$$\mathbf{r}_1 \times \mathbf{r}_2 = \|\mathbf{r}_1\|\|\mathbf{r}_2\|\sin(\beta-\alpha)\,\mathbf{e}_3$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ r_1\cos\alpha & r_1\sin\alpha & 0 \\ r_2\cos\beta & r_2\sin\beta & 0 \end{vmatrix}$$

$$= \mathbf{e}_3 r_1 r_2(\cos\alpha\sin\beta - \sin\alpha\cos\beta)$$
$$\therefore \sin\ (\beta-\alpha) = \cos\alpha\sin\beta - \sin\alpha\cos\beta$$

1.10 The diagonals of a parallelogram are given by the vectors, $3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Find the area of the parallelogram.

a

Assume vectors $\mathbf{D}_1$and $\mathbf{D}_2$ are the diagonals. The sides are $\frac{1}{2}(\mathbf{D}_1 + \mathbf{D}_2)$, and $\frac{1}{2}(\mathbf{D}_1 - \mathbf{D}_2)$. The required area is therefore, $\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \mathbf{e}_1(3-2) + \mathbf{e}_2(-6-1) + \mathbf{e}_3(4+1)$

The magnitude of this is $\sqrt{(1 + 49 + 25)} = 5\sqrt{3}$

1.11	Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, using the result, $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$, show that $[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2$
a	From the given result, $$\begin{aligned}[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] &= -(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{u}) \\ &= -(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \otimes \mathbf{w})(\mathbf{v} \times \mathbf{u}) \\ &= (\mathbf{u} \times \mathbf{v})\big((\mathbf{w} \cdot \mathbf{u} \times \mathbf{v})\mathbf{w}\big) \\ &= [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2\end{aligned}$$

1.12	Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$
	$$\begin{aligned}(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} - [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{w}]\mathbf{v} \\ &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} \\ &= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{w} \\ &= (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})\end{aligned}$$ Consequently, $$\begin{aligned}[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] &= \mathbf{e} \cdot [(\mathbf{e} \times \mathbf{u}) \times (\mathbf{e} \times \mathbf{v})] \\ &= \mathbf{e} \cdot [(\mathbf{e} \otimes \mathbf{e})(\mathbf{u} \times \mathbf{v})] \\ &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{e} \otimes \mathbf{e})\mathbf{e} \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e} = [\mathbf{e}, \mathbf{u}, \mathbf{v}]\end{aligned}$$ making use of the symmetry of $(\mathbf{e} \otimes \mathbf{e})$.
1.13	Given that u, v and w are vectors, find the values of scalars α and β in the equation, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$

$$\mathbf{u}\times\mathbf{v} = e_{ijk}u_i v_j \mathbf{e}_k = s_k \mathbf{e}_k$$

Expanding the full equation, we have that

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= e_{klm}s_k w_l \mathbf{e}_m \\ &= e_{klm}e_{ijk}u_i v_j w_l \mathbf{e}_m \\ &= \left(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi}\right)u_i v_j w_l \mathbf{e}_m \\ &= e_{ijk}u_i v_j w_i \mathbf{e}_j - e_{ijk}u_i v_j w_j \mathbf{e}_i \\ &= (u_i w_i)v_j \mathbf{e}_j - \left(v_j w_j\right)u_i \mathbf{e}_i \\ &= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{v}\end{aligned}$$

Clearly, $\alpha = -(\mathbf{v}\cdot\mathbf{w})$ and $\beta = (\mathbf{u}\cdot\mathbf{w})$

1.14 Given that $\mathbf{n}$ is a unit vector, use the fact that $\mathbf{n}\cdot\mathbf{u}$ is the projection of the vector $\mathbf{u}$ in the direction of $\mathbf{n}$ to represent $\mathbf{u}$ as $(\mathbf{n}\cdot\mathbf{u})\mathbf{n} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$ or $(\mathbf{n}\otimes\mathbf{n})\mathbf{u} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$.

By simple vector addition, we can represent $\mathbf{u}$ as $(\mathbf{n}\cdot\mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n}\cdot\mathbf{u})\mathbf{n}$.

Since $\mathbf{n}$ is a unit vector, $\mathbf{n}\cdot\mathbf{n} = 1$. Therefore,

$$\begin{aligned}\mathbf{u} &= (\mathbf{n}\cdot\mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n}\cdot\mathbf{u})\mathbf{n} \\ &= (\mathbf{n}\cdot\mathbf{u})\mathbf{n} + (\mathbf{n}\cdot\mathbf{n})\mathbf{u} - (\mathbf{n}\cdot\mathbf{u})\mathbf{n} \\ &= (\mathbf{n}\cdot\mathbf{u})\mathbf{n} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n}) \\ &= (\mathbf{n}\otimes\mathbf{n})\mathbf{u} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})\end{aligned}$$

1.15 Simplify the following by employing the substitution properties of the Kronecker Delta $(a)e_{ijk}\delta_{kn}$, $(b)e_{ijk}\delta_{is}\delta_{jm}$ $(c)e_{ijk}\delta_{is}\delta_{jm}$ $(d)\ a_{ij}\delta_{in}$ $(e)\delta_{ij}\delta_{jn}$ $(f)\delta_{ij}\delta_{jn}\delta_{ni}$

$(a)\ e_{ijn}$ $(b)e_{smk}$ $(c)e_{smk}$ $(d)a_{nj}(e)\delta_{in}$ $(f)\delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

1.16	Show that the sum of triple products, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}+(\mathbf{v}\times\mathbf{w})\times\mathbf{u}+(\mathbf{w}\times\mathbf{u})\times\mathbf{v}=\mathbf{o}$
	$(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ $(\mathbf{v}\times\mathbf{w})\times\mathbf{u}=(\mathbf{v}\cdot\mathbf{u})\mathbf{w}-(\mathbf{w}\cdot\mathbf{u})\mathbf{v}$ $(\mathbf{w}\times\mathbf{u})\times\mathbf{v}=(\mathbf{w}\cdot\mathbf{v})\mathbf{u}-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ Adding the three, we find that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}+(\mathbf{v}\times\mathbf{w})\times\mathbf{u}+(\mathbf{w}\times\mathbf{u})\times\mathbf{v}=\mathbf{o}$ This is the zero vector.
1.17	Given that, $I_{ij}=\iiint_V\left(x^m x^m\delta_{ij}-x^i x^j\right)\rho(x^1,x^2,x^3)dx^1dx^2dx^3$ is the moment of inertia along the axis $i-j$ where $x=x^1, y=x^2, z=x^3$ and $\varrho(x^1,x^2,x^3)$ is scalar density of the material find all the components of the tensor.
	$\mathrm{I}_{11}=\iiint_V(\mathrm{y}^2+\mathrm{z}^2)\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz},\quad \mathrm{I}_{21}=\mathrm{I}_{12}=\iiint_V \mathrm{xy}\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz},$ $\mathrm{I}_{22}=\iiint_V(\mathrm{z}^2+\mathrm{x}^2)\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz},\quad \mathrm{I}_{32}=\mathrm{I}_{23}=\iiint_V \mathrm{yz}\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz},$ $\mathrm{I}_{31}=\mathrm{I}_{13}=\iiint_V \mathrm{xy}\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz},\quad \mathrm{I}_{33}=\iiint_V(\mathrm{x}^2+\mathrm{y}^2)\,\varrho(\mathrm{x},\mathrm{y},\mathrm{z})\mathrm{dxdydz}$
1.18	Write (a) in the long form, (b) In direct notation, $a_i=\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}$
	We can see that in this equation, the there is one free index, that is I and it occurs once in every term on both sides. There is a dummy index, that is, j appearing repeated in one term. Accordingly, $a_i=\frac{\partial v_i}{\partial t}+v_1\frac{\partial v_i}{\partial x_1}+v_2\frac{\partial v_i}{\partial x_2}+v_3\frac{\partial v_i}{\partial x_3}$ Which are, indeed, three equations one each for $i=1, i=2$ and $i=3$ as follows:

	$$a_1 = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}$$ $$a_2 = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}$$ $$a_3 = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}$$ The equation comes from the natural law of the indestructibility of masses – or mass balance. It is often inaccurately called a continuity equation in some texts.
1.19	Given that λ and μ are scalar constants, and that the identity tensor, $\mathbf{I} = \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, $\mathbf{E} = E_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and $\boldsymbol{\sigma} = \sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ write the equation, $\boldsymbol{\sigma} = \lambda\mathbf{I}\,\mathrm{tr}\,\mathbf{E} + 2\mu\mathbf{E}$ in component form.
a	$$\mathrm{tr}\,\mathbf{E} = E_{\alpha\beta}\mathrm{tr}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta) = E_{\alpha\beta}\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = E_{\alpha\beta}\delta_{\alpha\beta} = E_{\alpha\alpha}$$ The given equation, in component form can be written as, $$\sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \lambda\delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta E_{kk} + 2\mu E_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ Using the common bases, we can write this in terms of components only : $$\sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = (\lambda\delta_{\alpha\beta}E_{kk} + 2\mu E_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$\sigma_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$$
1.20	If $\sigma_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$, show that, (a) $\sigma_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$ and (b) $\sigma_{ij}\sigma_{ij} = (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$
a	Multiplying both sides by E_{ij} we have, $$\sigma_{ij}E_{ij} = \lambda\delta_{ij}E_{kk}E_{ij} + 2\mu E_{ij}E_{ij}$$ By the substitution nature of the Kronecker Delta, we have that, $\delta_{ij}E_{ij} = E_{jj} = E_{kk}$ because j as well as k are dummy indices here. Consequently, $$\sigma_{ij}E_{ij} = \lambda E_{kk}E_{jj} + 2\mu E_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$$ Squaring both sides of the equation,

	$= \lambda^2 \delta_{ii}(E_{kk})^2 + 2\lambda\mu E_{jj} + 2\lambda\mu\delta_{ij}E_{ij} + 4\mu^2 E_{ij}E_{ij}$ $= 3\lambda^2(E_{kk})^2 + 4\lambda\mu E_{jj} + 4\mu^2 E_{ij}E_{ij}$ $= (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$
1.21	Given that $a_{mn}x^m x^n = 0$. Show that $a_{mn}, m, n = 1,2,3$ is antisymmetric
a	Given that $a_{mn}x^m x^n = 0$ for arbitrary values of $x^n, n = 1,2,3$ then we can write, $$a_{mn}x^m x^n = -a_{mn}x^m x^n$$ because zero is also a negative of itself. Swapping the roles of x^m and x^n on the RHS of the above, we can write, $$\begin{aligned} a_{mn}x^m x^n &= -a_{mn}x^m x^n \\ &= -a_{mn}x^n x^m \\ &= -a_{nm}x^n x^m \end{aligned}$$ after swapping the roles of the two dummy indices. We therefore consolidate on the LHS by writing, $$\begin{aligned} a_{mn}x^m x^n + a_{nm}x^n x^m &= 0 \\ (a_{mn} + a_{nm})x^m x^n &= 0 \end{aligned}$$ Notice that the quantity in the parenthesis is always symmetric. And also note the contraction of two symmetric tensors can only vanish if one or both tensors vanish. Here, $x^m x^n$ is a product of arbitrary tensors. We are left with the fact that $$a_{mn} + a_{nm} = 0$$ or, $$a_{mn} = -a_{nm}$$ which is the definition of anti-symmetry.
1.22	Given that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are linearly dependent.

a	Suspose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that, $$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = e_{ijk}a_i b_j c_k = e_{ijk}(\alpha b_i + \beta c_i) b_j c_k$$ $$= \alpha e_{ijk} b_i b_j c_k + \beta e_{ijk} c_i b_j c_k$$ $$= 0$$ Note that $b_i b_j c_k$ is symmetric in i and j, $c_i b_j c_k$ is symmetric in i and k ande_{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.
1.23	Show that the product of a symmetric and an antisymmetric object vanishes.
a	Consider the product sum, $e_{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and e_{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here. $$e_{ijk} b_i b_j c_k = -e_{ijk} b_i b_j c_k = -e_{ijk} b_j b_i c_k = -e_{ijk} b_i b_j c_k = 0$$ The first equality on account of the antisymmetry of e_{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself. This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.
1.24	Define $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$
a	In component form, $$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = e_{ijk} a_i b_j c_k$$ Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield, $$e_{ijk} b_j c_k a_i = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = e_{ijk} b_i c_j a_k$$ $$= e_{ijk} c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = e_{ijk} c_i a_j b_k$$

	$$[\mathbf{b},\mathbf{c},\mathbf{a}] = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = -\mathbf{b}\cdot\mathbf{a}\times\mathbf{c} = -\,[\mathbf{b},\mathbf{a},\mathbf{c}]$$ In a similar way, $[\mathbf{a},\mathbf{b},\mathbf{c}] = -[\mathbf{a},\mathbf{c},\mathbf{b}]$, and $[\mathbf{c},\mathbf{a},\mathbf{b}] = -[\mathbf{c},\mathbf{b},\mathbf{a}]$
1.25	Write in indicial notations (a) $s = A_1^2 + A_2^2 + A_3^2$ (b) $\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$
a	(a) In order to avoid confusion between squaring and simple indexing, we separate the two as follows: $$s = A_1^2 + A_2^2 + A_3^2 = A_1A_1 + A_2A_2 + A_3A_3 = A_iA_i$$ In a similar way, the Laplacian operator for a scalar function can be expressed as follows: $$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = \frac{\partial^2\phi}{\partial x_1\partial x_1} + \frac{\partial^2\phi}{\partial x_2\partial x_2} + \frac{\partial^2\phi}{\partial x_3\partial x_3} = \frac{\partial^2\phi}{\partial x_i\partial x_i}$$
1.26	Given the vectors $\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Find the dyad $\mathbf{a}\otimes\mathbf{b}$, and (b) Find $\mathrm{tr}(\mathbf{a}\otimes\mathbf{b})$
a	$$\begin{aligned}\mathbf{a}\otimes\mathbf{b} &= a_ib_j\mathbf{e}_i\otimes\mathbf{e}_j\\ &= [\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\otimes[b_1,b_2,b_3]\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}\\ &= [\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]\begin{bmatrix}3\times1 & 3\times(-3) & 3\times4\\ 1 & -3 & 4\\ (-2)\times1 & -2\times(-3) & -2\times4\end{bmatrix}\otimes\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}\\ &= [\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]\begin{bmatrix}3 & -9 & 12\\ 1 & -3 & 4\\ -2 & 6 & -8\end{bmatrix}\otimes\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix}\end{aligned}$$
	$$\mathrm{tr}(\mathbf{a}\otimes\mathbf{b}) = a_ib_i = 3 - 3 - 8 = -8$$
1.27	Given that $\mathbf{e}_i\times\mathbf{e}_j = e_{ijk}\mathbf{e}_k$, Find expressions for $\mathbf{e}_i\times\mathbf{e}_j\cdot\mathbf{e}_k$ and $\mathbf{e}_i\cdot\mathbf{e}_j\times\mathbf{e}_k$. Demonstrate the equality of the two expressions

a	$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k = \mathbf{e}_i \times \mathbf{e}_j = e_{ij\alpha}\mathbf{e}_\alpha$$ Taking the scalar product of the above vector with $\mathbf{e}_k$, $$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = e_{ij\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_k = e_{ij\alpha}\delta_{\alpha k} = e_{ijk}$$ Starting with $\mathbf{e}_j \times \mathbf{e}_k = e_{jk\alpha}\mathbf{e}_\alpha$ we can also take the scalar product with $\mathbf{e}_i$ and write, $$\mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = \mathbf{e}_j \times \mathbf{e}_k \cdot \mathbf{e}_i = e_{jk\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_i = e_{jk\alpha}\delta_{\alpha i} = e_{jki} = e_{ijk}$$ As a double swap does not alter sign.
1.28	Show that Cylindrical Polar basis vectors, $\mathbf{e}_r(r,\phi)$, $\mathbf{e}_\phi(r,\phi)$ and $\mathbf{e}_z$ constitute an orthonormal system. Hint: Show that they have unit magnitudes and are mutually orthogonal.
1	$$\|\mathbf{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$ $$\|\mathbf{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$ $$\|\mathbf{e}_z\|^2 = 1$$ They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs: $$\mathbf{e}_r \cdot \mathbf{e}_\phi = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$ $$\mathbf{e}_\phi \cdot \mathbf{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$ $$\mathbf{e}_z \cdot \mathbf{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$ So that they are pairwise orthogonal.
1.29	Show that Normalized Spherical Polar basis vectors, $$\mathbf{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3$$ $$\mathbf{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\mathbf{e}_1 + \cos\theta\sin\phi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3$$ $\mathbf{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$ constitute an orthonormal system. Hint: Show that they have unit magnitudes and are mutually orthogonal.

	$$\left\|\mathbf{e}_\rho\right\|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$ $$= \sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta = 1$$ $$\|\mathbf{e}_\theta\|^2 = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta$$ $$= \cos^2\theta\,(\cos^2\phi + \sin^2\phi) + \sin^2\theta = 1$$ $$\left\|\mathbf{e}_\phi\right\|^2 = \sin^2\phi + \cos^2\phi = 1$$ They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs: $$\mathbf{e}_{\boldsymbol{\rho}}\cdot\mathbf{e}_{\boldsymbol{\theta}} = \cos^2\phi\,(\sin\theta\cos\theta) + \sin^2\phi\,(\sin\theta\cos\theta) - \sin\theta\cos\theta = 0$$ $$\mathbf{e}_\phi\cdot\mathbf{e}_\rho = -\sin\theta\sin\phi\cos\phi + \sin\theta\sin\phi\cos\phi + 0 = 0$$ $$\mathbf{e}_\theta\cdot\mathbf{e}_\phi = -\sin\phi\cos\theta\cos\phi + \sin\phi\cos\theta\cos\phi + 0 = 0$$ So that they are pairwise orthogonal.
1.30	Begin with the Cartesian position vector, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$**,** use the transformation equations to find the Spherical position vector, $\mathbf{R} = \rho\mathbf{e}_\rho(\theta,\phi)$,where $\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$. (b) By a direct differentiation, partially with respect to the coordinate variables produces the set $\left\{\frac{\partial\mathbf{R}}{\partial\rho}, \frac{\partial\mathbf{R}}{\partial\theta}, \frac{\partial\mathbf{R}}{\partial\phi}\right\}$.and show that it is a set of orthogonal vectors.
	$$\frac{\partial\mathbf{R}}{\partial\rho} = \mathbf{e}_\rho,$$ $$\frac{\partial\mathbf{R}}{\partial\theta} = \rho\frac{\partial\mathbf{e}_\rho}{\partial\theta} = \rho\frac{\partial}{\partial\theta}(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})$$ $$= \rho(\cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}) \equiv \rho\mathbf{e}_\theta.$$ $$\frac{\partial\mathbf{R}}{\partial\phi} = \rho\frac{\partial\mathbf{e}_\rho}{\partial\phi} = \rho\frac{\partial}{\partial\phi}(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})$$ $$= \rho(-\sin\theta\sin\phi\,\mathbf{i} + \sin\theta\cos\phi\,\mathbf{j})$$ $$\equiv \rho\sin\theta\,\mathbf{e}_\phi.$$ From these, we can see that $\left\{\frac{\partial\mathbf{R}}{\partial\rho}, \frac{\partial\mathbf{R}}{\partial\theta}, \frac{\partial\mathbf{R}}{\partial\phi}\right\} = \{\mathbf{e}_\rho, \rho\mathbf{e}_\theta, \rho\sin\theta\,\mathbf{e}_\phi\}$. Obviously, the magnitudes are $\{1, \rho, \rho\sin\theta\}$ respectively. Consequently, this basis set can be normalized to $\{\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi\}$

1.31	The dot from the left. Given that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}$, Show that $\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{w} \otimes \mathbf{v})\mathbf{u}$
	The result on both sides is a vector. Testing a scalar product with $\mathbf{y} \in \mathbb{E}$: observe that $$\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w})\mathbf{y} = (\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{y})$$ and, $$[(\mathbf{w} \otimes \mathbf{v})\mathbf{u}] \cdot \mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \cdot \mathbf{v})$$ The two are equal on account of the commutativity of the dot products.
1.32	Given that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}$, Show that $\mathbf{u} \times (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$
	First observe that the final result will be a tensor. Operating it on $\mathbf{y} \in \mathbb{E}$, we have, $$[\mathbf{u} \times (\mathbf{v} \otimes \mathbf{w})]\,\mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \times \mathbf{v})$$ And on the right, we have $$[(\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}]\mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \times \mathbf{v})$$ As was required.

Advanced Topics

Reciprocal Base Systems

We have seen that we can use any set of linearly independent vectors a basis for our coordinate systems. So far, we have restricted ourselves to the Venerable Cartesian scheme of selecting the basis vectors in a restrictive way:

1. The vectors are spatial constants;
2. They are orthogonal
3. They have unit magnitudes.

We sometimes encounter simple situations where complications arise from this restrictive choice of coordinates. Problems of pipes such as blood vessels, shafts that transmit motion or forces and curved bars are simple cases where the use of the simple Cartesian systems significantly complicates matter. In such cases, we resort to a less restrictive coordinate system with a different set of basis vectors. The penalty for this choice is that we lose the simplicity of computing the coordinates scalars of those bases in the way we are used to.

Reciprocal base systems is designed to simplify these computations using an ingenious method. However, we will have to deal with two sets of base vectors each time rather than the single set, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we have got accustomed to.

Note at the outset that we can do this from the simple fact that we can have many different basis vectors in any situation. Any set of linearly independent vectors will do the job. In Cartesian we select a constant Orthonormal set. What happens when we remove the restriction of constancy, orthogonality and normality?

We MUST still ensure that the set we select are linearly independent. Suppose our basis vectors $\mathbf{g}_i$, $i = 1,2,3$ are not only not unit in magnitude, but in addition are NOT orthogonal. The only assumption we are making is that $\mathbf{g}_i \in \mathbb{E}$, $i = 1,2,3$ are linearly independent vectors.

With respect to this basis, we can express vectors $\mathbf{v}, \mathbf{w} \in \mathrm{E}$ in terms of the basis as,

$$\mathbf{v} = v^i \mathbf{g}_i \;\; \mathbf{w} = w^i \mathbf{g}_i$$

Where each v^i is called the contravariant component of $\mathbf{v}$. The basis vectors here are also functions of the coordinate variables:

$$\mathbf{g}_i = \mathbf{g}_i(x^1, x^2, x^3)$$

Even if we do not explicitly say so, assume that a variation exists unless we have specific information otherwise. By the linearity of the vector space,

$$\mathbf{v} + \mathbf{w} = (v^i + w^i)\mathbf{g}_i$$

Multiplication by scalar rule implies that if $\alpha \in \mathbb{R}$, $\forall \mathbf{v} \in \mathbb{E}$,

$$\alpha \mathbf{v} = (\alpha v^i)\mathbf{g}_i$$

It turns out that for any Linearly Independent set we select as our basis vectors, that is another set, related to it called the dual basis. The interplay between these two creates an interesting relationship that is easily exploited to reduce the penalty of the difficulty we would have in evaluating the components at this time when the orthonormality assumptions have been removed.

For any basis vectors $\mathbf{g}_i \in \mathbb{E}$, $i = 1,2,3$ there is a dual (or reciprocal) basis defined by the reciprocity relationship:

$$\mathbf{g}^i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}^i = \delta_j^i$$

Where δ_j^i is the mixed Kronecker delta

$$\delta_j^i = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Apart from the fact that one index is raised while the other is lowered, the meaning remains essentially the same as we had previously.

We call bases and components that have lower indices "covariant" while those with raised indices are "contravariant". Furthermore, covariant indices have contravariant components, and vice versa. Any vector has two possible representations: One with covariant bases and contravariant components and the other with contravariant bases with covariant components. When an object has both kinds of indices, we say they are mixed.

For any $\forall \mathbf{v} \in \mathbb{E}$,

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i$$

Are two related representations in the reciprocal bases. Taking the inner product of the above equation with the basis vector $\mathbf{g}_j$, we have

$$\mathbf{v} \cdot \mathbf{g}_j = v^i \mathbf{g}_i \cdot \mathbf{g}_j = v_i \mathbf{g}^i \cdot \mathbf{g}_j$$

Which gives us the *covariant* component,

$$\mathbf{v} \cdot \mathbf{g}_j = v^i g_{ij} = v_i \delta_j^i = v_j$$

The substitution property of the mixed Kronecker delta remains the same as before.

In the same easy manner, we may evaluate the contravariant components of the same vector by taking the dot product of the same equation with the contravariant base vector $\mathbf{g}^j$:

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \mathbf{g}_i \cdot \mathbf{g}^j = v_i \mathbf{g}^i \cdot \mathbf{g}^j$$

So that,

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \delta_i^j = v_i g^{ij} = v^j$$

The nine scalar quantities, g^{ij} as well as the nine related quantities g_{ij} play important roles in the coordinate system spanned by these arbitrary reciprocal set of basis vectors as we shall see. They are called metric coefficients because they metrize the space defined by these bases.

Similar to the way we obtained the equation,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$$

We can obtain the relationship,

$$\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$$

This comes from the fact that $\mathbf{g}^1$is orthogonal to $\mathbf{g}_2$ and $\cdot\, \mathbf{g}_3$, $\mathbf{g}^2$is orthogonal to $\mathbf{g}_3$ and $\mathbf{g}_1$, and $\mathbf{g}^3$is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}_2$, it follows that $\mathbf{g}_2 \times \mathbf{g}_3 = \mathbf{g}^1, \mathbf{g}_3 \times \mathbf{g}_1 = \mathbf{g}^2$, and $\mathbf{g}_1 \times \mathbf{g}_2 = \mathbf{g}^3$. Equation (.) captures these cases with the other six that vanish in a single expression as we saw in the ONB case.

Given that $g = \det g_{ij}$ of the covariant metric coefficients, It is not difficult to prove that

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} \equiv \sqrt{g}e_{ijk}$$

The dual of the expression, the equivalent contravariant equivalent also follows from the fact that,

$$\mathbf{g}^1 \times \mathbf{g}^2 \cdot \mathbf{g}^3 = \epsilon^{ijk} = \frac{1}{\sqrt{g}}.e^{ijk}$$

1.40	Given that, $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ are three linearly independent vectors and satisfy $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, show that $\mathbf{g}^1 = \frac{1}{V}\mathbf{g}_2 \times \mathbf{g}_3$, $\mathbf{g}^2 = \frac{1}{V}\mathbf{g}_3 \times \mathbf{g}_1$, and $\mathbf{g}^3 = \frac{1}{V}\mathbf{g}_1 \times \mathbf{g}_2$, where $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$
a	It is clear, for example, that $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$ as well as to $\mathbf{g}_3$ (an obvious fact because $\mathbf{g}^1 \cdot \mathbf{g}_2 = 0$ and $\mathbf{g}^1 \cdot \mathbf{g}_3 = 0$), we can say that the vector $\mathbf{g}^1$ must necessarily lie on the cross product $\mathbf{g}_2 \times \mathbf{g}_3$ of $\mathbf{g}_2$ and $\mathbf{g}_3$. It is therefore correct to write, $$\mathbf{g}^1 = \frac{1}{V}\mathbf{g}_2 \times \mathbf{g}_3$$ Where $V^{-1}$is a constant we will now determine. We can do this right away by taking the dot product of both sides of the equation with $\mathbf{g}_1$ we immediately obtain, $$\mathbf{g}_1 \cdot \mathbf{g}^1 = V^{-1}\,\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = 1$$ So that, $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$ the volume of the parallelepiped formed by the three vectors $\mathbf{g}_1, \mathbf{g}_2$, and $\mathbf{g}_3$ when their origins are made to coincide.

Suppose you have a function $f(x, y, z)$ of variables x, y and z. Let us assume there are some variables r, ϕ and Z such that, the original variables are themselves functions $x = x(r, \phi, Z)$, $y = y(r, \phi, Z)$, and $z = z(r, \phi, Z)$. A simple example is the polar coordinate transformation: $x = r\cos\phi$, $y = r\sin\phi$, $z = Z$. We can always get a new function $\mathrm{F}(r, \phi, Z) = f(x, y, z)$ by doing a coordinate transformation using these equations. It is a well known fact that the transformation equations are invertible provided that the Jacobian of the transformation,

$$\frac{\partial(x, y, z)}{\partial(r, \phi, Z)} \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{vmatrix} \neq 0$$

We prefer to use indexed variables. Hence instead of x, y and z, we prefer $x^i = x^i(u^1, u^2, u^3)$ where $i = 1,2,3$ as you can obviously see that instead of r, ϕ, Z, we are now talking about u^1, u^2, u^3. As before, we can say that the transformation will have an inverse provided the Jacobian, $\left|\frac{\partial x^k}{\partial u^i}\right|$ does not vanish. Therefore to say that the transformation is invertible ensures that $\left|\frac{\partial x^k}{\partial u^i}\right| \neq 0$.

Recall that in Cartesian coordinates, the vector connecting an arbitrary point to the origin, also called a position vector can be written as

$$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x^i\mathbf{e}_i$$

Or, in order to emphasize the functional dependencies,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i$$

First notice that once you have a correct expression for your position vector for an arbitrary location, you can, by partial differentiation obtain an alternative representation for your basis vectors. It is elementary, for example to see clearly that,

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial x}$$

And in general Cartesian coordinates, using index notation,

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}, \quad i = 1,2,3$$

Natural Bases

We generalize the result now in terms of natural bases that arise in coordinate transformations from the Cartesian:

In the curvilinear system (u^1, u^2, u^3) obtained from the transformation $x^i = x^i(u^1, u^2, u^3)$ from Cartesian coordinates, let $\mathbf{g}_i = \frac{\partial \boldsymbol{r}}{\partial u^i}$ and let $\mathbf{g}^j$ be the corresponding dual basis. Show that $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}$. If $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$ and $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$, show that $v\,V = 1$. Show also that $\mathbf{g}_\mathrm{i} \cdot \mathbf{g}_\mathrm{j} \times \mathbf{g}_\mathrm{k} = \epsilon_{ijk} = \sqrt{g}e_{ijk}$.

The position vector $\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i$ where $\mathbf{e}_i$, $i = 1,2,3$ are unit vectors that are orthonormal in the Euclidean space.

Changing variables, we can write that,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i = \mathbf{r}(u^1, u^2, u^3)$$

So that we have new coordinates $u^k, k = 1,2,3$. In this new system, the differential of the position vector $\mathbf{r}$ is,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i}\, du^i \equiv \mathbf{g}_i du^i$$

the above equation, as we shall soon show, defines the natural basis vectors in the new coordinate system. The vectors $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ are not necessarily unit vectors but they form a basis of the new system provided,

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \neq 0$$

Clearly, the reciprocal basis vectors are

$$\mathbf{g}^1 = V^{-1}\,\mathbf{g}_2 \times\, \mathbf{g}_3$$
$$\mathbf{g}_2 = V^{-1}\,\mathbf{g}_3 \times\, \mathbf{g}_1$$
$$\mathbf{g}^3 = V^{-1}\,\mathbf{g}_1 \times\, \mathbf{g}_2$$

(dot the first with $\mathbf{g}_1$ to see) Now we are given that $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$. Using the above relations, we can write,

$$\begin{aligned}\mathbf{g}^2 \times \mathbf{g}^3 &= (V^{-1}\,\mathbf{g}_3 \times\, \mathbf{g}_1) \times (V^{-1}\,\mathbf{g}_1 \times\, \mathbf{g}_2)\\ &= V^{-2}[(\,\mathbf{g}_3 \times\, \mathbf{g}_1 \cdot\, \mathbf{g}_2)\mathbf{g}_1 - (\,\mathbf{g}_3 \times\, \mathbf{g}_1 \cdot\, \mathbf{g}_1)\mathbf{g}_2]\\ &= V^{-2}(\,\mathbf{g}_1 \times\, \mathbf{g}_2 \cdot\, \mathbf{g}_3)\mathbf{g}_1 = V^{-1}\mathbf{g}_1\end{aligned}$$

We can now write,

$$
\begin{aligned}
v &= \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3 \\
&= \mathbf{g}^1 \cdot V^{-1}\mathbf{g}_1 \\
&= V^{-1}\mathbf{g}^1 \cdot \mathbf{g}_1 = V^{-1}
\end{aligned}
$$

Showing that, $v\,V = 1$ as required.

Clearly, the determinant of g_{ij}

$$
\begin{aligned}
g \equiv |g_{ij}| &= |\mathbf{g}_i \cdot \mathbf{g}_j| \\
&= \left|\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}\right| = \left|\frac{\partial x^k}{\partial u^i}\right|^2 \\
&= V^2
\end{aligned}
$$

Since the determinant of a product of matrices is the product of the determinants.

This means, $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \left|\frac{\partial x^i}{\partial u^j}\right| = \sqrt{g}$. We can therefore write,

$$\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = e_{123}\sqrt{g}$$

Swapping indices 2 and 3, we have,

$$\mathbf{g}_1 \cdot \mathbf{g}_3 \times \mathbf{g}_2 = -\sqrt{g} = e_{132}\sqrt{g} = \mathbf{g}_1 \times \mathbf{g}_3 \cdot \mathbf{g}_2$$

The second equality coming from the fact that swapping the cross with the dot changes nothing.

Lastly, swapping 1 and 3 in the last equation shows that,

$$\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2 = -(-\sqrt{g}) = e_{312}\sqrt{g}.$$

These three expressions together imply that,

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g}e_{ijk}$$

as required.

1.41	Show that $\mathbf{g}^j = g^{ij}\mathbf{g}_i = g^{ji}\mathbf{g}_i$ and establish the relation, $g_{ij}g^{jk} = \delta_i^k$

a	First expand $\mathbf{g}^j$ in terms of the $\mathbf{g}_i$s: $$\mathbf{g}^j = \alpha\mathbf{g}_1 + \beta\mathbf{g}_2 + \gamma\mathbf{g}_3$$ Dotting with $\mathbf{g}^1 \Rightarrow \mathbf{g}^j \cdot \mathbf{g}^1 = \alpha\mathbf{g}_1 \cdot \mathbf{g}^1 + \beta\mathbf{g}_2 \cdot \mathbf{g}^1 + \gamma\mathbf{g}_3 \cdot \mathbf{g}^1 = \mathrm{g}^{j1} = \alpha$. In the same way we find that $\beta = \mathrm{g}^{j2}$ and $\gamma = \mathrm{g}^{j3}$so that, $$\mathbf{g}^j = g^{j1}\mathbf{g}_1 + g^{j2}\mathbf{g}_2 + g^{j3}\mathbf{g}_3 = g^{ji}\mathbf{g}_i.$$ Similarly, $\mathbf{g}_i = g_{i\alpha}\mathbf{g}^\alpha$. Recall the reciprocity relationship: $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k$. Using the above, we can write $$\mathbf{g}_i \cdot \mathbf{g}^k = (g_{i\alpha}\mathbf{g}^\alpha) \cdot \left(g^{k\beta}\mathbf{g}_\beta\right) = g_{i\alpha}g^{k\beta}\mathbf{g}^\alpha \cdot \mathbf{g}_\beta = g_{i\alpha}g^{k\beta}\delta_\beta^\alpha = \delta_i^k$$ which shows that $$g_{i\alpha}g^{k\alpha} = g_{ij}g^{jk} = \delta_i^k$$ As required. This shows that the tensor g_{ij} and g^{ij} are inverses of each other.
1.42	
a	
1.43	

a	
1.44	
a	
1.45	

a

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