Liu et. al. Problems 2.1-2.18

2.1 Given $[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ and $[a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$, (e) (a) S_{ii}, (f) $S_{mn}a_m a_n$, (g) $S_{nm}a_m a_n$,

(a) Because the subscript index is repeated, summation is implied for the full range of acceptable values – that is, 1,2 and 3; therefore, $S_{ii} = S_{11} + S_{22} + S_{33} = 1 + 1 + 3 = 5$.

(b) In this case, two different indices are repeated. There is summation on both of them. To get it right, we must apply such one by one. We do it, starting with the first index, i, and later, after that is fully completed, we take the second index j, as follows:

$$S_{ij}S_{ij} = S_{1j}S_{1j} + S_{2j}S_{2j} + S_{3j}S_{3j}$$

$$
\begin{aligned}
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{2j}S_{2j} + S_{3j}S_{3j} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} \\
&\quad + S_{3j}S_{3j} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} \\
&\quad + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33} \\
&= 1 \times 1 + 0 \times 0 + 2 \times 2 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 0 \\
&\quad + 3 \times 3 \\
&= 28
\end{aligned}
$$

(c) In this case, just like the example above, two different indices are repeated. There is summation on both of them. We must similarly apply the summation one by one. To be consistent, even though we do not have to, we start with the first index, j, and later, after that is fully completed, we take the second index i, as follows:

$$
\begin{aligned}
S_{ji}S_{ji} &= S_{1i}S_{1i} + S_{2i}S_{2i} + S_{3i}S_{3i} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{2i}S_{2i} + S_{3i}S_{3i} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} \\
&\quad + S_{3i}S_{3i} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} \\
&\quad + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33} \\
&= 1 \times 1 + 0 \times 0 + 2 \times 2 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 0 \\
&\quad + 3 \times 3 \\
&= 28
\end{aligned}
$$

If you take a good look at this example and the one immediately before it, you see that we have obtained the same answers when the two dummy indices became swapped. The example here may look trivial, but the principle resulting in the equal answers we have obtained is profound and exceedingly useful. Master this and you will see that your understanding of subsequent concepts will become greatly increased.

(d) Proceeding as in the earlier two examples, we write,

$$
\begin{aligned}
S_{jk}S_{kj} &= S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{2k}S_{k2} + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} \\
&\quad + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} \\
&\quad + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} \\
&\quad + S_{31}S_{13} + S_{32}S_{23} + S_{33}S_{33} \\
&= 1 \times 1 + 0 \times 0 + 2 \times 3 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 2 \\
&\quad + 3 \times 3 \\
&= 23
\end{aligned}
$$

(e) Here, again, one index is repeated. A summation over all the allowable values of that index is implied. Accordingly,

$$a_m a_m = a_1 a_1 + a_2 a_2 + a_3 a_3 = 1 \times 1 + 2 \times 2 + 3 \times 3 = 14$$

(f) This is another example of a double summation. We proceed as we have done previously:

$$\begin{aligned} S_{mn} a_m a_n &= S_{1n} a_1 a_n + S_{2n} a_2 a_n + S_{3n} a_3 a_n \\ &= S_{11} a_1 a_1 + S_{12} a_1 a_2 + S_{13} a_1 a_3 + S_{2n} a_2 a_n \\ &+ S_{3n} a_3 a_n \\ &= S_{11} a_1 a_1 + S_{12} a_1 a_2 + S_{13} a_1 a_3 + S_{21} a_2 a_1 + S_{22} a_2 a_2 \\ &+ S_{23} a_2 a_3 + S_{31} a_3 a_1 + S_{32} a_3 a_2 + S_{33} a_3 a_3 \\ &= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 2 \times 1 \times 3 + 0 \times 2 \times 1 \\ &+ 1 \times 2 \times 2 + 2 \times 2 \times 3 + 3 \times 3 \times 1 + 0 \times 3 \times 2 \\ &+ 3 \times 3 \times 3 \\ &= 59 \end{aligned}$$

(g) As in the above example, everything unchanged except that location of m and n indices in the first term are now revered. It is good to work this out fully manually and draw lessons from the result. This can have far reaching effects on your understanding of other materials later.

$$S_{nm}a_m a_n = S_{n1}a_1 a_n + S_{n2}a_2 a_n + S_{n3}a_3 a_n$$
$$= S_{11}a_1 a_1 + S_{21}a_1 a_2 + S_{31}a_1 a_3 + S_{n2}a_2 a_n + S_{n3}a_3 a_n$$
$$= S_{11}a_1 a_1 + S_{21}a_1 a_2 + S_{31}a_1 a_3 + S_{12}a_2 a_1 + S_{22}a_2 a_2 + S_{32}a_2 a_3 + S_{13}a_3 a_1 + S_{23}a_3 a_2 + S_{33}a_3 a_3$$
$$= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 3 \times 1 \times 3 + 0 \times 2 \times 1 + 1 \times 2 \times 2 + 0 \times 2 \times 3 + 2 \times 3 \times 1 + 2 \times 3 \times 2 + 3 \times 3 \times 3$$
$$= 59$$

The fact that the last two results gave the same answer is NOT a coincidence. This example may look like some easy problem that is merely tedious. However, it strikes at the very heart of understanding the skills involved in the summation convention. These skills are not elementary nor are they trivial. We pause a little moment to look again at the problems 2.1f and 2.1g. By the time it fully sinks in, you will see that it should not be necessary for you to do the tedious arithmetic to see that they MUST give the same answer. Here is the proof:

$$S_{ij}a_ia_j = S_{ij}a_ja_i$$

Is true for the simple fact that multiplying a_i and a_j will always give us the same answer no matter in what order the operands are given: multiplication is Commutative. The fact that i and j in the above equations are repeated means that they are dummy variables. They can therefore be exchanged for any other set of dummy variables provided we are consistent. Accordingly, replace i by m and j by n on the left-hand side, and replace i by n and j by m on the right-hand side, we obtain,

$$S_{mn}a_m a_n = S_{nm}a_m a_n$$

Never forget that we are only able to arbitrarily replace variables on a side without doing exactly the same at each object because we are here dealing with variables that have repeated themselves and are dummy variables.

2.2 The repetition of the index j on the RHS term implies summation. Accordingly,

$$a_i = Q_{ij}a'_j = Q_{i1}a'_1 + Q_{i2}a'_2 + Q_{i3}a'_3$$

This represents three equations: one each for $i = 1, i = 2$ and $i = 3$ as follows:

$$
\begin{aligned}
a_1 &= Q_{11}a'_1 + Q_{12}a'_2 + Q_{13}a'_3 \\
a_2 &= Q_{21}a'_1 + Q_{22}a'_2 + Q_{23}a'_3 \\
a_3 &= Q_{31}a'_1 + Q_{32}a'_2 + Q_{33}a'_3
\end{aligned}
$$

We can write each of the three equations here to fully see what the equivalencies are:

(a) $a_p = Q_{pm}a'_m = Q_{p1}a'_1 + Q_{p2}a'_2 + Q_{p3}a'_3$ which represents three equations for $p = 1, p = 2$ and $p = 3$. These are:

$$
\begin{aligned}
a_1 &= Q_{11}a'_1 + Q_{12}a'_2 + Q_{13}a'_3 \\
a_2 &= Q_{21}a'_1 + Q_{22}a'_2 + Q_{23}a'_3 \\
a_3 &= Q_{31}a'_1 + Q_{32}a'_2 + Q_{33}a'_3
\end{aligned}
$$

An exact replica of what we had originally; confirms identity with the previous equation.

(b) $a_p = Q_{qp}a'_q = Q_{1p}a'_1 + Q_{2p}a'_2 + Q_{3p}a'_3$ is already looking different. If we proceed to write the three equations this represents for $p = 1, p = 2$ and $p = 3$, we find that,

$$a_1 = Q_{11}a'_1 + Q_{21}a'_2 + Q_{31}a'_3$$
$$a_2 = Q_{12}a'_1 + Q_{22}a'_2 + Q_{32}a'_3$$
$$a_3 = Q_{13}a'_1 + Q_{23}a'_2 + Q_{33}a'_3$$

Which equations have different coefficients from the previous set.

(c) $a_m = a'_n Q_{mn} = a'_1 Q_{m1} + a'_2 Q_{m2} + a'_3 Q_{m3} = Q_{m1}a'_1 + Q_{m2}a'_2 + Q_{m3}a'_3$ where in the second equation we have used the fact that multiplication of any two numbers is commutative. This then represents three equations for $m = 1, m = 2$ and $m = 3$. These are:

$$a_1 = Q_{11}a'_1 + Q_{12}a'_2 + Q_{13}a'_3$$

$$a_2 = Q_{21}a'_1 + Q_{22}a'_2 + Q_{23}a'_3$$
$$a_3 = Q_{31}a'_1 + Q_{32}a'_2 + Q_{33}a'_3$$

An exact replica of what we had originally; confirms identity with the previous equation.

2.3

Given that $[Q_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ and $[a'_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ demonstrate the equivalency of the subscripted equations and the corresponding matrix equations: (a) $a_i = Q_{ij}a'_j$ and $[a] = [Q][a']$ (b) $r = Q_{ij}a'_i a'_j$ and $r = [a']^{\mathrm{T}}[Q][a']$

(a) The equation, $a_i = Q_{ij}a'_j = Q_{i1}a'_1 + Q_{i2}a'_2 + Q_{i3}a'_3$ This represents three equations: one each for $i = 1, i = 2$ and $i = 3$ as follows:

$$a_1 = Q_{11}a'_1 + Q_{12}a'_2 + Q_{13}a'_3 = 1 \times 1 + 0 \times 2 + 2 \times 3 = 7$$

$$a_2 = Q_{21}a'_1 + Q_{22}a'_2 + Q_{23}a'_3 = 0 \times 1 + 1 \times 2 + 2 \times 3 = 8$$
$$a_3 = Q_{31}a'_1 + Q_{32}a'_2 + Q_{33}a'_3 = 3 \times 1 + 0 \times 2 + 3 \times 3 = 12$$

Using Mathematica, the matrix multiplication can be obtained:[Lai2.3]

```
In[1]:= Q = {{1, 0, 2}, {0, 1, 2}, {3, 0, 3}};
        a′ = {1, 2, 3};

In[3]:= MatrixForm[Q]

Out[3]//MatrixForm=
        (1 0 2)
        (0 1 2)
        (3 0 3)

In[4]:= MatrixForm[a′]

Out[4]//MatrixForm=
        (1)
        (2)
        (3)

In[5]:= Q.a′

Out[5]= {7, 8, 12}
```

$$
\begin{aligned}
Q_{ij}a_i'a_j' &= Q_{1j}a_1a_j + Q_{2j}a_2a_j + Q_{3j}a_3a_j \\
&= Q_{11}a_1a_1 + Q_{12}a_1a_2 + Q_{13}a_1a_3 + Q_{2j}a_2a_j + Q_{3j}a_3a_j \\
&= Q_{11}a_1a_1 + Q_{12}a_1a_2 + Q_{13}a_1a_3 + Q_{21}a_2a_1 + Q_{22}a_2a_2 \\
&\quad + Q_{23}a_2a_3 + Q_{31}a_3a_1 + Q_{32}a_3a_2 + Q_{33}a_3a_3 \\
&= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 2 \times 1 \times 3 + 0 \times 2 \times 1 \\
&\quad + 1 \times 2 \times 2 + 2 \times 2 \times 3 + 3 \times 3 \times 1 + 0 \times 3 \times 2 \\
&\quad + 3 \times 3 \times 3 \\
&= 59
\end{aligned}
$$

Mathematica code for the matrix multiplication is:

```
In[7]:= a′.Q.a′

Out[7]= 59
```

2.4 Write in indicial notation, the matrix equation, (a) $[A] = [B][C]$, (b) $[D] = [B]^{\mathrm{T}}[C]$ (c) $[E] = [B]^{\mathrm{T}}[C][F]$

(a) $A_{ij} = B_{i\alpha}C_{\alpha j}$ (b) $D_{ij} = B_{\alpha i}C_{\alpha j}$ (c) $E_{ij} = B_{\alpha i}C_{\alpha\beta}F_{\beta j}$

2.5 Write in indicial notations (a) $s = A_1^2 + A_2^2 + A_3^2$ (b) $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$

(a) In order to avoid confusion between squaring and simple indexin, we separate the two as follows:

$$s = A_1^2 + A_2^2 + A_3^2 = A_1 A_1 + A_2 A_2 + A_3 A_3 = A_i A_i$$

(b) In a similar way, the Laplacian operator for a scalar function can be expressed as follows:

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_1} + \frac{\partial^2 \phi}{\partial x_2 \partial x_2} + \frac{\partial^2 \phi}{\partial x_3 \partial x_3} = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$$

2.6 Given that $S_{ij} = a_i a_j$ and $S'_{ij} = a'_i a'_j$, where $a'_i = Q_{mi} a_m$ and $a'_j = Q_{nj} a_n$ and $Q_{ki} Q_{kj} = \delta_{ij}$ Show that $S'_{ii} = S_{ii}$

We are given that,

$$S'_{ij} = a'_i a'_j = (Q_{mi} a_m)(Q_{nj} a_n)$$
$$= Q_{mi} Q_{nj} a_m a_n$$

The indices i and j can be equated to either of the two. These are free indices so that we must do the same for each term in which they appear with exactly the same replacement. Contrast this with what freedom you would have with dummy indices:

$$S'_{ii} = Q_{mi} Q_{ni} a_m a_n = \delta_{mn} a_m a_n = a_m a_m = a_i a_i = S_{ii}$$

2.7 Write in the long form:

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

We can see that in this equation, the there is one free index, that is I and it occurs once in every term on both sides. There is a dummy index, that is, j appearing repeated in one term. Accordingly,

$$a_i = \frac{\partial v_i}{\partial t} + v_1 \frac{\partial v_i}{\partial x_1} + v_2 \frac{\partial v_i}{\partial x_2} + v_3 \frac{\partial v_i}{\partial x_3}$$

Which are, indeed, three equations one each for $i = 1, i = 2$ and $i = 3$ as follows:

$$a_1 = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3}$$

$$a_2 = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}$$

$$a_3 = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}$$

The equation comes from the natural law of the indestructibility of masses – or mass balance. It is often inaccurately called a continuity equation in some texts.

2.8 Given that

$$T_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$$

Show that, (a) $T_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$ and (b) $T_{ij}T_{ij} = (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$

(a) Multiplying both sides by E_{ij} we have,

$$T_{ij}E_{ij} = \lambda\delta_{ij}E_{kk}E_{ij} + 2\mu E_{ij}E_{ij}$$

By the substitution nature of the Kronecker Delta, we have that, $\delta_{ij}E_{ij} = E_{jj} = E_{kk}$ because j as well as k are dummy indices here. Consequently,

$$T_{ij}E_{ij} = \lambda E_{kk}E_{kk} + 2\mu E_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$$

(b) Squaring both sides of the equation,

$$\begin{aligned} T_{ij}T_{ij} &= \left(\lambda\delta_{ij}E_{kk} + 2\mu E_{ij}\right)\left(\lambda\delta_{ij}E_{kk} + 2\mu E_{ij}\right) \\ &= \lambda^2\delta_{ii}(E_{kk})^2 + 2\lambda\mu\delta_{ij}E_{ij} + 2\lambda\mu\delta_{ij}E_{ij} + 4\mu^2E_{ij}E_{ij} \\ &= 3\lambda^2(E_{kk})^2 + 4\lambda\mu E_{jj} + 4\mu^2E_{ij}E_{ij} \end{aligned}$$

$$= (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$$

2.9 Given that $a_i = T_{ij}b_j$ and $a_i' = T_{ij}'b_j'$ where $a_i = Q_{im}a_m'$ and $a_j' = Q_{nj}a_n$ and $T_{ij} = Q_{im}Q_{jn}T_{mn}'$ (a) show that $Q_{im}T_{mn}'b_n' = Q_{im}Q_{jn}T_{mn}'b_j$ and (b) if $Q_{ki}Q_{mi} = \delta_{mn}$, then $T_{kn}'\left(b_n' - Q_{jn}b_j\right)$

(a) Note that,

$$Q_{\alpha j}T_{mj}'b_\alpha = T_{mj}'Q_{\alpha j}b_\alpha = T_{mj}'b_j'$$

From which we easily see that, $b_j' = Q_{\alpha j}b_\alpha$. Now, we are given that,

$$Q_{im}T_{mn}'b_n' = Q_{im}T_{mn}'Q_{\alpha n}b_\alpha = Q_{im}T_{mn}'Q_{jn}b_j = Q_{im}Q_{jn}T_{mn}'b_j$$

(b) From the equation above, we have,

$$Q_{im}T_{mn}'b_n' - Q_{im}Q_{jn}T_{mn}'b_j = 0$$

$$\Rightarrow Q_{im}T_{mn}'\left(b_n' - Q_{jn}b_j\right) = 0$$

$$\Rightarrow T_{kn}'\left(b_n' - Q_{jn}b_j\right) = 0$$

Because we know that the matrix $[Q_{im}] \neq [0]$ since we are given that it is an orthogonal matrix. Note we could not remove T'_{mn} from the equation in the same way because the latter is arbitrary and we cannot exclude the fact that it can vanish.

2.10

$$\mathbf{a} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = [b_1, b_2, b_3] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

This is the same, using the summation convention, as $\mathbf{a} = a_i \mathbf{e}_i$ and $\mathbf{b} = b_j \mathbf{e}_j$. We can conclude here that

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= e_{ijk} a_i b_j \mathbf{e}_k \end{aligned}$$

$$= \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Either way the vector is fully represented in its compionent form. And the difference between a vector and its matrix form can be seen clearly.

The dyad $\mathbf{a} \otimes \mathbf{b} = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$ which can be given in its full component form as,

$$\mathbf{a} \otimes \mathbf{b} = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

$$= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

$$= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Again, as you can see, the matrix, just like in the vector case, is not exactly the same as the tensor.

2.11

(a)

$$e_{ijk}T_{ij} = 0$$

One index is free. This gives three equations.

i. $k = 1$. In this case $i = 2, j = 3$ and $i = 3, j = 2$ are the only nonzero cases. Hence we have,

$$e_{ij1}T_{ij} = e_{231}T_{23} + e_{321}T_{32} = 0$$
$$\Rightarrow T_{23} - T_{32} = 0 \Rightarrow \; T_{23} = T_{32}$$

ii. $k = 2$. In this case $i = 3, j = 1$ and $i = 1, j = 3$ are the only nonzero cases. Hence we have,

$$e_{ij2}T_{ij} = e_{312}T_{31} + e_{132}T_{13} = 0$$
$$\Rightarrow T_{31} - T_{13} = 0 \Rightarrow \; T_{31} = T_{13}$$

iii. $k = 3$. In this case $i = 1, j = 2$ and $i = 2, j = 1$ are the only nonzero cases. Hence we have,

$$e_{ij3}T_{ij} = e_{123}T_{12} + e_{213}T_{21} = 0$$
$$\Rightarrow T_{12} - T_{21} = 0 \Rightarrow T_{12} = T_{21}$$

(b)

$$\delta_{ij}e_{ijk} = \delta_{ji}e_{ijk} = -\delta_{ji}e_{jik} = -\delta_{ij}e_{ijk} = 0$$

Note that the second equation comes from swapping i and j inside the alternating symbol, while the last comes from a change of roles between i and j in the expression.

2.12

Full proof already done

2.13

$$e_{ijk}e_{klm} = \delta_{ij}\delta_{jm} - \delta_{im}\delta_{jj}$$

Setting $j = m$,

$$\begin{aligned} e_{ijk}e_{klj} &= \delta_{ij}\delta_{jj} - \delta_{ij}\delta_{jl} \\ &= 3\delta_{ij} - \delta_{il} = 2\delta_{il} \end{aligned}$$

Setting $i = l$ in the last equation, we have,

$$e_{ijk}e_{kij} = e_{ijk}e_{ijk} = 2\delta_{ii} = 2 \times 3 = 6.$$

2.14

$$\mathbf{b} \times \mathbf{c} = e_{\alpha\beta\gamma} b_\alpha c_\beta \mathbf{e}_\gamma = e_{\alpha\beta j} b_\alpha c_\beta \mathbf{e}_j$$

Consequently,

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= e_{kij} a_i e_{\alpha\beta j} b_\alpha c_\beta \mathbf{e}_k \\ &= e_{\alpha\beta j} e_{kij} a_i b_\alpha c_\beta \mathbf{e}_k \\ &= \left(\delta_{\alpha k}\delta_{\beta i} - \delta_{\alpha i}\delta_{\beta k}\right) a_i b_\alpha c_\beta \mathbf{e}_k \\ &= a_i b_k c_i \mathbf{e}_k - a_\alpha b_\alpha c_k \mathbf{e}_k \\ &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \end{aligned}$$

2.15

$$T_{ij} a_i a_j = T_{ij} a_j a_i = T_{ji} a_i a_j = -T_{ij} a_i a_j = 0$$

The first equation is simply a permutation of $a_i$and a_j since multiplication is commutative. In the second, we just interchanged the roles of i and j. Finally, the negative sign comes from the given relation and we conclude it is zero because it is negative of itself.

2.16

$$T_{ji} = \frac{1}{2}\left(S_{ji} + S_{ij}\right) = T_{ij}$$

$$R_{ji} = \frac{1}{2}\left(S_{ji} - S_{ij}\right) = -\frac{1}{2}\left(S_{ij} - S_{ji}\right) = -R_{ij}$$

$$T_{ij} + R_{ij} = \frac{1}{2}\left(S_{ij} + S_{ji} + S_{ij} - S_{ji}\right) = \frac{1}{2} \times 2S_{ij} = S_{ij}$$

2.17

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = \frac{\partial f}{\partial x_i} dx_i$$

$$dv_i = \frac{\partial v_i}{\partial x_1} dx_1 + \frac{\partial v_i}{\partial x_2} dx_2 + \frac{\partial v_i}{\partial x_3} dx_3 = \frac{\partial v_i}{\partial x_j} dx_j$$

2.18 From elementary vector analysis, we are used to the fact that the scalar triple product, by our usual notation of $\mathbf{a} = a_i \mathbf{e}_i$, $\mathbf{b} = b_j \mathbf{e}_j$, $\mathbf{c} = c_k \mathbf{e}_k$ is given by,

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

For the purpose of this question, we utilize double indices for the components to allow us to use the same symbol for the three vectors. In particular,

$$\mathbf{a}_1 \equiv \mathbf{a} = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3 = a_{1i}\mathbf{e}_i$$
$$\mathbf{a}_2 \equiv \mathbf{b} = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3 = a_{2j}\mathbf{e}_j$$
$$\mathbf{a}_3 \equiv \mathbf{c} = a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3 = a_{3k}\mathbf{e}_k$$

Clearly therefore,

$$\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a_3} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Simply by a change in our notation. We now proceed to compute the same triple product using the summation convention:

$$\begin{aligned} \mathbf{a}_1 \times \mathbf{a}_2 &= e_{ijk} a_{1i} a_{2j} \mathbf{e}_k \\ \mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a_3} &= \left(e_{ijk} a_{1i} a_{2j} \mathbf{e}_k\right) \cdot (a_{3\alpha} \mathbf{e}_\alpha) \\ &= e_{ijk} a_{1i} a_{2j} a_{3\alpha} \mathbf{e}_k \cdot \mathbf{e}_\alpha = e_{ijk} a_{1i} a_{2j} a_{3\alpha} \delta_{k\alpha} \\ &= e_{ijk} a_{1i} a_{2j} a_{3k} \\ &= e_{ijk} a_{i1} a_{j2} a_{k3} \end{aligned}$$

as the transpose operation does not alter the determinant. Note that, on the second line, we avoided the quadruple occurrence of a dummy index, k, by a timely change in the index used.

Given that $\mathbf{A}$ and $\mathbf{B}$ are tensors, show that the product $\mathbf{AB}$ obtained by a successive application of $\mathbf{B}$ and $\mathbf{A}$ is also a tensor transformation.

Given a vector $\mathbf{v}$, clearly, $\mathbf{w} \equiv \mathbf{Bv}$ is a vector.

Consequently,

$$\mathbf{ABv} = \mathbf{A}(\mathbf{Bv}) = \mathbf{Aw}$$

is a vector, and the two transformations are linear. Hence the result.

For the tensors $\mathbf{A}$ and $\mathbf{B}$, use the definition of the determinant to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$

Select linearly independent tensors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent. Now,

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

Show that $e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$

The definition of e_{ijk} and of δ_{ij} immediately shows that,

$e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$, and $e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$

The product, $e_{rst}e_{ijk} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$

$$= \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Given that

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Show that $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{rj}\delta_{si}$

Clearly, $e_{rsk}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$

Expanding the equation, we have:

$$e_{rsk}e_{ijk} = \delta_{ki}\begin{vmatrix}\delta_{rj} & \delta_{rk}\\ \delta_{sj} & \delta_{sk}\end{vmatrix} - \delta_{kj}\begin{vmatrix}\delta_{ri} & \delta_{rk}\\ \delta_{si} & \delta_{sk}\end{vmatrix} + 3\begin{vmatrix}\delta_{ri} & \delta_{rj}\\ \delta_{si} & \delta_{sj}\end{vmatrix}$$

$$= \delta_{ki}\left(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}\right) - \delta_{kj}(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk})$$

$$+ 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)$$

$$= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3(\delta_{ri}\delta_{sj}$$

$$- \delta_{si}\delta_{rj})$$

$$= -2(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}) + 3(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj})$$

$$= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$$

Given that $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$Show that $e_{rjk}e_{ijk} = 2\delta_{ri}$

In the equation, $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$ set $s = j$, we have,

$$e_{rjk}e_{ijk} = \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} = 3\delta_{ri} - \delta_{ri} = 2\delta_{ri}$$