Introduction to Tensor Analysis

Text: Introduction to Continuum Mechanics
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Summary

Scalars. You can get scalars from vectors in at least five different ways:

1. One Vector. Get its components (referred to a particular set of coordinates); It's magnitude; Direction;
2. From two vectors, produce the scalar product
3. From three vectors, the vector triple product

Vectors

1. Add (subtract?) two vectors; Multiply a single vector by a scalar; Vector product of two vectors;
2. Vector product of three vectors.

Summary

Tensor

1. Create a dyad from two vectors
2. Create a vector cross from a single vector

Invariants

1. Intrinsic properties of tensors
2. Obtain them in component form
3. Special Tensors and Operations

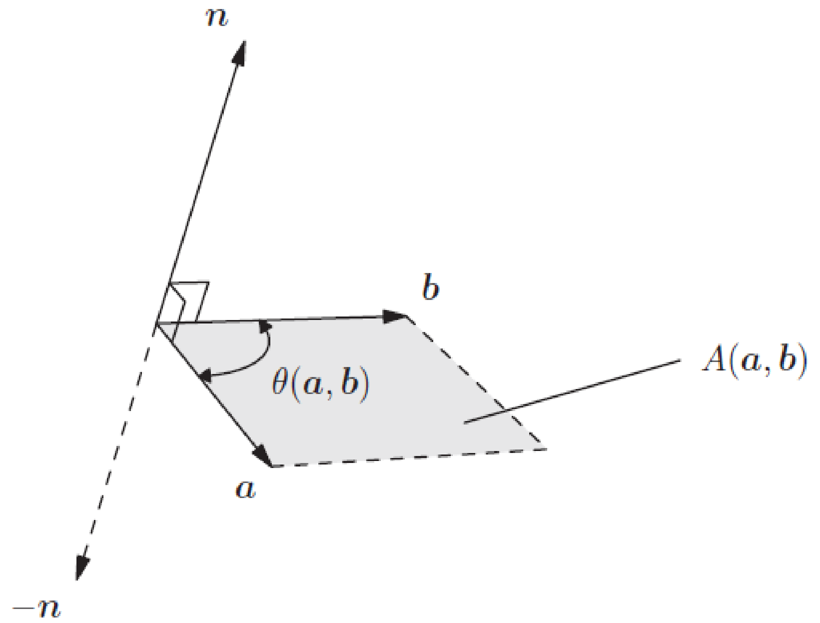
General Rule: To be successful, always ask yourself, what kind of object am I dealing with? What operation am I performing? What result type should I get?

Cross Product

Without any further ado, our definition of cross product is exactly the same as what you already know from elementary texts. We simply repeat a few of these for emphasis:

1. The magnitude
   $$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\theta \qquad (0 \le \theta \le \pi)$$
   of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is the area $A(\boldsymbol{a}, \boldsymbol{b})$ spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This is the area of the parallelogram defined by these vectors. This area is non-zero only when the two vectors are linearly independent.
2. θ is the angle between the two vectors.
3. The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Cross Product



The area $A(\boldsymbol{a}, \boldsymbol{b})$ spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The unit vector $\boldsymbol{n}$ in the direction of the cross product can be obtained from the quotient, $\frac{\boldsymbol{a}\times\boldsymbol{b}}{\|\boldsymbol{a}\times\boldsymbol{b}\|}$.

Cross Product

The cross product is bilinear and anti-commutative:

Given $\alpha \in \mathscr{R}, \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathscr{V}$,

$$(\alpha \boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \alpha(\boldsymbol{a} \times \boldsymbol{c}) + \boldsymbol{b} \times \boldsymbol{c}$$

$$\boldsymbol{a} \times (\alpha \boldsymbol{b} + \boldsymbol{c}) = \alpha(\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{a} \times \boldsymbol{c}$$

So that there is linearity in both arguments.

Furthermore, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Tripple Products

The trilinear mapping,

$$[\ ,\ ,] : \mathscr{V} \times \mathscr{V} \times \mathscr{V} \rightarrow \mathscr{R}$$

From the product set $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to real space is defined by:

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \equiv \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$$

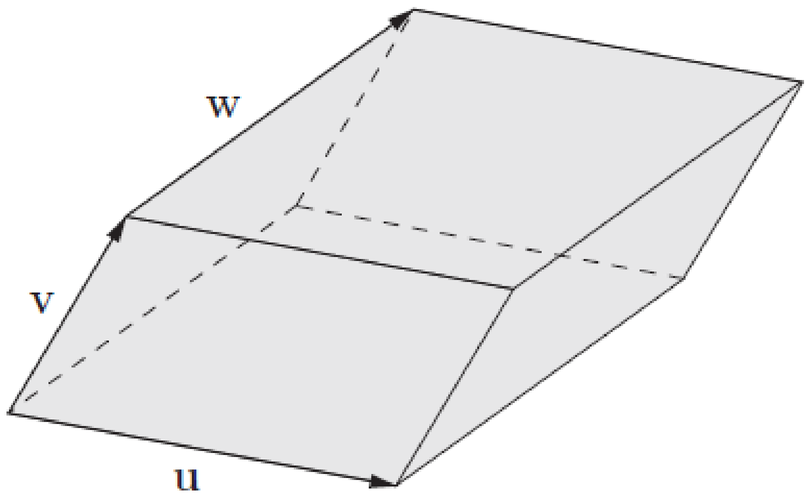
Has the following properties:

1. $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=[\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}]=[\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}]=-[\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}]=-[\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}]=-[\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}]$

HW: Prove this

1. Vanishes when ***a, b*** and ***c*** are linearly dependent.
2. It is the volume of the parallelepiped defined by ***a, b*** and ***c***

Tripple product



Parallelepiped defined by **u, v** and **w**

Summation Convention

* We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we deal with higher ranked objects, such notational conveniences become even more important. We shall often deal with coordinate transformations.

Summation Convention

* When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

Summation Convention

* Consider transformation equations such as,

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

* We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j$$

$$y_2 = \sum_{j=1}^{n} a_{2j}x_j$$

$$y_3 = \sum_{j=1}^{n} a_{2j}x_j$$

Summation Convention

* In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j$$
$$y_2 = a_{2j}x_j$$
$$y_3 = a_{2j}x_j$$

* Finally, we observe that $y_1, y_2,$ and y_3 can be represented as we have been doing by $y_i,\ i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \qquad i = 1,2,3$$

Summation Convention

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij} x_j$$

to represent compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

Summation Convention

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

for $i = 1, k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1, k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1, k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2, k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2, k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2, k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3, k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3, k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3, k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

Summation Convention

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}$$

In this same way, we could have also proved that,

$$a_{ij} a_{kj} = b_{ik}$$

* Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

Vector Components

Suppose our basis vectors $\mathbf{e}_i, i = 1,2,3$ are not only not unit in magnitude, but in addition are NOT orthogonal. The only assumption we are making is that $\mathbf{e}_i \in \mathscr{V}, i = 1,2,3$ are linearly independent vectors.

With respect to this basis, we can express vectors $\mathbf{v}, \mathbf{w} \in \mathscr{V}$ in terms of the basis as,

$$\mathbf{v} = v_i \mathbf{e}_i, \mathbf{w} = w_i \mathbf{e}_i$$

Where each v_i is called the component of $\mathbf{v}$, while w_i is called the component of $\mathbf{w}$

Vector Components

Clearly, addition and linearity of the vector space ⇒

$$\mathbf{v} + \mathbf{w} = (\,v_i + w_i)\mathbf{e}_i$$

Multiplication by scalar rule implies that if $\alpha \in \mathscr{R}$, $\forall \mathbf{v} \in \mathscr{V}$,

$$\alpha\,\mathbf{v} = (\alpha v_i)\mathbf{e}_i$$

Kronecker Delta

Kronecker Delta: δ_{ij}, δ^{ij} or δ_j^i has the following properties:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$
$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$
$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

As is obvious, these are obtained by allowing the indices to attain all possible values in the range. The Kronecker delta is defined by the fact that when the indices explicit values are equal, it has the value of unity. Otherwise, it is zero. The above nine equations can be written more compactly as,

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Vector components

- For any $\forall \mathbf{v} \in \mathscr{V}$,

$$\mathbf{v} = v_i \mathbf{e}_i$$

Are two related representations in the reciprocal bases. Taking the inner product of the above equation with the basis vector $\mathbf{e}_j$, we have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{e}_j &= v_i \mathbf{e}_i \cdot \mathbf{e}_j = v_i \delta_{ij} \\ &= v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} \\ &= v_j \end{aligned}$$

Levi Civita Symbol

* The Levi-Civita Symbol: e_{ijk}
* $e_{111} = 0, e_{112} = 0, e_{113} = 0, e_{121} = 0, e_{122} = 0, e_{123} = 1, e_{131} = 0, e_{132} = -1, e_{133} = 0$
  $e_{211} = 0, e_{212} = 0, e_{213} = -1, e_{221} = 0, e_{222} = 0, e_{223} = 0, e_{231} = 1, e_{232} = 0, e_{233} = 0$
  $e_{311} = 0, e_{312} = 1, e_{313} = 0, e_{321} = -1, e_{322} = 0, e_{323} = 1, e_{331} = 0, e_{332} = 0, e_{333} = 0$

Levi Civita Symbol

* While the above equations might look arbitrary at first, a closer look shows there is a simple logic to it all. In fact, note that whenever the value of an index is repeated, the symbol has a value of zero. Furthermore, we can see that once the indices are an even arrangement (permutation) of 1,2, and 3, the symbols have the value of 1, When we have an odd arrangement, the value is -1. Again, we desire to avoid writing twenty seven equations to express this simple fact. Hence we use the index notation to define the Levi-Civita symbol as follows:
* $$e_{ijk} = \begin{cases} 1 \text{ if } i, j \text{ and } k \text{ are an even permutation of } 1{,}2 \text{ and } 3 \\ -1 \text{ if } i, j \text{ and } k \text{ are an odd permutation of } 1{,}2 \text{ and } 3 \\ 0 \ \text{ In all other cases} \end{cases}$$

Kronecker Delta & Levi Civita Symbol

* The following relationship between the Kronecker Delta and the Levi-Civita Symbol provides the background for other important results:

$$e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$

* Recall that a determinant vanishes if two or more of its columns are equal, and that swapping rows or columns negate the sign.
* These facts (more instructively) demonstrate the equality even without expanding the determinant.

Kronecker Delta & Levi Civita Symbol

* Similarly, for the same reasons, we can easily see that,

$$e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$$

* And taking the product of these, we can easily show that,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Cross Product of Basis Vectors

The definition of Levi Civita Symbol implies $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$.

i	j	$e_{ijk}\mathbf{e}_k$
1	3	$e_{13k}\mathbf{e}_k = e_{131}\mathbf{e}_1 + e_{132}\mathbf{e}_2 + e_{133}\mathbf{e}_3 = -\mathbf{e}_2$
1	2	$e_{12k}\mathbf{e}_k = e_{121}\mathbf{e}_1 + e_{122}\mathbf{e}_2 + e_{123}\mathbf{e}_3 = \mathbf{e}_3$
2	3	$e_{23k}\mathbf{e}_k = e_{231}\mathbf{e}_1 + e_{232}\mathbf{e}_2 + e_{233}\mathbf{e}_3 = \mathbf{e}_1$
3	1	$e_{31k}\mathbf{e}_k = e_{311}\mathbf{e}_1 + e_{312}\mathbf{e}_2 + e_{313}\mathbf{e}_3 = \mathbf{e}_2$
1	1	$e_{11k}\mathbf{e}_k = e_{111}\mathbf{e}_1 + e_{112}\mathbf{e}_2 + e_{113}\mathbf{e}_3 = 0$
2	1	$e_{21k}\mathbf{e}_k = e_{211}\mathbf{e}_1 + e_{212}\mathbf{e}_2 + e_{213}\mathbf{e}_3 = -\mathbf{e}_1$

Examples

* This relationship also implies that,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k = e_{kij}\mathbf{e}_k$$

* Because a double swap does not alter the value of the alternating symbol.
* Confirm each example in the above table as we shall use this formula frequently for vectors and we shall expand its use for tensors.

Components of Products

Consider $\mathbf{a} = a_i \mathbf{e}_i$, and $\mathbf{b} = b_j \mathbf{e}_j$. Apart from scaling a vector – the same thing as multiplying it with a scalar, how many binary products of vectors are you familiar with?

* **Scalar product.** A binary product of two vectors; The result is a scalar quantity. It's operation is represented by a "dot". It is therefore sometimes called a "dot product".

Components of Products

- **Vector product.** A binary product of two vectors; The result is a vector quantity. It's operation is represented by a "times" or "cross". It is therefore sometimes called a "cross product".
  - Note that we derive the name of the product, either from the result produced, or from the symbol used to represent the operation.

$$\mathbf{a}\cdot\mathbf{b} = (a_i\mathbf{e}_i)\cdot\left(b_j\mathbf{e}_j\right) = a_ib_j\mathbf{e}_i\cdot\mathbf{e}_j$$
$$= a_ib_j\delta_{ij} = a_ib_i$$
$$\mathbf{a}\times\mathbf{b} = (a_i\mathbf{e}_i)\times\left(b_j\mathbf{e}_j\right) = a_ib_j\mathbf{e}_i\times\mathbf{e}_j$$
$$= e_{ijk}a_ib_j\mathbf{e}_k$$

Produce a Tensor

* We are used to producing scalars or vectors by taking a product of two vectors. One exceedingly important object that you can also produce from taking such a binary product is a **Tensor**. Naturally, we shall call such a product a “Tensor Product”.
* Its symbol, $\otimes$, is not a dot or a cross. It is a symbol that may look strange. That symbol combines the product sign and a circle. It is called a dyad operator. Therefore, as before, a tensor product also has a nickname, “the Dyad”, or a “Dyad Product”.

Dyad Basis

* The most elementary tensor you can get is the dyad product of two base vectors: $\mathbf{e}_i \otimes \mathbf{e}_j$
* The product of two vectors can be expressed in terms of this dyad base:

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

  * The summation convention still applies so that it is easy to see that the above expression contains nine components.
* Observe immediately that, in 3D, just as you express a vector in terms of three basis vectors, there are nine base dyads for expressing every tensor: $\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \mathbf{e}_1 \otimes \mathbf{e}_3, \mathbf{e}_2 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_2 \otimes \mathbf{e}_3, \mathbf{e}_3 \otimes \mathbf{e}_1, \mathbf{e}_3 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3$
  * Finding the components of a tensor is to find nine scalar coefficients to these base dyads.

Dyad Properties

- To explain what a dyad is, we use an operational definition of the way it operates on a vector. We shall obtain an operational definition of what a tensor in general is from this result:
- Given a dyad, $\mathbf{a} \otimes \mathbf{b}$, Suppose we take its operation on a vector, What do we get?

DEFINITION: Given a dyad $\mathbf{a} \otimes \mathbf{b}$, and a vector $\mathbf{c}$, the operation,

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} \equiv (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

OBSERVATION: Note that the quantity on the RHS is a vector. That means that the operation of a dyad on a vector produces a *vector in the direction of the first operand*.

What is a Tensor

* It is easy for us to define a tensor here. We have already seen a tensor: A dyad is a tensor. We have already observed how it operates on a vector. We know it produces another vector. We now proceed to give an operational definition of a tensor in general:

DEFINITION: A **linear transformation** of a vector into a vector is what we call a tensor.

Tensor

- We have seen that the dyad $\mathbf{a} \otimes \mathbf{b}$ is a tensor. We supply the vector $\mathbf{c}$, it produces the vector, $(\mathbf{b} \cdot \mathbf{c})\mathbf{a}$.
- To prove that is is a tensor, all that is left is to establish its linearity. A tensor not only transforms vectors to vectors, the operation must be linear.
- What do we mean by an "Operation is linear"?

Vector Class

* We will formally express the fact that a vector class is closed under addition and scaling. Here are the reasons before we state them formally.
  1. If you add two vectors, you get a vector by the parallelogram law.
  2. If you scale a vector (or multiply it with a scalar- means exactly the same thing) you get another vector in the same direction.
* Clearly, if $\mathbf{c}$ and $\mathbf{d}$ are vectors, and α and β are scalars, then, if $\alpha\mathbf{c} + \beta\mathbf{d}$ is certainly a vector.

Formality

We can formalize what we already know about vectors:

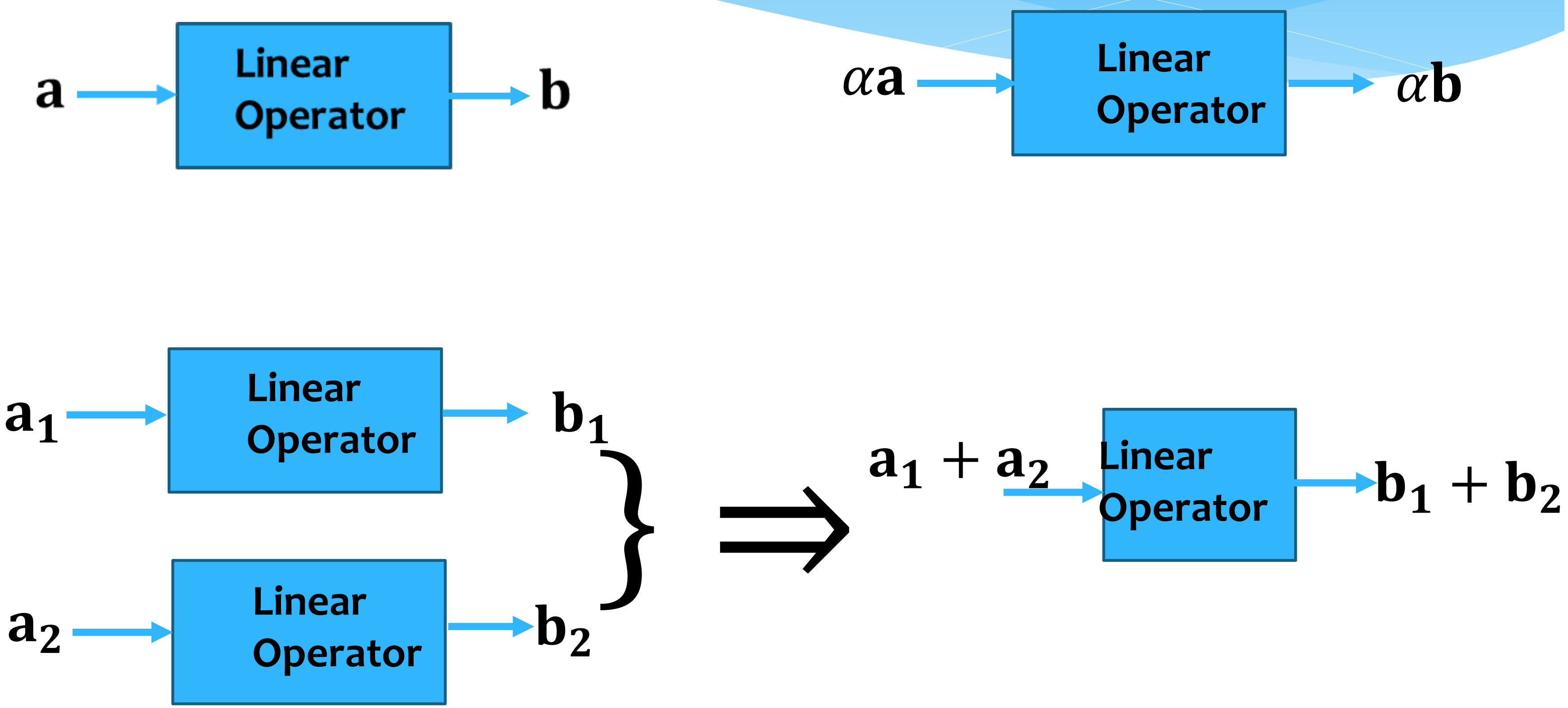
* A set of objects in such a way that addition is defined, is commutative and associative in such a way that the set is closed under the addition operation.
* It is closed and associative under scaling.
* There is an additive inverse
* The scalar product is defined

Linearity

Note that NOT all transformations of a vector into a vector is a tensor. To be a tensor, a transformation must first transform a vector into a vector; The transformation MUST secondly be LINEAR.

* When a transformation is linear, there is a simple relationship between such a transformation and the transformation of the same vector when is is scaled.
* There is also a simple relationship between the transformation of two vectors and the transformation of their sum.
* A combination of these two imply linearity

Scaled & Added Transformations



Linear Transformation

* From the previous diagram, it is clear that the transformation of a scaled vector is the same scale of the initial vector transformation. And, the linear transformation of the addition of two vectors is the addition of the two transformations.
* Consequently, given a tensor $\mathbf{T}$, scalars α and β, with vectors $\mathbf{a}$ and $\mathbf{b}$, linearity implies,

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{Ta} + \beta\mathbf{Tb}$$

Transformation, Operator, Function

* You may have noticed that we have been rather careless in the use of these terminologies. Usually, they mean slightly different things.
  * An advanced mathematician can easily fault us.
  * However, I am using these terms in an operational way; Here, in the definition of linear transformations, they mean essentially the same thing. We usually express the tensor transformation on a vector in the operator notation; simply placing the vector to be transformed to the right of the tensor, for example, $\mathbf{Tu}$.
  * We could have expressed this in a functional way, for example, $\mathbf{T}(\mathbf{u})$.
* The meanings are unchanged. They refer to the same linear transformation sharing domains and co-domains.

Examples

Show that a vector that transforms every vector to the vector $\mathbf{e}_1 + 2\mathbf{e}_2$ cannot be a tensor.

* Call this transformation $\mathbf{F}$. Consider the vector $\mathbf{a}$. We are given that, $\mathbf{Fa} = \mathbf{e}_1 + 2\mathbf{e}_2$. Vector $\mathbf{b} = 3\mathbf{a}$ is another vector which is a scaled version of $\mathbf{a}$.
* $\mathbf{Fb} = \mathbf{e}_1 + 2\mathbf{e}_2 \neq 3\mathbf{e}_1 + 6\mathbf{e}_2$ violating linearity. Hence, $\mathbf{F}$, despite the fact that it transforms vectors to vectors, is not a tensor because it is not a linear transformation.

Linearity of the Dyad

Show that the Dyad, defined previously is a linear transformation.

- Given vectors $\mathbf{a}$ and $\mathbf{b}$, the dyad of these is $\mathbf{a} \otimes \mathbf{b}$, Consider,

$$
\begin{aligned}
(\mathbf{a} \otimes \mathbf{b})(\alpha \mathbf{c} + \beta \mathbf{d}) & \\
&= \mathbf{a}[\mathbf{b} \cdot (\alpha \mathbf{c} + \beta \mathbf{d})] \\
&= \mathbf{a}[\alpha \mathbf{b} \cdot \mathbf{c} + \beta \mathbf{b} \cdot \mathbf{d}] \\
&= \alpha \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) + \beta \mathbf{a}(\mathbf{b} \cdot \mathbf{d}) \\
&= \alpha(\mathbf{a} \otimes \mathbf{b})\mathbf{c} + \beta(\mathbf{a} \otimes \mathbf{b})\mathbf{d}
\end{aligned}
$$

We invoked the distributive property of the dot product. And we have proved that the dyad not only transforms a vector to a vector, it does so linearly.

Homework 2.1

1. For any tensor $\mathbf{S}$, show that, $(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha = \mathbf{S}$
2. Given vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, establish the identities:
   a) $(\mathbf{u}\times)(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$
   b) $(\mathbf{u} \otimes\ )(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \otimes \mathbf{v}) \otimes \mathbf{w}$
3. Show that that if the tensor $\mathbf{T}$ is invertible, for any vector $\mathbf{k}$, $\mathbf{T}\mathbf{k} = \mathbf{o}$ automatically means that $\mathbf{k} = \mathbf{o}$.
4. Show that if the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent and $\mathbf{T}$ is invertible, then the vectors $\mathbf{T}\mathbf{u}, \mathbf{T}\mathbf{v}$ and $\mathbf{T}\mathbf{w}$ are also linearly independent.

Due March 30, 2019

Solutions

1. In the dyad, $(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha$ observe that $\mathbf{S}\mathbf{e}_\alpha$ is a vector. Operate this dyad on a given vector $\mathbf{v}$, we have,
$$[(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha]\mathbf{v} = (\mathbf{S}\mathbf{e}_\alpha)v_\alpha = \mathbf{S}(v_\alpha \mathbf{e}_\alpha) = \mathbf{S}\mathbf{v}$$
because every tensor is a linear operator; from which it is obvious that $(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha = \mathbf{S}$

2. **a.** $$(\mathbf{u}\times)(\mathbf{v}\otimes\mathbf{w}) = (\mathbf{u}\times)\left(v_j w_k \mathbf{e}_j \otimes \mathbf{e}_k\right)$$
$$= e_{\alpha\beta\gamma}u_\beta \mathbf{e}_\alpha \otimes \mathbf{e}_\gamma\left(v_j w_k \mathbf{e}_j \otimes \mathbf{e}_k\right)$$
$$= e_{\alpha\beta\gamma}u_\beta v_j w_k\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\gamma\right)\left(\mathbf{e}_j \otimes \mathbf{e}_k\right)$$
$$= e_{\alpha\beta\gamma}u_\beta v_j w_k \mathbf{e}_\alpha \otimes \mathbf{e}_k \delta_{\gamma j}$$
$$= e_{\alpha\beta j}\, u_\beta v_j \mathbf{e}_\alpha \otimes \mathbf{e}_k$$
$$= (\mathbf{u}\times\mathbf{v}) \otimes \mathbf{w}$$

b. $$(\mathbf{u}\otimes)(\mathbf{v}\otimes\mathbf{w}) = u_i v_j w_k \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = (\mathbf{u}\otimes\mathbf{v})\otimes\mathbf{w}$$

Solutions Cont'd

3. Simply premultiply $\mathbf{Tk} = \mathbf{o}$ by $\mathbf{T}^{-1}$. Clearly, since is invertible, $\mathbf{T}^{-1}$ exists, and we can write,

$$\mathbf{T}^{-1}\mathbf{Tk} = \mathbf{T}^{-1}\mathbf{o} = \mathbf{o}$$

Which shows that $\mathbf{k}$ must vanish once its transformation by an invertible tensor is the zero vector.

4. Let us assume that $\mathbf{Tu}$, $\mathbf{Tv}$ and $\mathbf{Tw}$ are not linearly independent. In particular, we can find α such that,

$$\mathbf{Tu} = \alpha(\mathbf{Tv} + \mathbf{Tw})$$

In that case, given that $\mathbf{T}$ is invertible, there is a tensor $\mathbf{T}^{-1}$ such that,

$$\mathbf{T}^{-1}\,\mathbf{Tu} = \alpha(\mathbf{T}^{-1}\mathbf{Tv} + \mathbf{T}^{-1}\mathbf{Tw})$$

That is, $\mathbf{u} = \alpha(\mathbf{v} + \mathbf{w})$ showing that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are not linearly independent; contradicting the initial premise.

Special Tensors

Notation.

It is customary to write the tensor mapping in the operator form; without the parentheses. Hence, we can write,

$$\mathbf{Tu} \equiv \mathbf{T}(\mathbf{u})$$

For the mapping by the tensor $\mathbf{T}$ on the vector variable and dispense with the parentheses unless when needed.

Zero Tensor or Annihilator

The annihilator $\mathbf{O}$ is defined as the tensor that maps all vectors to the zero vector, $\mathbf{o}$:

$$\mathbf{O}\mathrm{u} = \mathbf{o}, \qquad \forall \mathbf{u} \in \mathscr{V}$$

The Identity

The identity tensor $\mathbf{I}$ is the tensor that leaves every vector unaltered. $\forall \mathbf{u} \in \mathscr{V}$,

$$\mathbf{Iu} = \mathbf{u}$$

Furthermore, $\forall \alpha \in \mathscr{R}$, the tensor, $\alpha\mathbf{I}$ is called a spherical tensor.

The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\mathbf{T} \in \mathscr{T}$ and $\mathbf{u} \in \mathscr{V}$, the mapping $\mathbf{w} \equiv \mathbf{Tu}$ produces a vector.

The Inverse

Consider a linear mapping that, operating on $\mathbf{w}$, produces our original argument, $\mathbf{u}$, if we can find it:

$$\mathbf{Yw} = \mathbf{u}$$

As a linear mapping, operating on a vector, clearly, $\mathbf{Y}$ is a tensor. It is called the inverse of $\mathbf{T}$ because,

$$\mathbf{Yw} = \mathbf{YTu} = \mathbf{u}$$

So that the composition $\mathbf{YT} = \mathbf{I}$, the identity mapping. For this reason, we write,

$$\mathbf{Y} = \mathbf{T}^{-1}$$

Inverse

It is easy to show that if $\mathbf{YT} = \mathbf{I}$, then $\mathbf{TY} = \mathbf{YT} = \mathbf{I}$.

- ***HW: Show this.***

The set of invertible sets is closed under composition. It is also closed under inversion. It forms a group with the identity tensor as the group's neutral element

Transposition of Tensors

Given $\mathbf{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\mathbf{A}^{\mathrm{T}}$ satisfying

$$\mathbf{w} \cdot \left(\mathbf{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\mathbf{A}\mathbf{w})$$

Is called the transpose of A.

A tensor indistinguishable from its transpose is said to be symmetric.

Invariants

There are certain mappings from the space of tensors to the real space. Such mappings are called Invariants of the Tensor. Three of these, called Principal invariants play key roles in the application of tensors to design and analysis. We shall define them shortly.

The definition given here is free of any association with a coordinate system. It is a good practice to derive any other definitions from these fundamental ones:

The Trace

If we write

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

* where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors.

For any second order tensor $\mathbf{T}$, and linearly independent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the linear mapping $I_1: \mathscr{T} \rightarrow \mathscr{R}$

$$I_1(\mathbf{T}) \equiv \mathrm{tr}(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Is independent of the choice of the basis vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is called the First Principal Invariant of $\mathbf{T}$ or Trace of $\mathbf{T} \equiv \mathrm{tr}(\mathbf{T}) \equiv I_1(\mathbf{T})$

The Trace

Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary independent vectors let us choose the Cartesian Basis vectors, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

For any second order tensor $\boldsymbol{T}$,

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = [\mathbf{Ti}, \mathbf{j}, \mathbf{k}] + [\mathbf{i}, \mathbf{Tj}, \mathbf{k}] + [\mathbf{i}, \mathbf{j}, \mathbf{Tk}]$$

Since $[\mathbf{i}, \mathbf{j}, \mathbf{k}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1$.

The Trace

The trace is a linear mapping. It is easily shown that $\alpha, \beta \in \mathscr{R}$, and $\mathbf{S}, \mathbf{T} \in \mathscr{T}$

$$\text{tr}(\alpha\mathbf{S} + \beta\mathbf{T}) = \alpha\text{tr}(\mathbf{S}) + \beta\text{tr}(\mathbf{T})$$

HW. Show this by appealing to the linearity of the vector space.

While the trace of a tensor is linear, the other two principal invariants are nonlinear. We now proceed to define them

Square of the trace

The second principal invariant $I_2(\mathbf{S})$ is related to the trace. In fact, you may come across books that define it so. However, the most common definition is that

$$I_2(\mathbf{S}) = \frac{1}{2}\left[I_1^2(\mathbf{S}) - I_1(\mathbf{S}^2)\right]$$

Independently of the trace, we can also define the second principal invariant as,

Second Principal Invariant

The Second Principal Invariant, $I_2(\mathbf{T})$, using the same notation as above is

$$\frac{[(\mathbf{Ta}),(\mathbf{Tb}),\mathbf{c}]+[\mathbf{a},(\mathbf{Tb}),(\mathbf{Tc})]+[(\mathbf{Ta}),\mathbf{b},(\mathbf{Tc})]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

This quantity remains unchanged for any arbitrary selection of linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

The Determinant

The third mapping from tensors to the real space underlying the tensor is the determinant of the tensor. While you may be familiar with that operation and can easily extract a determinant from a matrix, it is important to understand the definition for a tensor that is independent of the component expression. The latter remains relevant even when we have not expressed the tensor in terms of its components in a particular coordinate system.

The Determinant

As before, For any second order tensor $\mathbf{T}$, and any linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

* The determinant of the tensor $\mathbf{T}$,

$$\det(\mathbf{T}) = \frac{[(\mathbf{Ta}), (\mathbf{Tb}), (\mathbf{Tc})]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

(In the special case when the chosen vectors are orthonormal, the denominator is unity)

Other Principal Invariants

- It is good to note that there are other principal invariants that can be defined. The ones we defined here are the ones you are most likely to find in other texts.
- An invariant is a scalar derived from a tensor that remains unchanged in any coordinate system. Mathematically, it is a mapping from the tensor space to the real space. Or simply **a scalar valued function of the tensor**.

Deviatoric Tensors

* Given any tensor $\mathbf{S}$, A deviatoric tensor may be created from $\mathbf{S}$ by the following process:
$$\mathbf{S}_0 \equiv \operatorname{dev}\mathbf{S} \equiv \mathbf{S} - \frac{1}{3}(\operatorname{tr}\mathbf{S})\mathbf{1} = \mathbf{S} - s\mathbf{1}$$
* So that $s = \frac{1}{3}(\operatorname{tr}\mathbf{S})$; $s\mathbf{1}$ is called the spherical part, and $\mathbf{S}_0$ as defined here is called the deviatoric part of $\mathbf{S}$. Every tensor thus admits the decomposition,
$$\mathbf{S} = \mathbf{S}_0 + s\mathbf{1}$$
* When the trace of a tensor is zero, the tensor is said to be traceless. A deviatoric stress is traceless.

Symmetric, Antisymmetric Parts

Every second order tensor can be split into its symmetric and antisymmetric parts:

$$\frac{1}{2}\left(\mathbf{S}+S^{\mathrm{T}}\right)+\frac{1}{2}\left(\mathbf{S}-\mathbf{S}^{\mathrm{T}}\right) \equiv \operatorname{sym} S+\operatorname{skw} S$$

This decomposition is unique. The component representation of these two parts will be given shortly.

Inner Product of Tensors

The trace provides a simple way to define the inner product of two second-order tensors. Given $\mathbf{S}, \mathbf{T} \in \mathscr{T}$

The trace,

$$\text{tr}\left(\mathbf{S}^{\text{T}}\mathbf{T}\right) = \text{tr}\left(\mathbf{S}\mathbf{T}^{\text{T}}\right)$$

Is a scalar, independent of the coordinate system chosen to represent the tensors. This is defined as the inner or scalar product of the tensors $\mathbf{S}$ and $\mathbf{T}$. That is,

$$\mathbf{S}:\mathbf{T} \equiv \text{tr}\left(\mathbf{S}^{\text{T}}\mathbf{T}\right) = \text{tr}\left(\mathbf{S}\mathbf{T}^{\text{T}}\right)$$

The Tensor Product

A product mapping from two vector spaces to $\mathscr{T}$ is defined as the tensor product. It has the following properties:

$$"\otimes": \mathscr{V} \times \mathscr{V} \to \mathscr{T}$$

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

It is an ordered pair of vectors. It acts on any other vector by creating a new vector in the direction of its first vector as shown above. This product of two vectors is called a tensor product or a simple dyad.

Example

* $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1 \quad y_2 \quad y_3) = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}$$
* $$(\mathbf{x} \otimes \mathbf{y})\mathbf{u} = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1 \quad y_2 \quad y_3)\right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1u_1 + x_1y_2u_2 + x_1y_3u_3 \\ x_2y_1u_1 + x_2y_2u_2 + x_3y_1u_1 \\ x_3y_1u_1 + x_3y_2u_2 + x_3y_3u_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1u_1 + y_2u_2 + y_3u_3) = \mathbf{x}(\mathbf{y} \cdot \mathbf{u})$$

Dyad Properties

The tensor product is linear in its two factors.

Based on the obvious fact that for any tensor $\mathbf{T}$ and

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathscr{V}$$

$$\mathbf{T}(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{T}\mathbf{u}(\mathbf{v} \cdot \mathbf{w}) = [(\mathbf{T}\mathbf{u}) \otimes \mathbf{v}]\mathbf{w}$$

It is clear that $\mathbf{T}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{T}\mathbf{u}) \otimes \mathbf{v}$

Furthermore, the contraction,

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{T} = \mathbf{u} \otimes \left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)$$

A fact that can be established by operating each side on the same vector.

Transpose of a Dyad

For $\mathbf{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\mathbf{A}^{\mathrm{T}}$ satisfying

$$\mathbf{w} \cdot \left(\mathbf{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\mathbf{A}\mathbf{w})$$

Is called the transpose of $\mathbf{A}$. Now let $\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$ a dyad.

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{A}\mathbf{w}) &= \\ &= \mathbf{v} \cdot [(\mathbf{a} \otimes \mathbf{b})\mathbf{w}] = \mathbf{v} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{w})] \\ &= (\mathbf{v} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{w}) = (\mathbf{w} \cdot \mathbf{b})(\mathbf{v} \cdot \mathbf{a}) \\ &= \mathbf{w} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{v} \end{aligned}$$

So that $(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$

Showing that the transpose of a dyad is simply a reversal of its operands.

Composition with Tensors

Operate on the vector $\mathbf{z}$ and let $\mathbf{Tz} = \mathbf{w}$. On the LHS, $(\mathbf{u} \otimes \mathbf{v})\mathbf{Tz} = (\mathbf{u} \otimes \mathbf{v})\mathbf{w}$. On the RHS, we have:

$$\left(\mathbf{u} \otimes \left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)\right)\mathbf{z} = \mathbf{u}\left(\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) \cdot \mathbf{z}\right) = \mathbf{u}\left(\mathbf{z} \cdot \left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\mathbf{u} \otimes \left(\mathbf{z} \cdot \left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)\right) = \mathbf{u} \otimes \left(\mathbf{v} \cdot (\mathbf{Tz})\right)$$

follows from the definition of transposition. Hence,

$$\left(\mathbf{u} \otimes \left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right)\right)\mathbf{z} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \otimes \mathbf{v})\mathbf{w}$$

Dyad on Dyad Composition

For $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathscr{V}$, We can show that the dyad composition,

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \otimes \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$$

Again, the proof is to show that both sides produce the same result when they act on the same vector. Let $\mathbf{y} \in \mathscr{V}$, then the LHS on $\mathbf{y}$ yields:

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x})\mathbf{y} = (\mathbf{u} \otimes \mathbf{v})[\mathbf{w}(\mathbf{x} \cdot \mathbf{y})]$$
$$= \mathbf{u}(\mathbf{v} \cdot \mathbf{w})(\mathbf{x} \cdot \mathbf{y})$$

Which is obviously the result from the RHS also.

This therefore makes it straightforward to contract dyads by breaking and joining as seen above.

Trace of a Dyad

Show that the trace of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v}$.

Given any three independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, (No loss of generality in letting the three independent vectors be the curvilinear basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$). Using the above definition of trace, we can write that,

Trace of a Dyad

$$\mathrm{tr}(\mathbf{u}\otimes\mathbf{v})$$

$$=\frac{[\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_1\},\mathbf{e}_2,\mathbf{e}_3]+[\mathbf{e}_1,\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_2\},\mathbf{e}_3]+[\mathbf{e}_1,\mathbf{e}_2,\{(\mathbf{u}\otimes\mathbf{v})\mathbf{e}_3\}]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,]}$$

$$=\frac{1}{e_{123}}[\{v_1\mathbf{u}\},\mathbf{e}_2,\mathbf{e}_3]+[\mathbf{e}_1,\{v_2\mathbf{u}\},\mathbf{e}_3]+[\mathbf{e}_1,\mathbf{e}_2,\{\boldsymbol{v}_3\mathbf{u}\}]$$

$$=\frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{23i}\mathbf{e}_i)+(e_{31i}\mathbf{e}_i)\cdot(v_2\,\mathbf{u})+(e_{12i}\mathbf{e}_i)\cdot(v_3\mathbf{u})\}$$

$$=\frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{231}\mathbf{e}_1)+(e_{312}e_2)\cdot(v_2\mathbf{u})+(e_{123}e_3)\cdot(v_3\mathbf{u})\}=\ \mathbf{u}\cdot\mathbf{v}$$

Other Invariants of a Dyad

* It is easy to show that for a tensor product

$$\mathbf{D} = \mathbf{u} \otimes \mathbf{v} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathscr{V}$$

$$\mathbf{I}_2(\mathbf{D}) = \mathbf{I}_3(\mathbf{D}) = 0$$

HW. Show that this is so.

We proved earlier that $\mathbf{I}_1(\mathbf{D}) = \mathbf{u} \cdot \mathbf{v}$

Furthermore, if $\boldsymbol{T} \in \mathscr{T}$, then,

$$\mathrm{tr}(\boldsymbol{Tu} \otimes \boldsymbol{v}) = \mathrm{tr}(\boldsymbol{w} \otimes \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{Tu} \cdot \boldsymbol{v}$$

Component Representation

$$\mathbf{T} = \mathrm{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

* The coefficient T_{ij} can be found operating each side above of the above with $\mathbf{e}_\beta$,

$$\mathbf{T}\mathbf{e}_\beta = \mathrm{T}_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_\beta$$
$$= \mathrm{T}_{ij}\mathbf{e}_i\delta_{j\beta} = T_{i\beta}\mathbf{e}_i$$

Taking the dot product of the above vector equation with $\mathbf{e}_\alpha$,

$$\mathbf{e}_\alpha \cdot \mathbf{T}\mathbf{e}_\beta = T_{i\beta}\mathbf{e}_\alpha \cdot \mathbf{e}_i = T_{i\beta}\delta_{\alpha i} = T_{\alpha\beta}$$

For the identity tensor, it is easy to show that,

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Showing that the Kronecker deltas are actually the coefficients of the identity tensor.

Component Representation

$$\mathbf{T} = \mathrm{T}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- The coefficient T_{ij} can be found by,

$$\begin{aligned}\mathrm{T}_{ij} &= \mathbf{T}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\\ &= \mathrm{tr}\left(\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right)\\ &= \mathrm{tr}\left(\left(\mathbf{T}\mathbf{e}_j\right) \otimes \mathbf{e}_i\right)\\ &= \left(\mathbf{T}\mathbf{e}_j\right)\cdot\mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j\end{aligned}$$

- Give you three minutes to try this. How do you avoid multiple repeats of dummy variables?

Symmetry

For tensor $\boldsymbol{T}$ in component form,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

The transpose,

$$\mathbf{T}^{\mathrm{T}} = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$$
$$= T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

If the tensor is symmetrical,

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}$$
$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{T}^{\mathrm{T}} = T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that symmetry implies that,

$$T_{ij} = T_{ji}$$

AntiSymmetry

- A tensor is antisymmetric if its transpose is its negative. In product bases that are either covariant or contravariant, antisymmetry, like symmetry can be expressed in terms of the components:

If $\mathbf{T}$ is antisymmetric, then,

$$\mathbf{T} = -\mathbf{T}^{\mathrm{T}}$$

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -\mathbf{T}^{\mathrm{T}} = -T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that anti−symmetry implies that,

$$T_{ij} = -T_{ji}$$

Antisymmetric tensors are also said to be skew-symmetric.

Symmetric & Skew Parts of Tensors

For any tensor $\mathbf{T}$, define the symmetric and skew parts $\text{sym}\ \mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{\mathrm{T}}\right)$, and $\text{skw}\ \mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^{\mathrm{T}}\right)$. It is easy to show the following:

$\mathbf{T} = \text{sym}\ \mathbf{T} + \text{skw}\ \mathbf{T}$

$\text{skw}(\text{sym}\ \mathbf{T}) = \text{sym}(\text{skw}\ \mathbf{T}\ ) = 0$. We can also write that,

$$\text{sym}\ \mathbf{T} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

and

$$\text{skw}\ \mathbf{T} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

Composition

Composition of tensors in component form follows the rule of the composition of dyads.

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\begin{aligned}\mathbf{TS} &= \left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\\ &= T_{ij}S_{\alpha\beta}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\\ &= T_{ij}S_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\delta_{j\alpha}\\ &= T_{ij}S_{j\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\\ &= T_{i\alpha}S_{\alpha j}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

Addition

* Addition of two tensors of the same order is the addition of their components provided they are refereed to the same product basis.

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\mathbf{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\mathbf{T} + \mathbf{S} = (T_{ij}+S_{ij})\mathbf{e}_i \otimes \mathbf{e}_j,$$

Component Representation of Invariants

* Invoking the definition of the three principal invariants, we now find expressions for these in terms of the components of tensors in various product bases.
* First note that for $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, the triple product,

$$[\{\mathbf{T}\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3] = \left[\left\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\right\}, \mathbf{e}_2, \mathbf{e}_3\right]$$
$$= \left[\left\{T_{ij}\mathbf{e}_i\delta_{1j}\right\}, \mathbf{e}_2, \mathbf{e}_3\right] = T_{i1}[\mathbf{e}_i, \mathbf{e}_2, \mathbf{e}_3] = T_{i1}e_{i23}$$

The Trace

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\begin{aligned}\mathrm{tr}(\boldsymbol{T}) &= \frac{[\mathbf{Te}_1, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \\ &= \frac{\left[\left\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\right\}, \mathbf{e}_2, \mathbf{e}_3\right] + \left[\mathbf{e}_1, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_2, \mathbf{e}_3\right] + \left[\mathbf{e}_1, \mathbf{e}_2, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right]}{e_{123}} \\ &= T_{i1}e_{i23} + T_{i2}e_{1i3} + T_{i3}e_{12i} \\ &= T_{11} + T_{22} + T_{33} \\ &= T_{ii}\end{aligned}$$

A much easier way is to recall that the trace is a linear operator. Hence the trace of T,

$$\mathrm{tr}\,\mathbf{T} = T_{ij}\mathrm{tr}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = T_{ij}\delta_{ij} = T_{ii}$$

Because the trace of a dyad is simply the dot product of its operands.

Second Invariant

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{[\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

Which, in a similar way to the above, can be shown to be,

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji})$$

Which is half the square of the trace minus trace of the square of the tensor $\mathbf{T}$

Determinant

The third invariant,

$$\frac{[(\mathbf{Te_1}), (\mathbf{Te}_2), (\mathbf{Te}_3)]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} = e_{ijk}T_{i1}T_{j2}T_{k3}$$

$$= \det(\mathbf{T})$$

Dual Vectors

* Recall that a skew tensor, the negative of its transpose, satisfies,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$$

There two immediate consequences of this:

1. For a skew tensor, $T_{ij} = -T_{ji}$
   Only three of the nine components are independent as the others are either zero or negatives of one of the three. Is this obvious?
2. Following the above, ALL information contained in the tensor can be made into a vector. Such a vector exists for every antisymmetric tensor. It is the Dual vector.

The Vector Cross

The converse of the above situation: Creating a skew tensor from a vector. Such a tensor is called the Vector Cross. Given a vector $\mathbf{u} = u_i \mathbf{e}_i$, the tensor

$$\boldsymbol{\Omega} = (\mathbf{u} \times) \equiv e_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

is called a vector cross. The following relation is easily established between a the vector cross and its associated vector:

$$\forall \mathbf{v} \in \mathscr{V}, (\mathbf{u} \times)\mathbf{v} = \mathbf{u} \times \mathbf{v}$$

The vector cross is *traceless* and *antisymmetric*. *(HW. Show this)*

We created this tensor from a single vector!

Dual vector relationship

* Suppose we have been given a skew tensor,

$$\Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = (\mathbf{u}\times) = e_{i\alpha j}u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

We want to find the components u_α from the Ω_{ij}s.

Clearly,

$$\Omega_{ij} = e_{i\alpha j}u_\alpha$$

Multiplying both sides by e_{ijk}, we obtain,

$$\begin{aligned} e_{ijk}\Omega_{ij} &= e_{ijk}e_{i\alpha j}u_\alpha \\ &= -2\delta_{k\alpha}u_\alpha = -2u_k \end{aligned}$$

So that we can find the components of the dual vector from,

$$u_k = -\frac{1}{2}e_{ijk}\Omega_{ij}; \qquad \Omega_{ij} = e_{i\alpha j}u_\alpha$$

Examples

Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $(\mathbf{u}\times):(\mathbf{v}\times) = 2\mathbf{u}\cdot\mathbf{v}$. Hence show that $\|\mathbf{u}\times\| = \sqrt{2}\|\mathbf{u}\|$

Orthogonal Tensors

Given a Euclidean Vector Space $\mathcal{E}$, a tensor $\mathbf{Q}$ is said to be orthogonal if, $\forall \mathbf{a}, \mathbf{b} \in \mathcal{E}$,

$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{a} \cdot \mathbf{b}$$

Specifically, we can allow $\mathbf{a} = \mathbf{b}$, so that

$$(\mathbf{Qa}) \cdot (\mathbf{Qa}) = \mathbf{a} \cdot \mathbf{a}$$

Or

$$||\mathbf{Qa}|| = ||\mathbf{a}||$$

In which case the mapping leaves the magnitude unaltered.

Orthogonal Tensors

Let $\mathbf{q} = \mathbf{Qa}$

$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{q} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

By definition of the transpose, we have that,

$$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$

Clearly, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{1}$

A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse equal to its transpose.

Orthogonal

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) = \left(\det \mathbf{Q}^{\mathrm{T}}\right)(\det \mathbf{Q}) = (\det \mathbf{Q})^2 = 1$$

Which clearly shows that

$$(\det \mathbf{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "*proper orthogonal*".

Rotation & Reflection

A rotation is a proper orthogonal tensor while a reflection is not.

Rotation

* **Let Q be a rotation. For any pair of vectors u, v show that $\mathbf{Q}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Qu})\times(\mathbf{Qv})$**

This question is the same as showing that the cofactor of $\boldsymbol{Q}$ is $\boldsymbol{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\mathbf{T}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Qu})\times(\mathbf{Qv})$$

where

$$\mathbf{T} = \mathrm{cof}(\mathbf{Q}) = \det(\mathbf{Q})\,\mathbf{Q}^{-\mathrm{T}}$$

Now that $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = 1$, and

$$\mathbf{Q}^{-\mathrm{T}} = (\mathbf{Q}^{-1})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q}$$

This implies that $\mathbf{T} = \mathbf{Q}$ and consequently,

$$\mathbf{Q}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Qu})\times(\mathbf{Qv})$$

* Take another look at the antisymmetric tensor.