Constitutive Models

Linear Elasticity

General Laws & Constitutive Models

* In our study of Continuum Mechanics, we have been looking at the following:
  1. The geometry of deformation and motion. Concepts of strain, deformation gradient, etc.
  2. Concept of stress: Cauchy and other stress tensors
  3. Balance of Mass, Momentum, Energy and Entropy inequality
* These ideas apply to all materials whether solid, liquid or gaseous. In particular, Kinematics ignored, pro tempore, the forces that cause motion.

Balance Laws

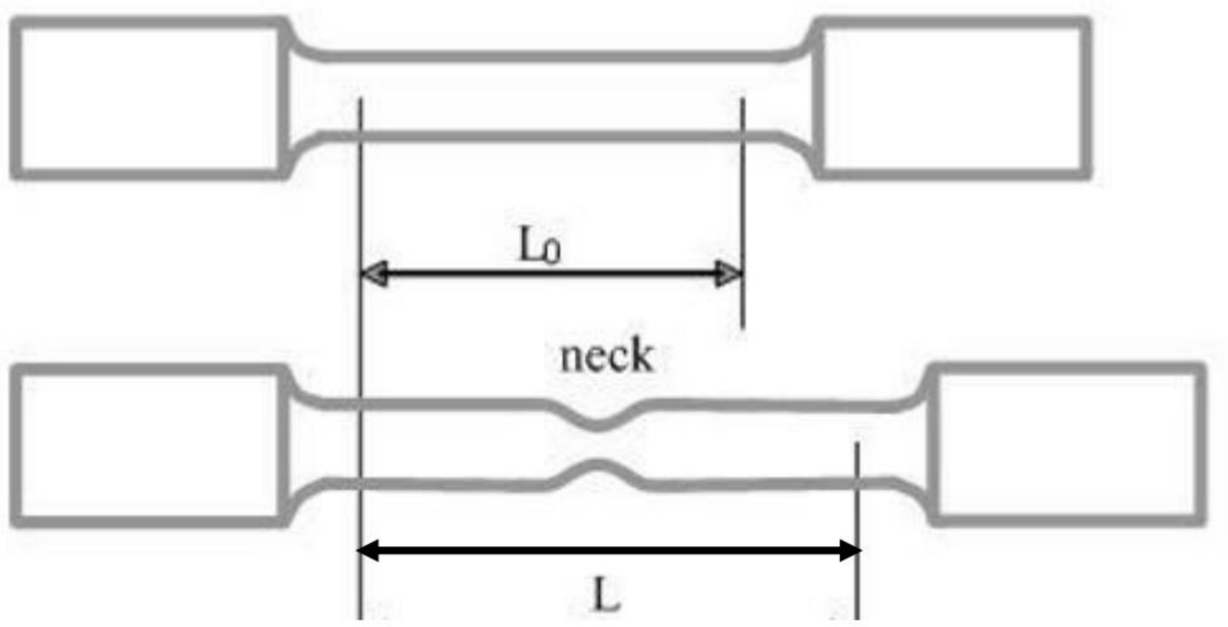
* In studying balance laws, we are studying the laws of nature in quantifiable forms that can be applied to systems of materials we may want to study
* Balance of mass is essentially affirming that, in non-relativistic terms, matter is neither created not destroyed
* Balance of momenta are simple restatements of the well-know laws of Newton to apply to actual bodies.
* Balance of energy are statements about energy conversion

Constitutive Laws

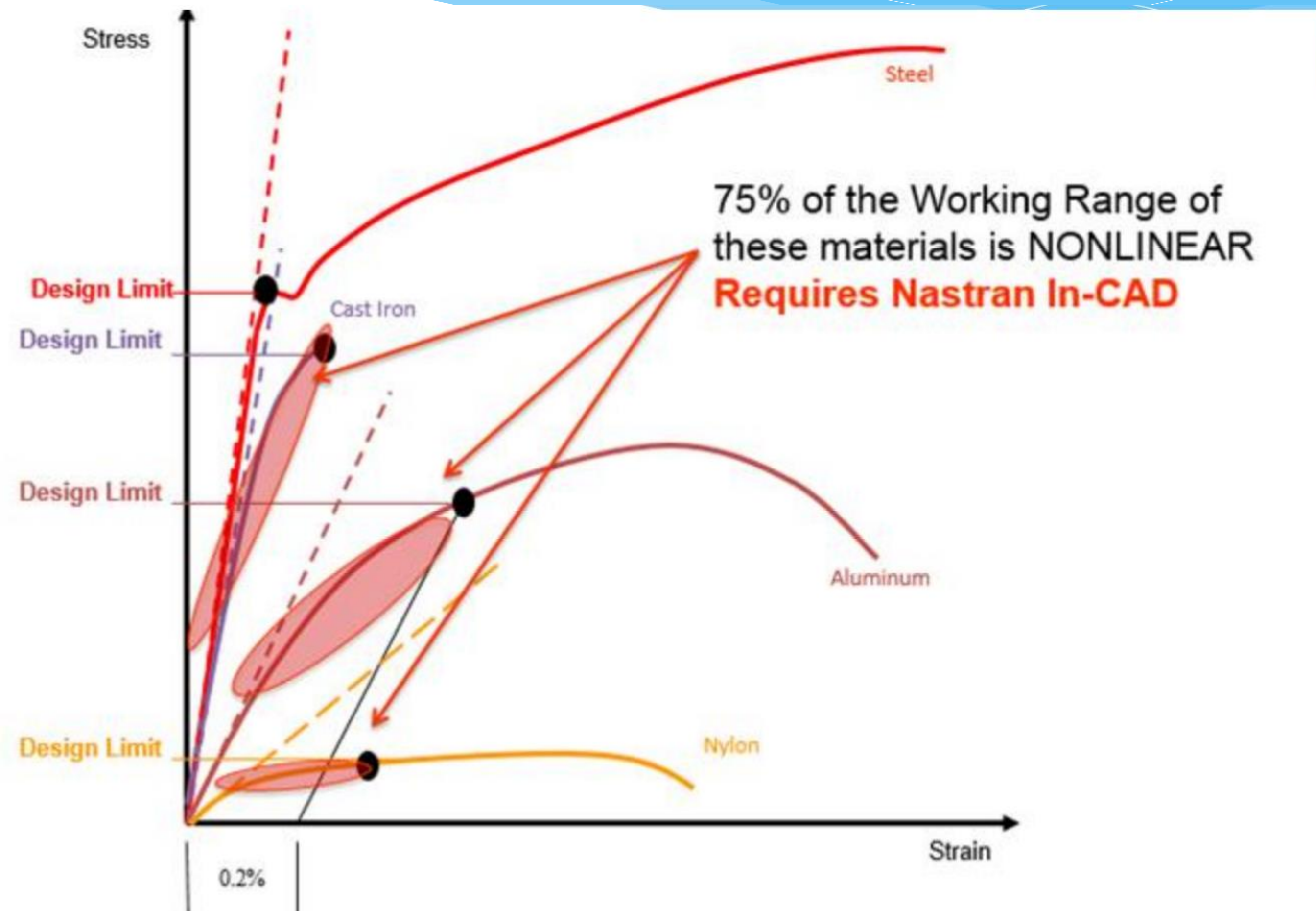
- A constitutive law applies to a specific material or material type.
- The response to external loads: Forces, Heat, Magnetic, etc. depends on particular constitutions of the bodies.
- This is why different materials are chosen to provide different responses.
- We may want a material to be strong, rigid and lightweight. There are occasions when we desire them to be flexible and compliant.

Uniaxial Test Specimens

In laboratory testing, we often use simple shapes and subject them to uniaxial loading. From here, we compute such things as elastic modulus, yield strength, ultimate strength, etc.



Stress Strain Curves



Uniaxial Test Results

* Most materials show an amount of linearity in the stress-strain relationship before behaving differently.
* Most solid materials are able to return to their original states for small strains. This ability to recover their original states is called elasticity.
* The above graphs show why it is easy to confuse linearity and elasticity. In many cases, the region of linearity and elasticity have significant overlaps.
* For the purpose of analysis, we make a clear distinction between the two concepts.

Small Deformations

* Guided by the fact that elasticity and linearity deformation ranges are small, and the fact that linear systems theory and superposition make such analysis comparatively simple, we shall first look at simple models of small deformation.
* We will need to redefine our strain tensors to reflect this fact as we make some modifications in the sequel.
* We shall introduce the displacement function and its derivatives

Displacement Function

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X}$$

is called the displacement function. Recall that the Langrangian strain function,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right)$$

Taking the material gradient of the displacement function, we have,

$$\mathbf{H} \equiv \mathbf{Grad}\,\mathbf{u}(\mathbf{X},t) = \mathbf{F} - \mathbf{I}$$

Hence,

$$\mathbf{F} = \mathbf{H} + \mathbf{I}$$

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$$\mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{H}^{\mathrm{T}}\mathbf{H} + \mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{I}$$

We focus on situations where $||\mathbf{u}(\mathbf{X}, t)|| \approx 0$, Hence products of the displacement gradients are ignorable. Therefore, we can write,

$$\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I} \approx \mathbf{H} + \mathbf{H}^{\mathrm{T}}$$

Small Strains

* For small strains therefore,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) \approx \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right)$$

In Cartesian Coordinates, the strain tensor, in terms of displacements is,

$$\mathbf{E} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

Linear Stress Strain Laws

- Moving from the simple uniaxial experiment that plots uniaxial strain to uniaxial stress, from our theory of stress and strain, we know that we are dealing with tensor quantities.
- A typical second-order tensor has nine components. The idea that stresses are proportional to strains naturally leads to the equation,

$$\boldsymbol{\sigma} = \mathbf{CE}$$

$$\sigma_{ij} = c_{ijkl} e_{kl}$$

Where, as we can see, the coefficient is a fourth order tensor giving us 81 constants! Nine stresses proportional to nine strains!

Isotropy and homogeneity

* We make the simplifying assumption that the body we are dealing with is isotropic and homogeneous.
* Its properties are invariant with respect to location and direction within the body. In this case, we immediately see that the number of constants we are dealing with reduce greatly from 81.

Symmetry of Stress and Strain

* Recall that the strain tensor is symmetrical; We are therefore dealing with six components.
* Similarly, by the balance of angular momentum, we know that the stress tensor is also symmetrical. We are therefore relating six stresses to six strains.
* This gives us a maximum of 36 constants.
* We can further reduce the number of constants by the isotropy assumption. We can argue that the coefficient fourth-order tensor $\mathbf{C}$ must necessarily be isotropic.

Isotropic Tensors

* For a tensor to be isotropic, its components under any rotation must remain unchanged. All scalars are isotropic, there are no isotropic vectors.
* It can be shown that for second-order tensors, only spherical tensors are isotropic. We have already seen the action of an isotropic tensor in the hydrostatic pressure stress case that creates the same traction in every direction.
* For a third-order tensor, only the Levi-Civita tensor and its multiples are isotropic.
* For fourth-order tensors, there are three isotropic tensors.

Component Forms of Isotropic Tensors

Tensors $\delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$

$$\delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

$$\delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

are the only tensors of the fourth order that will not have their components altered in rotations.

- Isotropy implies that same stress strain relations are obeyed even when orientations change implies that the tensor,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

where λ, β and γ are the constants.

Isotropic Stress Strain Relation

* For a homogeneous, isotropic body,

$$\begin{aligned}\sigma_{ij} = c_{ijkl}e_{kl} &= \left(\lambda\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}\right)e_{kl} \\ &= \lambda\delta_{ij}e_{kk} + \beta\delta_{ik}e_{kj} + \gamma\delta_{il}e_{jl} \\ &= \lambda\delta_{ij}e_{kk} + \beta e_{ij} + \gamma e_{ji} \\ &= \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\end{aligned}$$

if we write, $\mu \equiv \frac{1}{2}(\beta + \gamma)$ and observe that the strain tensor is symmetrical. In general, coordinate-free form, we have,

$$\boldsymbol{\sigma} = \lambda(\text{tr}\,\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$

This is the **Generalized Hooke's Law** for linear elastic bodies undergoing small strains.

Generalized Hooke's Law in Cartesian Coordinates

* In Cartesian coordinates, the above equations can be expanded to obtain,

$$\sigma_x = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\epsilon_x$$
$$\sigma_y = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\epsilon_y$$
$$\sigma_z = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu\epsilon_z$$
$$\tau_{xy} = 2\mu\epsilon_{xy}$$
$$\tau_{yz} = 2\mu\epsilon_{yz}$$
$$\tau_{zx} = 2\mu\epsilon_{zx}$$

Inversion

- Taking a trace of the last equation,

$$\mathrm{tr}\,\boldsymbol{\sigma} = (3\lambda + 2\mu)\mathrm{tr}\,\mathbf{E}$$

The constitutive equation can be inverted by,

$$\mathbf{E} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda\,\mathrm{tr}\,\boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}\,\mathbf{I}$$

which we can write in component form as,

$$\epsilon_x = \frac{\sigma_x}{2\mu} - \frac{\lambda\,\mathrm{tr}\,\boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}, \qquad \epsilon_y = \frac{\sigma_y}{2\mu} - \frac{\lambda\,\mathrm{tr}\,\boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}$$

$$\epsilon_z = \frac{\sigma_z}{2\mu} - \frac{\lambda\,\mathrm{tr}\,\boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}, \qquad \epsilon_{xy} = \frac{\tau_{xy}}{2\mu}, \epsilon_{yz} = \frac{\tau_{yz}}{2\mu}, \epsilon_{zx} = \frac{\tau_{zx}}{2\mu}$$

The constants λ and μ will now be connected to constants that have physical significance:

Uniaxial Stress

* Recall that the uniaxial state of stress is the stress tensor is defined as,

$$\boldsymbol{\sigma} = \sigma_x \mathbf{e}_1 \otimes \mathbf{e}_1$$

while all other stress components, $\sigma_y = \sigma_z = \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$.

Clearly, in this case, $\mathrm{tr}\,\boldsymbol{\sigma} = \sigma_x$. Taking the trace of the stress-strain law, we find,

$$\begin{aligned}\mathrm{tr}\,\boldsymbol{\sigma} = \sigma_x &= \lambda(\mathrm{tr}\,\mathbf{E})\,\mathrm{tr}(\mathbf{I}) + 2\mu\,\mathrm{tr}\,\mathbf{E} \\ &= 3\lambda(\mathrm{tr}\,\mathbf{E}) + 2\mu\,\mathrm{tr}\,\mathbf{E} \\ &= (3\lambda + 2\mu)\mathrm{tr}\,\mathbf{E}\end{aligned}$$

Furthermore, in this situation, $\epsilon_{xy} = \frac{\tau_{xy}}{2\mu} = 0$, and, similarly,

$$\epsilon_{yz} = \epsilon_{zx} = 0$$

Uniaxial Stress

* But,

$$\epsilon_x = \frac{\sigma_x}{2\mu} - \frac{\lambda \operatorname{tr} \boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}$$
$$= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_x$$
$$= \frac{1}{\mu}\left(\frac{\lambda + \mu}{3\lambda + 2\mu}\right)\sigma_x$$

From which we can conclude that Modulus of Elasticity,

$$E \equiv \frac{\sigma_x}{\epsilon_x} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

under uniaxial stress.

Uniaxial Stress

- Normal strain in $\mathbf{e}_2$ direction:

$$\epsilon_y = \frac{\sigma_y}{2\mu} - \frac{\lambda \operatorname{tr} \boldsymbol{\sigma}}{2\mu(3\lambda + 2\mu)}$$

$$= -\frac{\lambda\, \sigma_x}{2\mu(3\lambda + 2\mu)}$$

$$= -\frac{\lambda\, \epsilon_x}{2\mu(3\lambda + 2\mu)} \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

$$= -\frac{\lambda}{2(\lambda+\mu)}\, \epsilon_x = \epsilon_z$$

From which we can conclude that the Poisson Ratio,

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

Uniaxial Stress

- The strain tensor corresponding to uniaxial stress can now be seen to be,

$$\mathbf{E} = \frac{\sigma_x}{E_Y}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \nu \mathbf{e}_2 \otimes \mathbf{e}_2 - \nu \mathbf{e}_3 \otimes \mathbf{e}_3)$$

showing that uniaxial stress does not produce uniaxial strain in an isotropic, homogeneous elastic body.

Equations in terms of measured parameters

* Usual parameters we measure in a laboratory experiment are the Elastic Modulus and Poisson Ratio. A simple inversion of the relationships we have derived is effected by the following code:

```
Solve[{λ / (2 (λ + μ)) - ν == 0,
   EE (λ + μ) - 3 λ μ - 2 μ μ == 0}, {λ, μ}]
```

$$\left\{\left\{\lambda \to -\frac{EE\,\nu}{-1+\nu+2\,\nu^2},\ \mu \to \frac{EE}{2\,(1+\nu)}\right\}\right\}$$

Stress Strain Relations

* We therefore have that,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{2(1+\nu)}$$

Which allows us to express the linear stress-strain relations in terms of E and ν

Home Work

* Use the Mathematica code here as a guide. Express the following equations in cylindrical and spherical coordinates:

1. Small-Strain Displacement relations: $\mathbf{E} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right)$
2. Linear-Elastic Stress-Strain Law: $\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$
3. Linear-Elastic Strain-Stress Law: $\mathbf{E} = \frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda\,\mathrm{tr}\,\boldsymbol{\sigma}}{2\mu(3\lambda+2\mu)}\,\mathbf{I}$

Claude-Louis Navier

* You have heard about the Navier Stokes Equations of Fluid Mechanics! Meet [Mr Navier on Wikipedia here]
* It may surprise you that the original equations were addressed to the motion, not of fluids but of solids!
* Navier's equations are a restatement of Cauchy's equations of motion in terms of displacements rather than stresses of Cauchy.
* The famous Navier Stokes, as you will later discover, are essentially an amendment of the same equations, using the salient viscosity of fluids and making the fundamental variables strain rates rather than strains.

Navier's Equations

- We will need to compute the divergence of the product of a scalar and the identity tensor at some point. It is a good idea to see how we can do that right now:
- Given a scalar ϕ, what is the divergence of $\phi\mathbf{I}$ given that $\mathbf{I}$ is the identity tensor?

$$\begin{aligned}\operatorname{div}(\phi\mathbf{I}) &= \operatorname{tr}\big(\operatorname{grad}(\phi\mathbf{I})\big)\\ &= \operatorname{tr}\Big(\big(\phi\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j\big)_{,k}\otimes\mathbf{e}_k\Big)\\ &= \operatorname{tr}\big(\phi_{,k}\,\delta_{ij}\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\big)\\ &= \big(\phi_{,k}\,\delta_{ij}\mathbf{e}_i\delta_{jk}\big) = \phi_{,i}\,\mathbf{e}_i\\ &= \operatorname{grad}\phi\end{aligned}$$

Navier's Equations

* We are now in a position to look further at the stress strain relations:

$$\boldsymbol{\sigma} = \lambda(\text{tr } \mathbf{E})\mathbf{I} + 2\mu\mathbf{E}$$
$$= \lambda\,(\text{div } \mathbf{u})\mathbf{I} + \mu\left(\text{grad } \mathbf{u} + \text{grad}^{\text{T}}\mathbf{u}\right)$$

Taking the divergence of both sides, we have,

$$\text{div } \boldsymbol{\sigma} = \lambda(\text{grad div } \mathbf{u}) + \mu(\text{div grad } \mathbf{u} + \text{grad div } \mathbf{u})$$
$$= (\lambda + \mu)\text{grad div } \mathbf{u} + \mu \text{ div grad } \mathbf{u}$$

We can now rewrite Cauchy's equations now as,

$$\text{div } \boldsymbol{\sigma} + \mathbf{b} = \varrho\ddot{\mathbf{u}}$$

That is,

$$(\lambda + \mu)\text{grad div } \mathbf{u} + \mu \text{ div grad } \mathbf{u} + \mathbf{b} = \varrho\ddot{\mathbf{u}}$$

Navier’s Equations of Motion

* The second term is often called “grad squared”. Be careful what you say! What the textbooks call ∇^2or the Laplace or Laplacian operator is NOT the grad of grad. It is this quantity, divergence of grad. Notice that grad of grad is a tensor operator while div grad is a scalar operator. The latter is actually the trace of the former tensor. You need to bear this in mind because you will be using Mathematica to compute these quantities. Your immediate assignment now is to compute the Navier Equations of motion in Cylindrical, Cartesian and Spherical coordinates.

Home Work

1. Show that for a differentiable vector field $\mathbf{u}$, $\text{div grad}^{\text{T}}\mathbf{u} = \text{grad div } \mathbf{u}$
2. Show that for any differentiable scalar field, ϕ, $\text{div } (\phi\mathbf{I}) = \text{grad } \phi$. Hence show that $\text{div }\big((\text{div } \mathbf{u})\mathbf{I}\big) = \text{grad div } \mathbf{u}$.
3. From the two problems above, can we conclude that $\text{grad}^{\text{T}}\mathbf{u} = (\text{div } \mathbf{u})\mathbf{I}$? Why not? What then do the equality of their divergences tell us?
4. Use the Cauchy first law of motion to show that

$$(\lambda + \mu)\text{grad div } \mathbf{u} + \mu \text{ div grad } \mathbf{u} + \mathbf{b} = \varrho\ddot{\mathbf{u}}$$

   Express the above in Cartesian, Cylindrical and Spherical Polar coordinates.

Home Work

5. Show that $\text{tr}\,\mathbf{E} = \text{div}\,\mathbf{u}$.
6. Express the Stress-Strain relations in terms of Modulus of elasticity and the Poisson Ratio.
7. What is an isotropic tensor? Given that the Tensors,

$$\delta_{ij}\delta_{kl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$
$$\delta_{ik}\delta_{jl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$
$$\delta_{il}\delta_{jk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

are the only isotropic tensors of the fourth order, are the following tensors also isotropic? $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i$, $\mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_k$, $\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j$

Home Work

8. Show that in terms of engineering parameters ν and E, that Naviers' equations of motion can be written as,

$$\frac{E}{2(1+\nu)}\left[\text{grad div }\mathbf{u}+\frac{1}{1-2\nu}\text{ div grad }\mathbf{u}\right]+\mathbf{b}=\varrho\ddot{\mathbf{u}}$$

9. Use index notation to show that

$$\text{curl curl }\mathbf{u}=\text{grad div }\mathbf{u}-\text{div grad }\mathbf{u}$$

10. Using the above result, show that Navier's equation can be written as,

$$(\lambda+2\mu)\text{grad div }\mathbf{u}+\mu\text{ curl curl }\mathbf{u}+\mathbf{b}=\varrho\ddot{\mathbf{u}}$$

11. Why must the elastic tensor $\mathbf{C}$, in the equation $\boldsymbol{\sigma}=\mathbf{CE}$, be isotropic? What will the consequences be if it is not isotropic?

Number 9

* The key hint to this question is the relationship:

$$e_{ijk}e_{i\alpha\beta} = \delta_{j\alpha}\delta_{k\beta} - \delta_{j\beta}\delta_{k\alpha}$$

A proof of this can be found on the "300 problems" on this webpage. Let $\mathbf{v} = \text{curl}\ \mathbf{u} = e_{ijk}u_{k,j}\ \mathbf{e}_i$. Then we can write, $\text{curl}\ \mathbf{v} = e_{\alpha\beta i}v_{i,\beta}\ \mathbf{e}_\alpha$. All that is left is to recognize the fact that $v_i = e_{ijk}u_{k,j}$ so that,

$$\begin{aligned}\text{curl}\ \mathbf{v} = \text{curl curl}\ \mathbf{u} &= e_{\alpha\beta i}v_{i,\beta}\ \mathbf{e}_\alpha \\ &= e_{\alpha\beta i}\left(e_{ijk}u_{k,j}\right)_{,\beta}\mathbf{e}_\alpha = e_{\alpha\beta i}\ e_{ijk}u_{k,j\beta}\ \mathbf{e}_\alpha \\ &= \left(\delta_{j\alpha}\delta_{k\beta} - \delta_{j\beta}\delta_{k\alpha}\right)u_{k,j\beta}\ \mathbf{e}_\alpha \\ &= u_{k,kj}\ \mathbf{e}_j - u_{k,jj}\ \mathbf{e}_k \\ &= \text{grad div}\ \mathbf{u} - \text{div grad}\ \mathbf{u}\end{aligned}$$