Theory of Stress

Stress: Scalar, Vector & Tensor
Cauchy Stress Principle, Stress Tensors

Fundamental Principles

* Your first contact with material response is most likely the famous Hooke's law. It may therefore come as a surprise that this law and other similar laws governing more complicated materials are NOT the fundamental principles of mechanics. They are simply talking about the peculiar attributes of materials.
* Instead, we will look at the fundamental ideas – principally due to Cauchy
* In this installment, we see Cauchy in the following:
  1. The Euler-Cauchy Stress Principle (The principle of action/reaction) in real material bodies)
  2. Cauchy Stress Theorem (Existence of the stress tensor. Once you find it, you know everything about stress at the point.)
  3. Cauchy Laws of Motion (Motion laws beyond particles. Stress symmetry as a natural principle)

Meet the Man Cauchy (Wikipedia)

Baron **Augustin-Louis Cauchy** FRS FRSE (/koʊˈʃiː/;[1] French: [oɡystɛ̃ lwi koʃi]; 21 August 1789 – 23 May 1857) was a French mathematician, engineer and physicist who made pioneering contributions to several branches of mathematics, including: mathematical analysis and continuum mechanics. He was one of the first to state and prove theorems of calculus rigorously, rejecting the heuristic principle of the generality of algebra of earlier authors. He almost singlehandedly founded complex analysis and the study of permutation groups in abstract algebra.

A profound mathematician, Cauchy had a great influence over his contemporaries and successors;[2] Hans Freudenthal stated: "More concepts and theorems have been named for Cauchy than for any other mathematician (in elasticity alone there are sixteen concepts and theorems named for Cauchy)."[3] Cauchy was a prolific writer; he wrote approximately eight hundred research articles and five complete textbooks, on a variety of topics in the fields of mathematics and mathematical physics.

Augustin-Louis Cauchy

Cauchy around 1840. Lithography by Zéphirin Belliard after a painting by Jean Roller.

Stress

What is "Stress"?

- Stress is a measure of ***force intensity*** either within or on the bounding surface of a body subjected to loads.
- The Continuum Model takes a **macroscopic** approach:
  - **Measurable aggregate behavior** rather than the microscopic, atomistic activities that may in fact have led to them, and consequently, the
  - Standard **results of calculus** applicable in the case of limiting values of this quotient as the areas to which the forces are applied become very small.

What is Stress?

- The simple answer of force per unit area raises the following questions:
  - What force?
  - As the size and shape of the material in question change as motion evolves: What area:
  - What direction? Which surface? Which location?
- A more rigorous definition of stress is required to settle these and other matters of importance.

Three Stresses

In defining stress, care must be taken to note that sometimes we are talking about a

Tensor – the *Stress Tensor*.

- This completely characterizes the stress state at a particular location. That such a tensor exists is proved as Cauchy's theorem – a fundamental law in Continuum Mechanics.

Vector - the *Traction – Stress Vector*

- Intensity of resultant forces on a particular surface in a specific direction. This is, roughly speaking, what we have in mind when we say that stress is "force per unit area."

Scalar – the *scalar magnitude* of the traction vector or some other scalar function of the stress tensor.

Stress Examples

- **Stress Tensor**: Cauchy stress, Piola-Kirchhoff Stress, Spherical stress, etc.
- **Stress Vector. In these cases directions are either implicit or explicitly stated:** Normal stress, Shear stress, Surface traction
- **Stress Scalars**: Equivalent (Von-Mises) stress (second principal invariant of the deviatoric stress tensor), Yield stress, Ultimate tensile stress, hydrostatic pressure (the scalar multiplier of the identity tensor for the spherical stress tensor by the same name), etc.

Cauchy Resolution

* Cauchy's theorem connects all these views of stress and allows us to see the full picture.
* He postulated the existence of a tensor field whose value at any point tells us EVERYTHING we need to know at that point about the stress.

Two Forces

"… a distinction is established between two types of forces which we have called 'body forces' and 'surface tractions', the former being conceived as due to a direct action at a distance, and the latter to contact action." **AEH Love**

It is convenient to examine these forces by categorizing them as follows:

Body forces b (force per unit volume);

* These are forces originating from sources (fields of force usually) outside of the body that act on the volume (or mass) of the body.

Surface forces i.e.: **f**(force per unit area of surface across they which they act)

Body Forces

Body forces b (force per unit volume); These are forces originating from sources - usually outside of the body

* Fields of force
* Act on the volume (or mass) of the body.

- These forces arise from the placement of the body in force fields
  * **Definition:** A field of force is a *Euclidean Point Space* in which a force function is specified at every point
  * *Examples:* gravitational, electrical, magnetic or inertia
- As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed.
- Body forces are therefore assumed to be continuously distributed over the entire volume of the material.

Surface Forces

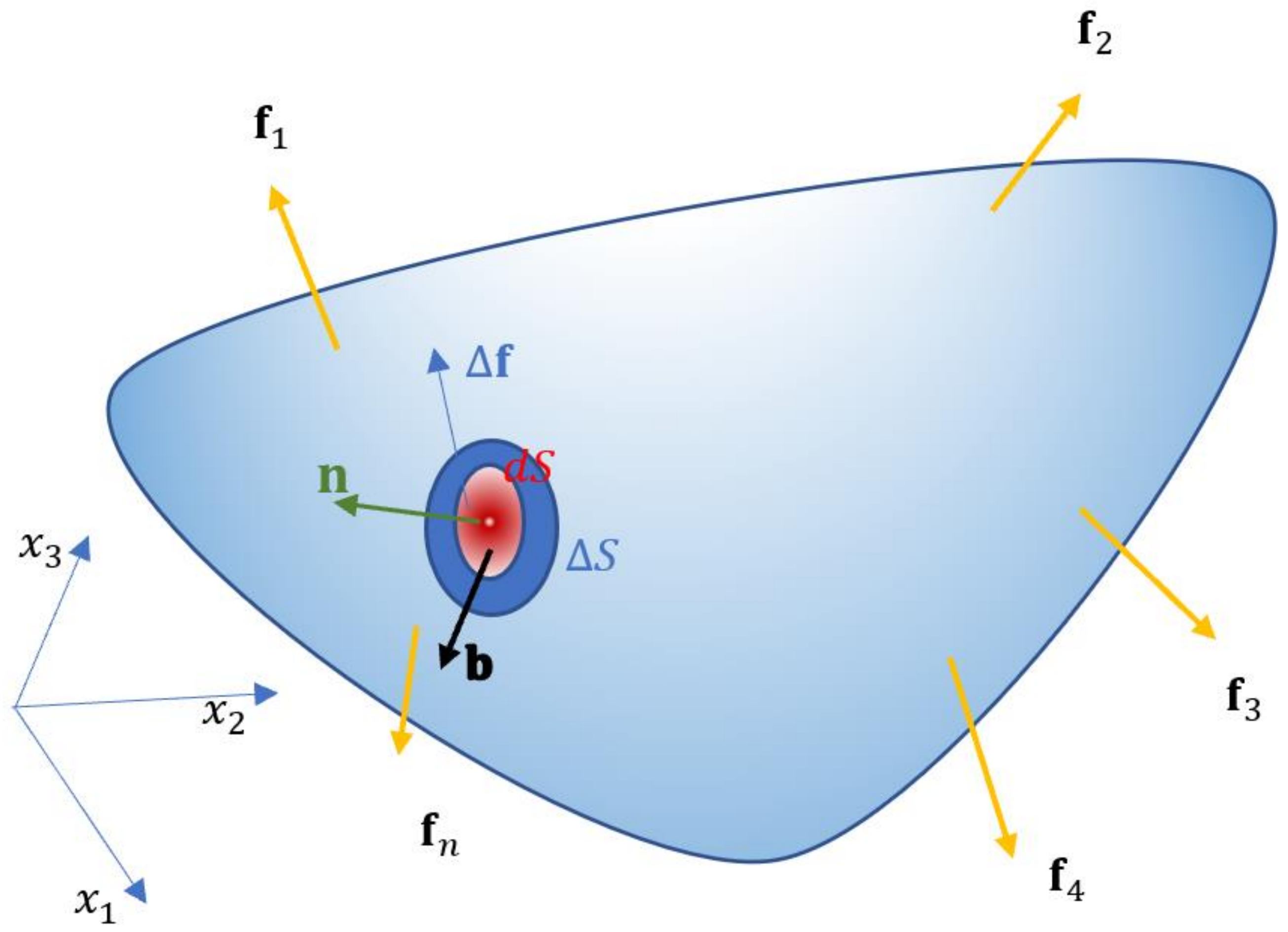
- Force per unit area of surface across which they act and are distributed in some fashion over a surface element of the body
- Element could be part of the bounding surface, or an arbitrary element of surface within the body;
  - Examples: Shear stresses, normal stresses such as hydrostatic pressure, Wind loading, contact with another solid etc.

Balance of forces

- Forces on an element V surrounding a point $P(x_1, x_2, x_3)$ in a body acted upon by the forces $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_n$.
- The resultant body force per unit volume is $\mathbf{b}(x_1, x_2, x_3)$.

* Consider a small portion of a body with volume ΔV shown in blue. Its surface ΔS has normal vectors pointing in different directions as we move from point to point on the surface.
* Let the resultant of all forces on the surface be $\Delta\mathbf{f}$.
* We take a portion of this small part that is sufficiently small that the surface has a specified normal, $\mathbf{n}$. The force intensity on this surface, we denote as $\mathbf{t}^{(\mathbf{n})}$ where the superscript is written to denote the (normal to the) surface on which this traction acts.

Forces on an Element



Surface Forces on an Element

* We can write that,

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} = \frac{d\mathbf{f}}{dS}$$

IMPORTANT NOTE: *The traction $\mathbf{t}^{(\mathbf{n})}$ gets its direction, not from $\mathbf{n}$ but from $\mathbf{f}$. $\mathbf{n}$ signifies the orientation of the surface on which it acts.*

Or, conversely that

$$\Delta \mathbf{f} = \int_{\Delta S} \mathbf{t}^{(\mathbf{n})} dS$$

In general, $\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3)$ and $\mathbf{n} = \mathbf{n}(x_1, x_2, x_3)$ as the surface itself is not necessarily a plane. It is only as the limit is approached that $\mathbf{n}$ is a fixed direction for the elemental area and $\Delta \mathbf{f}$ and $\mathbf{t}^{(\mathbf{n})}$ are in the same direction.

Body Forces on an Element

The density $\varrho = \varrho(x_1, x_2, x_3)$ as it varies over the whole body. If the resultant body force in the volume element ΔV is $\Delta \mathbf{B}$, we can compute the body force per unit volume

$$\mathbf{b} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{B}}{\Delta V} = \frac{d\mathbf{B}}{dV}$$

so that the body force on the element of volume

$$\Delta \mathbf{B} = \int_{\Delta V} \mathbf{b} dV .$$

Surface Traction

Vector intensity of the vector force on the surface as the surface area approaches a limit.

* It is defined for a specific surface with an orientation defined by the outward normal $\mathbf{n}$.
* This implies immediately that the traction at a given point is dependent upon the orientation of the surface. It is a vector that has different values at the same point depending upon the orientation of the surface we are looking at. There are multiple stress vectors at one point!
* The concept of a vector cannot resolve this ambiguity. Cauchy provided the world with the ideas to do so. He noted that the different stress vectors is actually the realizations, on different planes, of the same stress tensor.

Surface Traction

It is a function of the coordinate variables.

- It is therefore proper to write,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}(\mathbf{n}, x_1, x_2, x_3) \equiv \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3)$$

to make these dependencies explicitly obvious

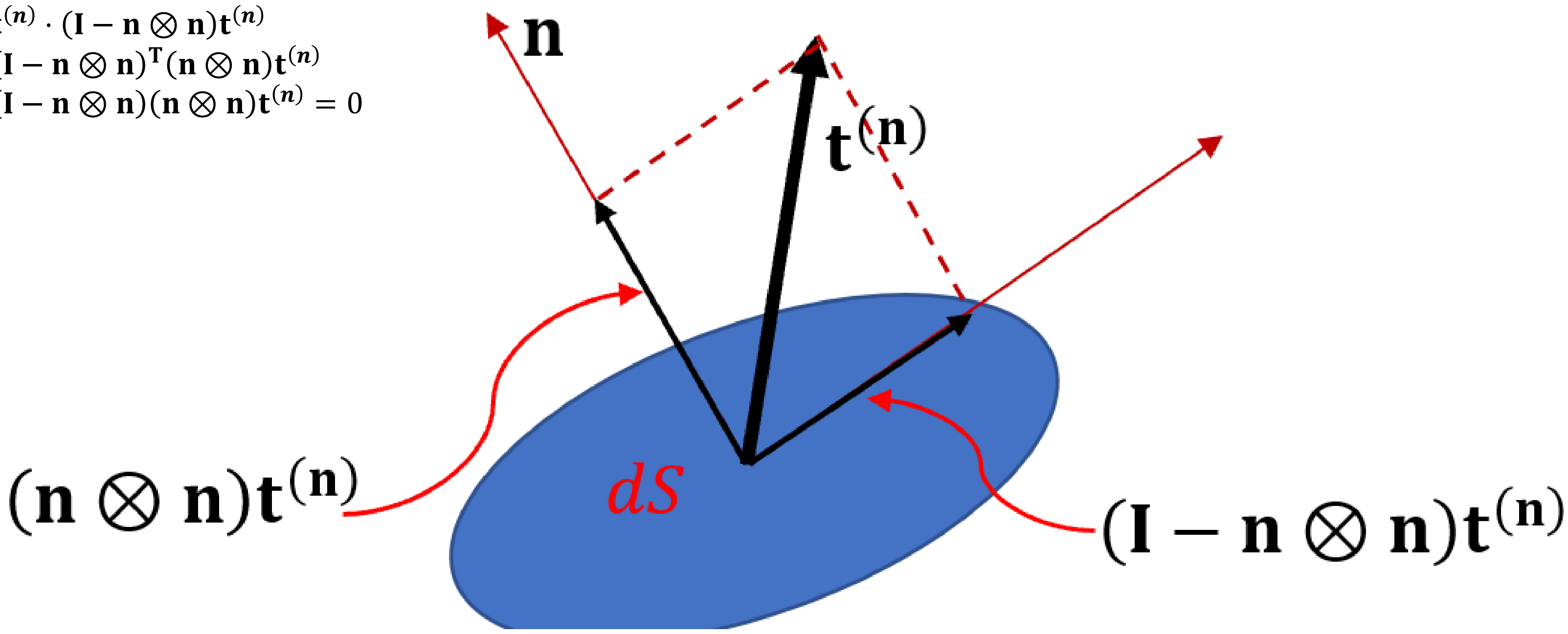
- In general, $\mathbf{t}$ and $\mathbf{n}$ are not in the same direction; that is, there is an angular orientation between the resultant ***stress vector*** and the surface outward normal.

Surface Traction is expressed as the vector sum of its projection $t_n \equiv \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$ along the normal $\mathbf{n}$ and $t_s \equiv \left\|\mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n}\right\|$ on the surface itself.

Normal & Tangential Tractions

For any unit vector $\mathbf{n}$, show that the vectors $(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)}$ and $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)}$ are perpendicular to each other. Show also that for any positive integer r, $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})^r = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$.

$$(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)} \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)}$$
$$= \mathbf{t}^{(n)} \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})^{\mathbf{T}}(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)}$$
$$= \mathbf{t}^{(n)} \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(n)} = 0$$



Normal & Shearing Stresses

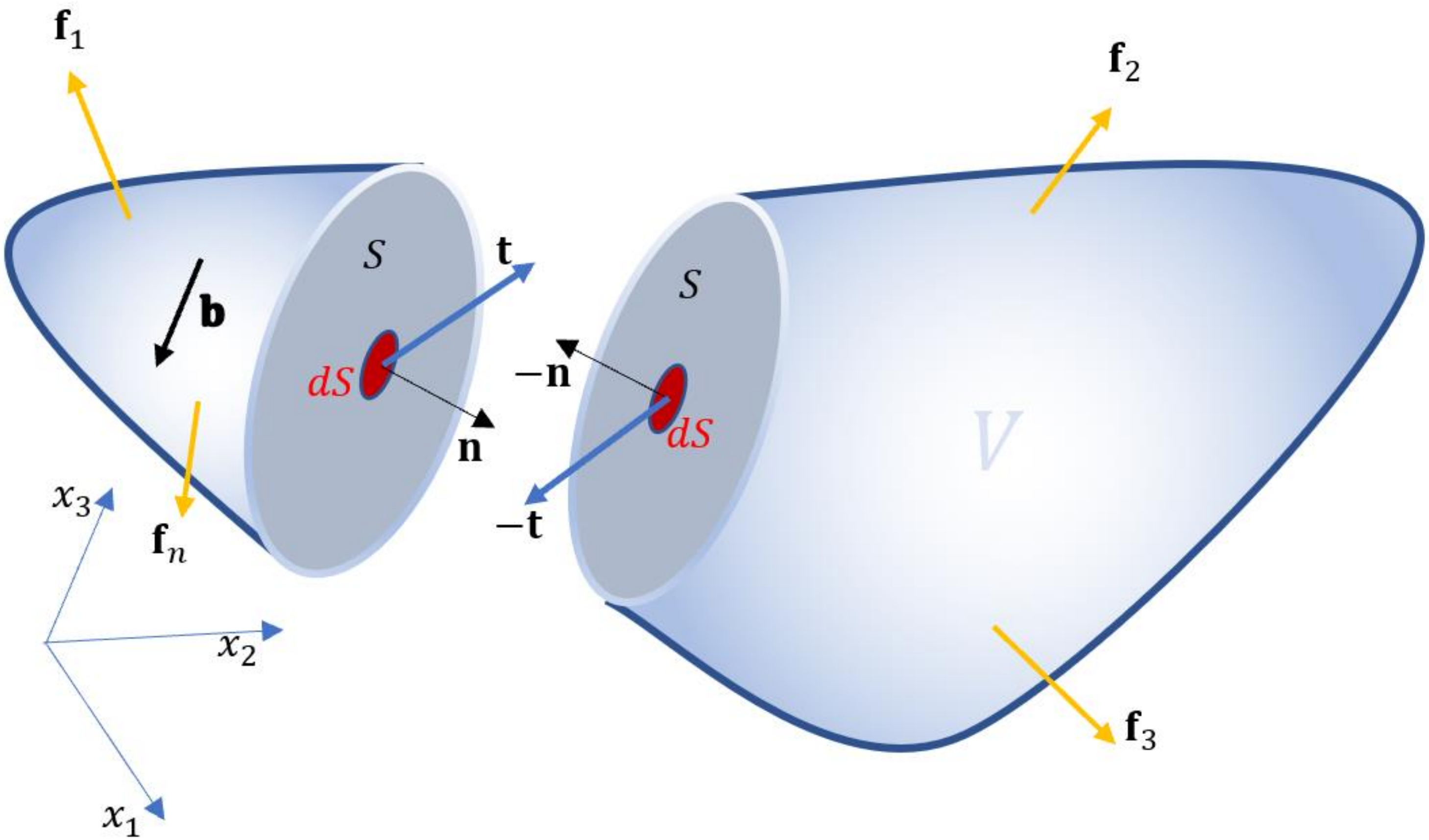
* It is easy to show that the normal stress vector is $(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})}$
* The shear stress vector is the surface projection of the resultant $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})}$.

  (Show these by simply opening the parentheses and interpreting)
* These normal and shearing components of the stress vector are called the *normal* and *shear tractions* respectively.
* t_n and t_s are the scalar magnitudes of the normal and tangential components of the reaction on any surface of interest.

Euler-Cauchy Stress Principle

* The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equipollent to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field $\mathbf{t}^{(\mathbf{n})}$. In view of Newton's third law of action and reaction, this principle can be expressed compactly in the equation,

$$\mathbf{t}^{(-\mathbf{n})} = -\mathbf{t}^{(\mathbf{n})}$$


$\mathbf{f}_1$
$\mathbf{f}_2$
S
$\mathbf{t}$
S
$\mathbf{b}$
dS
$-\mathbf{n}$
$\mathbf{n}$
dS
V
x_3
$-\mathbf{t}$
$\mathbf{f}_n$
x_2
$\mathbf{f}_3$
x_1

Cauchy's Theorem

- *Provided the stress vector* $\mathbf{t}^{(\mathbf{n})}$*acting on a surface with outwardly drawn unit normal* $\mathbf{n}$ *is a continuous function of the coordinate variables, there exists a second-order tensor field* $\boldsymbol{\sigma}(\mathbf{x})$, *independent of* $\mathbf{n}$, *such that* $\mathbf{t}^{(\mathbf{n})}$ *is a linear function of* $\mathbf{n}$ *such that:*

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$$

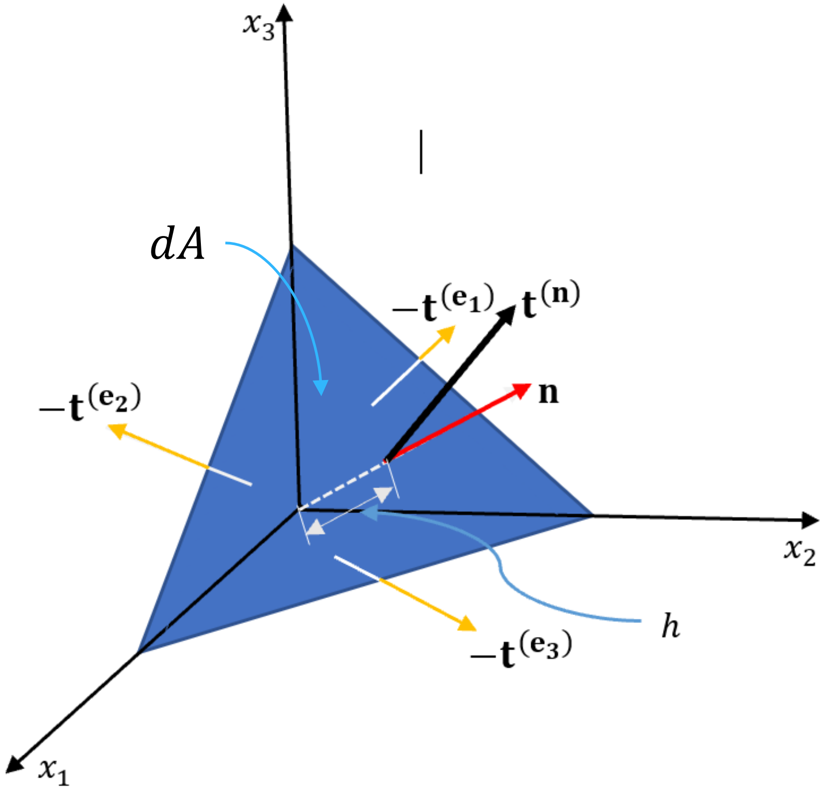
- The tensor $\boldsymbol{\sigma}$ in the above relationship is the tensor of proportionality and it is called *Cauchy Stress Tensor*. It is also the "true stress" tensor for reasons that will become clear later.

Cauchy Stress Theorem

* To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal base area dA oriented in an arbitrary direction specified by a normal vector $\mathbf{n}$ (Figure 6.3). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane $\mathbf{n}$. The stress vector on this plane is denoted by $\mathbf{t}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{t}^{(-\mathbf{e}_1)}$, $\mathbf{t}^{(-\mathbf{e}_2)}$ and $\mathbf{t}^{(-\mathbf{e}_3)}$ From equilibrium of forces, Newton's second law of motion, we have

$$\varrho\left(\frac{h}{3}dA\right)\mathbf{a} = \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_1)}dA_1 - \mathbf{t}^{(\mathbf{e}_2)}dA_2 - \mathbf{t}^{(\mathbf{e}_3)}dA_3$$

$$= \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_i)}dA_i$$



where the left-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ϱ is the density, $\mathbf{a}$ is the acceleration, and h is the height of the tetrahedron, considering the plane $\mathbf{n}$ as the base.

$$\varrho\left(\frac{h}{3}dA\right)\boldsymbol{a} = \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_i)}dA_i$$

Cauchy Theorem

The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face:

$$dA_i = (\mathbf{n} \cdot \mathbf{e}_i) dA = n_i dA$$

and then substituting into the equation to cancel out dA:

$$\mathbf{t}^{(\mathbf{n})} dA - \mathbf{t}^{(\mathbf{e}_i)} n_i dA = \varrho \left(\frac{h}{3} dA \right) \boldsymbol{a}$$

To consider the limiting case as the tetrahedron shrinks to a point, $h \to 0$, that is the height of the tetrahedron approaches zero. As a result, the right-hand-side of the equation approaches 0, so the equation becomes,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{e}_i)} n_i$$

Interpretation

We are now to interpret the components $\mathbf{t}^{(\mathbf{e}_i)}$ in this equation. Consider $\mathbf{t}^{(\mathbf{e}_1)}$ the value of the resultant stress traction on the first coordinate plane. Recall that the identity tensor, $\mathbf{I} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$ we have,

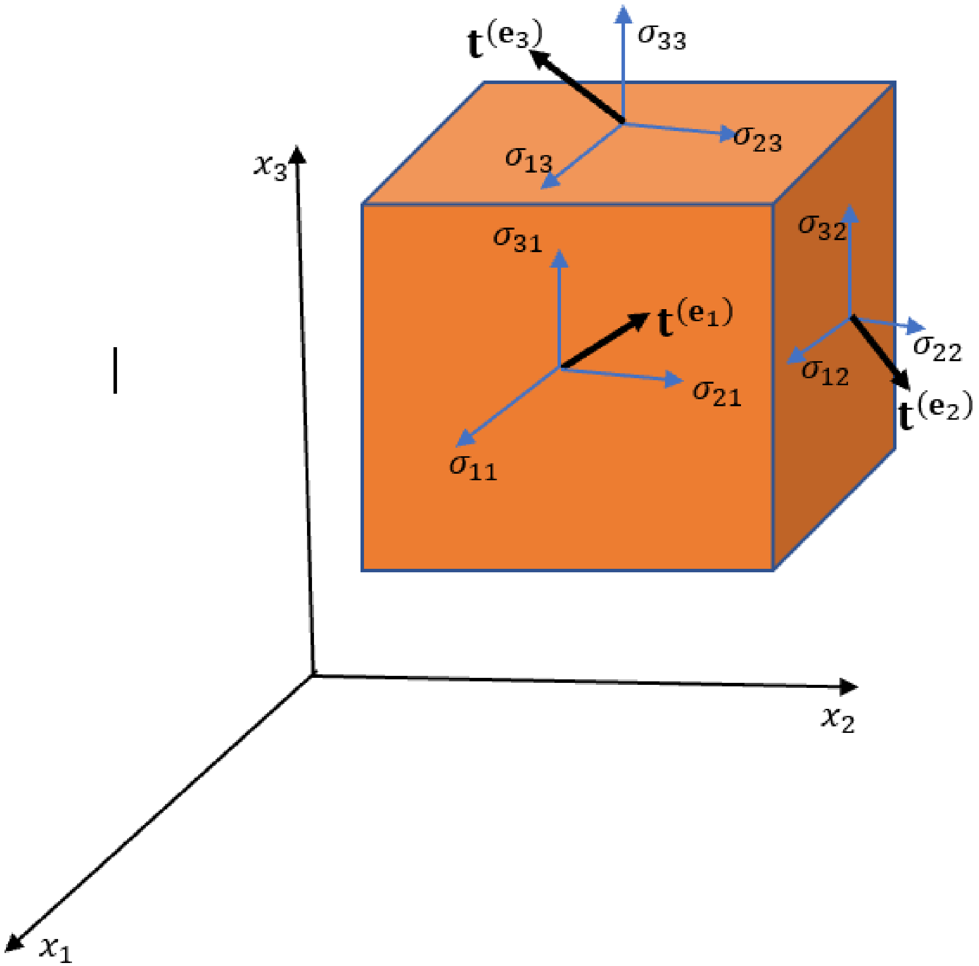
$$\mathbf{t}^{(\mathbf{e}_1)} = (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3)\mathbf{t}^{(\mathbf{e}_1)}$$
$$= \left(\mathbf{e}_j \otimes \mathbf{e}_j\right)\mathbf{t}^{(\mathbf{e}_1)} = \sigma_{j1}\mathbf{e}_j$$

Where the scalar quantity σ_{j1} is defined by the above equation as,

$$\sigma_{j1} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_1)}$$

the resolved component of the resultant traction on the surface perpendicular to $\mathbf{e}_1$ in the $\mathbf{e}_j$ direction, or in general, we write the components as,

$$\sigma_{ij} = \mathbf{e}_i \cdot \mathbf{t}^{(\mathbf{e}_j)} = \mathbf{t}^{(\mathbf{e}_j)} \cdot \mathbf{e}_i, i = 1,2,3$$



$\mathbf{t}^{(\mathbf{e}_3)}$
σ_{33}
σ_{13}
σ_{23}
x_3
σ_{31}
σ_{32}
$\mathbf{t}^{(\mathbf{e}_1)}$
σ_{22}
σ_{12}
σ_{21}
$\mathbf{t}^{(\mathbf{e}_2)}$
σ_{11}
x_2
x_1

Interpretation

* Above figure is a graphical depiction of this definition where we can see that $\sigma_{ij} = \mathbf{e}_i \cdot \mathbf{t}^{(\mathbf{e}_j)}$is the scalar component of the stress vector on the j coordinate plane in the i direction. For any coordinate plane therefore,

$$\mathbf{I}\,\mathbf{t}^{(\mathbf{e}_i)} = \left(\mathbf{e}_j \otimes \mathbf{e}_j\right)\mathbf{t}^{(\mathbf{e}_i)} = \mathbf{e}_j\left(\mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}\right)$$

* we may therefore write, $\mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ji}\mathbf{e}_j$, so that the stress or traction vector on an arbitrary plane determined by its orientation in the outward normal $\mathbf{n}$. But, using the wedge, we showed that,

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \mathbf{t}^{(\mathbf{e}_i)} n_i \\ &= \sigma_{ji}\mathbf{e}_j n_i\end{aligned}$$

* After substituting for the traction on each coordinate plane. Which is another way of saying that the component of the vector $\mathbf{t}^{(\mathbf{n})}$along the j coordinate direction is $\sigma_{ji}n_i$ which is the contraction, $\boldsymbol{\sigma}(\mathbf{x})\mathbf{n} = \mathbf{t}^{(\mathbf{n})}$. This proves Cauchy Stress Theorem.

Normal Stress Again

- Obviously, σ_{ij} are the components of the stress tensor in the coordinate system of computation that we have used so far. The Cauchy law, being a vector equation remains valid in all coordinate systems. We will then have to compute the different values of the stress tensor in the system of choice when for any reason we choose to work in not-Cartesian coordinates.
- Earlier on, we introduced the normal $t_n \equiv \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n}$ and shearing $\mathbf{t}^{(\mathbf{n})} - t_n\mathbf{n}$ components of the stress vector. We can compute these values now in terms of the scalar components of the stress tensor. Using Cauchy theorem, we have that,

$$\sigma = t_n \equiv \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = (\boldsymbol{\sigma}(\mathbf{x})\mathbf{n}) \cdot \mathbf{n} = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n}$$

Shear Stress

* This double contraction defines the scalar value which is the magnitude of the stress vector acting in the direction of the normal to the plane. The other important scalar quantity – the magnitude of the corresponding projection of the traction vector to the surface itself is obtained from Pythagoras theorem:
* $\tau = \left\| \mathbf{t}^{(\mathbf{n})} - t_n \mathbf{n} \right\| = \left\| \mathbf{t}^{(\mathbf{n})} - \sigma \mathbf{n} \right\|$

Nominal Stress

From last chapter, we recall that the vector current area in a deformed body $d\mathbf{a} = Jd\mathbf{A} \cdot \mathbf{F}^{-1}$ where $d\mathbf{A}$ is its image in the material coordinates. The resultant force acting on an area bounded by ΔS in the deformed coordinates can be obtained, using Cauchy stress theorem as,

$$dP = \int_{\Delta S} d\mathbf{a} \cdot \boldsymbol{\sigma} = \int_{\Delta S_0} J(d\mathbf{A} \cdot \mathbf{F}^{-1}) \cdot \boldsymbol{\sigma}$$

$$= \int_{\Delta S_0} d\mathbf{A} \cdot J\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}$$

$$= \int_{\Delta S_0} d\mathbf{A} \cdot \mathbf{N}$$

* where $\mathbf{N} \equiv J\mathbf{F}^{-1}\boldsymbol{\sigma}$ is called the *Nominal Stress* Tensor

States of Stress

We present here examples of states of stress as an illustration:

1. Hydrostatic Pressure
2. Uniaxial Tension
3. Equal Biaxial tension
4. Pure Shear

Hydrostatic Pressure

$$\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{1} = -pg^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j) = -p(\mathbf{g}_i \otimes \mathbf{g}^i)$$

In a Cartesian system, we have, $\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{1} = -p(\mathbf{e}_i \otimes \mathbf{e}_i)$.

For a surface whose outward normal is the unit vector $\mathbf{n}$, the traction

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} = -pg^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j)\mathbf{n} \\ &= -pg^{ij}\mathbf{g}_i(\mathbf{n} \cdot \mathbf{g}_j) = -pg^{ij}n_j\mathbf{g}_i \\ &= -pn^i\mathbf{g}_i = -p\mathbf{n}\end{aligned}$$

Furthermore, the scalar normal traction

$$\sigma = \mathbf{t}^{(\mathbf{n})} \cdot \mathbf{n} = -p\mathbf{n} \cdot \mathbf{n} = -p$$

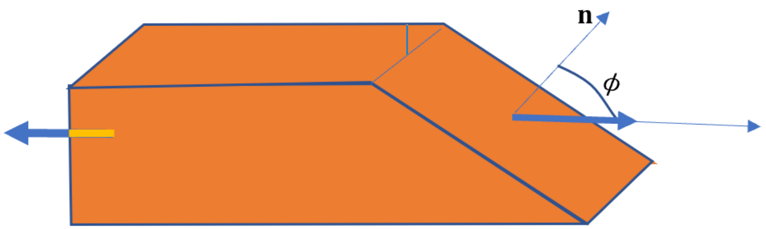
And the shear stress: the magnitude of the vector difference between the traction and the vector normal traction.

$$\tau = \left\| \mathbf{t}^{(\mathbf{n})} - \sigma\mathbf{n} \right\| = 0$$

Uniaxial Tension

Define uniaxial tension as a state where there is a normal traction t in a given direction (unit vector $\boldsymbol{\alpha}$)and zero traction in directions perpendicular to it. The Cauchy stress for this is $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})$. The traction on a surface with unit vector $\mathbf{n}$ at an angle ϕ to the stress direction, is,

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})\boldsymbol{n}$$
$$= t\boldsymbol{\alpha}(\boldsymbol{n} \cdot \boldsymbol{\alpha}) = t\boldsymbol{\alpha} \cos \phi$$



From which we can see that, under uniaxial stress, the traction is always directed along the vector $\boldsymbol{\alpha}$ no matter what the orientation of the surface might be.

Of course, when $\phi = \pi/2$, traction is zero.

Biaxial Traction

Consider the stress tensor, $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + \boldsymbol{\beta} \otimes \boldsymbol{\beta})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are perpendicular directions. The traction on an arbitrary plane $\mathbf{n}$

$$\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + \boldsymbol{\beta} \otimes \boldsymbol{\beta})\mathbf{n}$$
$$= t\boldsymbol{\alpha}(\mathbf{n} \cdot \boldsymbol{\alpha}) + t\boldsymbol{\beta}(\mathbf{n} \cdot \boldsymbol{\beta}) = t\boldsymbol{\alpha} \cos \phi + t\boldsymbol{\beta} \sin\phi$$

The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{t, t, 0\}$. $\boldsymbol{\sigma}(\mathbf{x})$ this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = t\mathbf{u}_1 \otimes \mathbf{u}_1 + t\mathbf{u}_2 \otimes \mathbf{u}_2$$

Pure Shear

Consider the stress tensor, $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are perpendicular directions. The traction on an arbitrary plane $\mathbf{n}$

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha})\mathbf{n} \\ &= t\boldsymbol{\alpha}(\mathbf{n} \cdot \boldsymbol{\beta}) + t\boldsymbol{\beta}(\mathbf{n} \cdot \boldsymbol{\alpha}) = t\boldsymbol{\alpha} \sin \phi + t\boldsymbol{\beta} \cos \phi\end{aligned}$$

The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{t, -t, 0\}$. Furthermore, $\boldsymbol{\sigma}$ in this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = t\mathbf{u}_1 \otimes \mathbf{u}_1 - t\mathbf{u}_2 \otimes \mathbf{u}_2$$

Examples

1. Consider two tractions $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{t}^{(\bar{\mathbf{n}})}$ on planes with normal vectors $\mathbf{n}$ and $\bar{\mathbf{n}}$ respectively. Show that
$$\bar{\mathbf{n}} \cdot \mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{t}^{(\bar{\mathbf{n}})}$$
if and only if the associated Cauchy stress tensor is symmetric.

Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor. It follows that,
$$\begin{aligned}\mathbf{n} \cdot \mathbf{t}^{(\bar{\mathbf{n}})} &= \mathbf{n} \cdot \boldsymbol{\sigma}\bar{\mathbf{n}} \\ &= \bar{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}\end{aligned}$$
But $\bar{\mathbf{n}} \cdot \mathbf{t}^{(\mathbf{n})} = \bar{\mathbf{n}} \cdot \boldsymbol{\sigma}\mathbf{n}$. This can be equal to $\bar{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}$ only if
$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$$
This is the definition of symmetry of the tensor.

Examples

2. Let the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 1 & 4 & -3 \\ 4 & 1 & 0 \\ -3 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

(a) Find the component of the traction on the plane $2x_1 + 3x_2 + x_3 = 5$.

(b) Find the stress tensor referred to the orthonormal bases $\boldsymbol{\xi}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2), \boldsymbol{\xi}_3 = \frac{1}{3}(2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)$

The unit normal to the given plane is the vector $\mathbf{n} = \frac{1}{\sqrt{14}}(2\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3)$. The traction vector is $\boldsymbol{\sigma}\,\mathbf{n}$.

Examples

(a) In the Mathematica code shown, the Cauchy stress tensor components are entered using the variable Sig. The normal to the surface is entered with the variable NorVec. The tensor operation on this normal as in Cauchy theorem is to take the product **σ n** as shown in symbolic form. Hence the result.

```
Sig = {{1, 4, -3}, {4, 1, 0}, {-3, 0, 0}}
```

{{1, 4, -3}, {4, 1, 0}, {-3, 0, 0}}

```
NorVec = {2 / Sqrt[14], 3 / Sqrt[14], 1 / Sqrt[14]}
```

$$\left\{\sqrt{\frac{2}{7}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right\}$$

```
Trac = Sig . NorVec // MatrixForm
```

itrixForm=

$$\begin{pmatrix} -\frac{3}{\sqrt{14}}+\sqrt{14} \\ 4\sqrt{\frac{2}{7}}+\frac{3}{\sqrt{14}} \\ -3\sqrt{\frac{2}{7}} \end{pmatrix}$$

Examples

(b) We need to refer the same tensor to another orthonormal system. First, we have only two of the three vectors $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$. We need to find $\boldsymbol{\xi}_1$.

$$\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 \times \boldsymbol{\xi}_3$$

With a small computation, we find that $\boldsymbol{\xi}_1 = \frac{1}{3\sqrt{2}}(-\mathbf{e}_1 - 4\mathbf{e}_2 - \mathbf{e}_3)$. Note that Mathematica computes this directly in Cross($\boldsymbol{\xi}_2, \boldsymbol{\xi}_3$).

It is necessary to rotate the Cauchy stress tensor and the appropriate rotation tensor is:

$$\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$$

* The Mathematica code below computes the rotations $\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathrm{T}}$ of the Cauchy tensor $\boldsymbol{\sigma}$ from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$:

Recall from last term that to rotate a vector, you only need to operate the relevant rotation tensor on it. However, in this case, you need to operate the rotation and its inverse – which is its transpose in order to obtain a rotation of the stress tensor to a new coordinate system as we have shown. The decimal floating points are computed to see the simple numbers.

```
e1 = {1, 0, 0}; e2 = {0, 1, 0}; e3 = {0, 0, 1};
ξ2 = {1 / Sqrt[2], 0, -1 / Sqrt[2]};
ξ3 = {2 / 3, -1 / 3, 2 / 3};
ξ1 = Cross[ξ2, ξ3]
```

$$\left\{-\frac{1}{3\sqrt{2}}, -\frac{2\sqrt{2}}{3}, -\frac{1}{3\sqrt{2}}\right\}$$

```
Q = TensorProduct[ξ1, e1] + TensorProduct[ξ2, e2] + TensorProduct[ξ3, e3]
```

$$\left\{\left\{-\frac{1}{3\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{2}{3}\right\}, \left\{-\frac{2\sqrt{2}}{3}, 0, -\frac{1}{3}\right\}, \left\{-\frac{1}{3\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{2}{3}\right\}\right\}$$

```
NewSig = Q.Sig.Transpose[Q] // MatrixForm
```

ixForm=

$$\begin{pmatrix} \frac{\sqrt{2}}{3} + \frac{\frac{1}{\sqrt{2}} - \frac{2\sqrt{2}}{3}}{\sqrt{2}} - \frac{-2 - \frac{1}{3\sqrt{2}} + 2\sqrt{2}}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} - \frac{2}{3}\sqrt{2}\left(-2 - \frac{1}{3\sqrt{2}} + 2\sqrt{2}\right) & \frac{\sqrt{2}}{3} - \frac{\frac{1}{\sqrt{2}} - \frac{2\sqrt{2}}{3}}{\sqrt{2}} - \frac{-2 - \frac{1}{3\sqrt{2}} + 2\sqrt{2}}{3\sqrt{2}} \\ -\frac{8}{3} + \frac{4\sqrt{2}}{3} - \frac{1 - \frac{2\sqrt{2}}{3}}{3\sqrt{2}} & -\frac{2\sqrt{2}}{3} - \frac{2}{3}\sqrt{2}\left(1 - \frac{2\sqrt{2}}{3}\right) & \frac{8}{3} + \frac{4\sqrt{2}}{3} - \frac{1 - \frac{2\sqrt{2}}{3}}{3\sqrt{2}} \\ \frac{\sqrt{2}}{3} - \frac{-2 - \frac{1}{3\sqrt{2}} - 2\sqrt{2}}{3\sqrt{2}} + \frac{\frac{1}{\sqrt{2}} - \frac{2\sqrt{2}}{3}}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} - \frac{2}{3}\sqrt{2}\left(-2 - \frac{1}{3\sqrt{2}} - 2\sqrt{2}\right) & \frac{\sqrt{2}}{3} - \frac{-2 - \frac{1}{3\sqrt{2}} - 2\sqrt{2}}{3\sqrt{2}} - \frac{\frac{1}{\sqrt{2}} - \frac{2\sqrt{2}}{3}}{\sqrt{2}} \end{pmatrix}$$

```
N[%] // MatrixForm
```

ixForm=

$$\begin{pmatrix} 0.165031 & -0.794529 & 0.498365 \\ -0.794529 & -0.996729 & 4.5388 \\ 0.498365 & 4.5388 & 2.8317 \end{pmatrix}$$

Example

3. Let the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} 5x_2x_3 & 3x_2^2 & 0 \\ 3x_2^2 & 0 & -x_1 \\ 0 & -x_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Find the component of the traction at the point $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 \end{pmatrix}$ on the surface $x_1^2 + x_2^2 + x_3 = 0$.

We first need to find the normal to the surface. The surface here is not plane. However, there is a tangent to the plane at the point of interest. We take the gradient of its equation to find the unit vector along this normal.

Example

```
ϕ[x1_, x2_, x3_] := x1^2 + x2^2 + x3

NormalVec = Grad[ϕ[x1, x2, x3], {x1, x2, x3}]
```

$\{2 x_1, 2 x_2, 1\}$ ← Notice here the normal changes from point to point

```
UnitNormalVec = Normalize[NormalVec] /. {x1 → 1/2, x2 → Sqrt[3]/2, x3 → -1}
```

$$\left\{\frac{1}{\sqrt{5}}, \sqrt{\frac{3}{5}}, \frac{1}{\sqrt{5}}\right\}$$

```
VarSigma[x1_, x2_, x3_] := {{5 x2 x3, 3 x2^2, 0}, {3 x2^2, 0, -x1}, {0, -x1, 0}}

TractionVec = VarSigma[1./2, Sqrt[3]/2, -1].UnitNormalVec // MatrixForm
```

atrixForm=

$$\begin{pmatrix} -0.193649 \\ 0.782624 \\ -0.387298 \end{pmatrix}$$

Example

* At a certain point of a body, Cauchy stress tensor is:

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 2 & 5 & 3 \\ 5 & 1 & 4 \\ 3 & 4 & 3 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

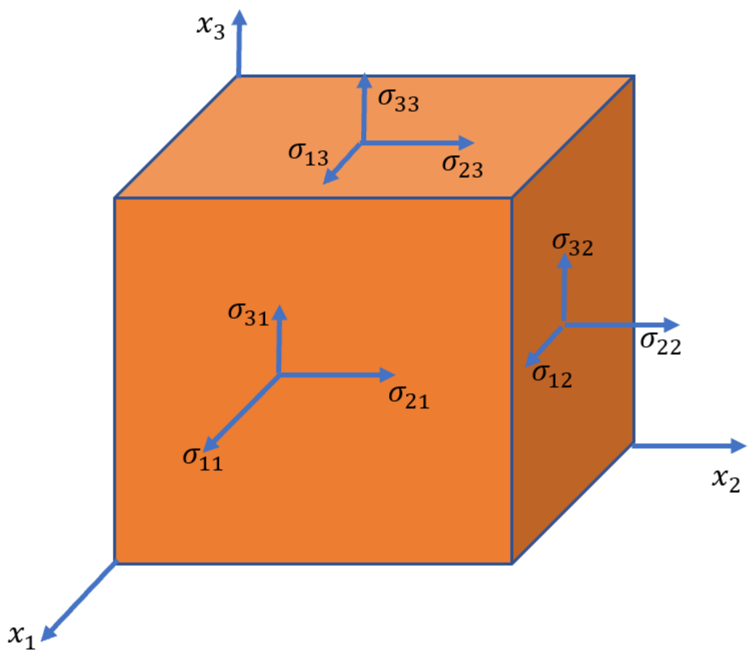
(a) Find the components of the traction vector at a point on the plane whose normal has direction ratios 3:1:-2

(b) Find the normal and shear components of this traction.

This is a very easy question. Obtain the unit normal $\mathbf{n}$ and operate the Cauchy stress $\boldsymbol{\sigma}$ on it. To find the normal and the shear stresses, invoke the dyad $(\mathbf{n} \otimes \mathbf{n})$ on slide 18 and the projection tensor $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$.

Principal Stresses and Directions

- We know from Cauchy's stress law that the components of the stress tensor can be depicted as shown in the figure below.
- We also know that by suitable rotations, we are able to transform the components of the stress tensor into other orthonormal systems of coordinates using rotation tensors.



The Eigenvalue Problem

* The eigenvalue problem for the stress tensor is simply this: Can we find suitable rotations such that the only stress components we have to deal with are the normal stresses? If so, what will those normal stresses be? What will those directions be?
* Given any tensor $\boldsymbol{\sigma}$ and a vector $\mathbf{n}$, the product $\boldsymbol{\sigma}\mathbf{n}$ is obviously a vector. When can we have the new vector to be such that,

$$\boldsymbol{\sigma}\mathbf{n} = \alpha\mathbf{n}$$

* Or, expressed in component form, $\sigma_{ij}n_j = \alpha n_i$.
* This will happen only when we can solve the equations,

$$|\sigma_{ij} - \alpha\delta_{ij}| = 0$$

Eigenvalue Problem

* Opening the determinant gives us the equation,

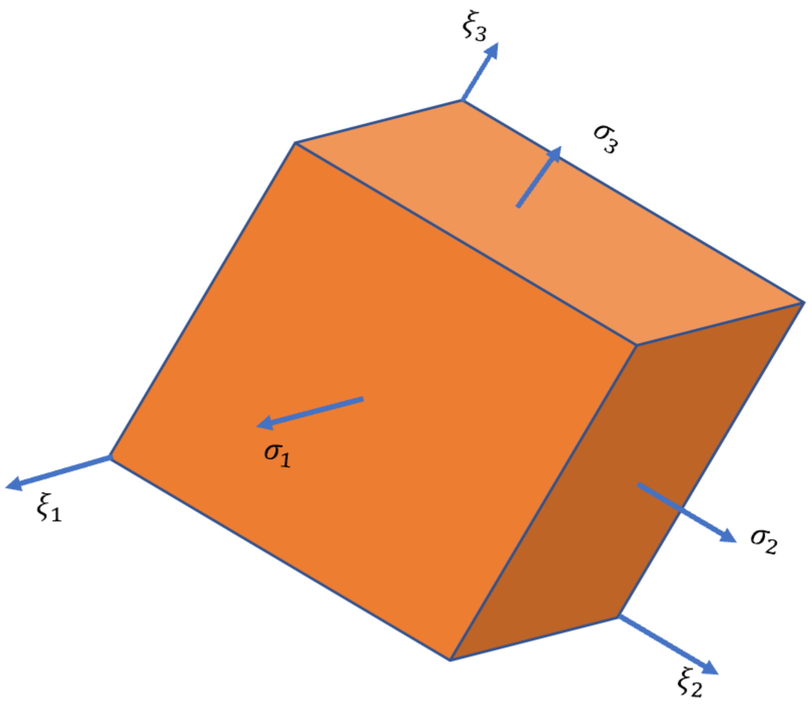
$$-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3 = 0$$

* This is the characteristic equation of the stress tensor. A most important equation in mechanical design as we shall see.
* The three solutions to this equation are the principal stresses, σ_1, σ_2 and σ_3. The principal directions can be obtained by substituting each back in the equation, $\sigma_{ij}n_j = \alpha n_i$ and solving for the three vector directions in each case.
* When we do that, we obtain the transformation shown in the next slide:

Principal stresses and planes

* We obtain the stresses and the directions as shown. As usual, we can refer to this new set of coordinates after the transforming
rotation tensor

* Given any stress tensor, you can use Mathematica to obtain the eigenvalues and principal directions as the following examples show.



Example

2. Given that the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Find principal stresses, principal directions and find the traction vector on a plane whose unit normal is $\frac{1}{\sqrt{2}}(0,1,1)$. Also rotate the stress tensor back to its principal directions

```
CauchyStr = {{3, 1, 1}, {1, 0, 2}, {1, 2, 0}};
NorVec = 1 / Sqrt[2] {0, 1, 1};
TracVec = CauchyStr.NorVec
```

$$\left\{\sqrt{2}, \sqrt{2}, \sqrt{2}\right\}$$

```
Eigenvalues[CauchyStr]
```

{4, -2, 1}

```
vecs = Eigenvectors[CauchyStr]
```

{{2, 1, 1}, {0, -1, 1}, {-1, 1, 1}}

```
e1 = {1., 0, 0}; e2 = {0, 1., 0}; e3 = {0., 0, 1.};
Q = TensorProduct[e1, Normalize[vecs[[1]]]] + TensorProduct[e2, Normalize[vecs[[2]]]] +
  TensorProduct[e3, Normalize[vecs[[3]]]]
```

{{0.816497, 0.408248, 0.408248}, {0., -0.707107, 0.707107}, {-0.57735, 0.57735, 0.57735}}

```
NewCauchy = Q. CauchyStr .Transpose[Q] // MatrixForm
```

trixForm=

$$\begin{pmatrix} 4. & 0. & 0. \\ 0. & -2. & 0. \\ -1.66533\times 10^{-16} & 0. & 1. \end{pmatrix}$$