Department of Systems Engineering

University of Lagos

Mid-Term Test; August 2, 2018

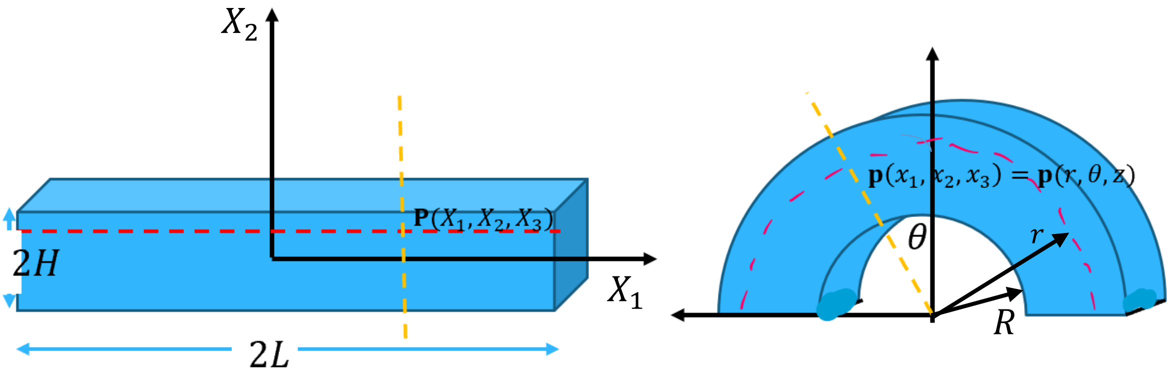
SSG 516: Continuum Mechanics: Coordinate Systems & Kinematics

Attempt all questions. Time Allowed: 9:15-10:45

---

1. What are the features of a curvilinear coordinate system? Coordinate surfaces are not plane, coordinate lines are not straight, different dual bases, bases vectors are spatially variable and have non-zero derivatives, (2) Explain, using concrete examples for an orthogonal curvilinear system,
   a. Orthonormal, natural and dual bases For orthonormal, each base vector is of unit magnitude and the unit vectors are mutually orthogonal. Natural bases are obtained from differentiating the position vector. If we pack the natural bases into the columns of a matrix, the dual bases components are the rows of the inverse of that same matrix. In Cylindrical, orthonormal bases are $\boldsymbol{e}_r \equiv \cos\phi\, \boldsymbol{e}_1 + \sin\phi\, \boldsymbol{e}_2$, $\mathbf{e}_\phi \equiv -\sin\phi\, \boldsymbol{e}_1 + \cos\phi\, \boldsymbol{e}_2$, and $\mathbf{e}_z = \mathbf{e}_3$ The natural bases are $\mathbf{e}_r, r\, \mathbf{e}_\phi$, and $\mathbf{e}_z$ while the dual bases are $\mathbf{e}_r, \frac{1}{r}\mathbf{e}_\phi$, and $\mathbf{e}_z$ (3)
   b. Coordinate surfaces, coordinate variables For cylindrical, the coordinate variables are $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$, and $z = z$. The coordinate surfaces are the surfaces with equations $r = const$, (Cylinders) $\phi = const$,Planes through the z-axis $z = const$.planes perpendicular to the z-axis (2)
   c. Scalar, vector and tensor fields Any time the value of a function is specified at all points in a subset of the Euclidean Point Space, it is called a field. If the function is a scalar, we have a scalar field, eg, temperature. If it is a vector, we have a vector field, e.g. velocity of a moving fluid in its spatial configuration, or a tensor field, e.g. strain or deformation gradient. (3)

2. The Figure below gives the transformation from a straight bar to a semicircular bar.
   a. Explain why, in general, the functions $r = r(X_2), \theta = \theta(X_1)$ and $z = z(X_3)$ describe the transformation. The radial distance of any point is dependent only on its X_2 component, independent of the other values. While the angular orientation is only dependent on its X_1 distance. Points in the z direction are dependent only on their original locations in the X_3 direction. (2)
   b. Find the deformation gradient when $r = R + \alpha X_2$, $\theta = \frac{\pi X_1}{2L}$, and $z = \beta X_3$. Find also the Right Cauchy Green Tensor and the Lagrangian strain. (2)



$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \alpha \mathbf{e}_\theta \otimes \mathbf{E}_1 + \frac{\pi r}{2L}\mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3$$

Clearly,

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

is the Right Stretch Tensor, (2) The Lagrange Strain function is:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}\left(\left(\frac{\pi r}{2L}\right)^2 - 1\right)\mathbf{E}_1 \otimes \mathbf{E}_1 + \frac{(\alpha^2 - 1)}{2}\mathbf{E}_2 \otimes \mathbf{E}_2 + \frac{(\beta^2 - 1)}{2}\mathbf{E}_3 \otimes \mathbf{E}_3$$ **(2)**

c. Explain what coordinate system options you have in analyzing the material state and the spatial state of this example. Could transform the equations to Cartesian Coordinates or analyse using cylindrical Polar (2) How do you arrive at the optimal to use? Polar is better for the spatial state because the equations are easier to deal with. Using Cartesian brings complicated expressions, which, though easy to get are quite tedious to analyze. (2)

3. Explain what the Polar decomposition of the deformation gradient achieves and tell why it is important. It separates the rigid body motion part from the deformation gradient. It allows us to have a function that can represent actual deformations. (2) State, without proof, the Polar decomposition theorem of the deformation gradient. For a given deformation gradient $\mathbf{F}$, there is a unique rotation tensor $\mathbf{R}$, and unique, positive definite symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ for which, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ (2)

(b) The Seth-Hill tensor strain functions take the form,

$$\mathbf{E} = \frac{1}{n}(\mathbf{U}^n - \mathbf{I}), \qquad n \neq 0$$

explain the terms in this tensor equation n is an integer while $\mathbf{U}$ is the right stretch tensor. (3) and give reasons why other tensors do not feature in the general definition of strain. They do not feature because we can easily prove that stretches and shape changes are governed by only the stretch tensors. Only they need to be featured in a proper strain definition.(3)

4. In a deformation given by the vector transformation function, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, if the deformation gradient of this deformation is $\mathbf{F}(\mathbf{X})$, the cofactor of the deformation gradient is $\mathbf{F}^c(\mathbf{X})$, and the scalar J is the determinant of the deformation gradient. Explain which of these quantities govern

   a. The change in areas between the referential and spatial configurations, Local area changes are governed by the cofactor of the deformation gradient $d\mathbf{a} = J\mathbf{F}^{-\mathrm{T}}d\mathbf{A} = \mathbf{F}^c d\mathbf{A}$ **(4)**

   b. Volume changes, Governed by the determinant of the deformation gradient $dv = JdV$ (3) and

   c. Length changes Governed by the deformation gradient $d\mathbf{x} = \mathbf{F}d\mathbf{X}$(3)

   Write relevant equations to justify your answers.

5. Which of the following tensors govern the change in angles between line elements from referential to spatial configurations: Left Stretch tensor, Right Stretch tensor or the Rotation Tensor? Right Stretch Tensor(4) Obtain an expression for the change transformed angle between two line elements to justify your answer.

In the referential configuration, the angle between the line elements, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ is,

$$\Theta = \cos^{-1}\left(\frac{d\mathbf{X}_1 \cdot d\mathbf{X}_2}{\|d\mathbf{X}_1\|\|d\mathbf{X}_2\|}\right)$$

To find the angle between any two elements in the spatial configuration we simply recall that the angle we seek is

$$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\|d\mathbf{x}_2\|}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\|\mathbf{U}d\mathbf{X}_1\|\|\mathbf{U}d\mathbf{X}_2\|}\right) \quad (6)$$