Balance Laws

Conservation of Mass, Momentum & Energy

Conservation Laws

Balance of Mass

- The principle of mass conservation

Balance of Momentum & Angular Momentum

- A reformulation of Newton's second law of motion –
- Emphasis on continuously distributed matter. Symmetry

* **Balance of Energy and the Work Principle.**
  - Conjugate Stress Analysis
  - Consistency in the scalar quantities Work and Energy.
* **Inbalance of Entropy** –
  - Statement of the second law of thermodynamics.
  - The principle of energy availability;
  - Implications on Processes & Efficiency

Balance of Mass

- The mass of a continuously distributed body is defined in basic physics as the total amount of substance or material contained in the body.
  - The basic idea behind the conservation law is that the mass of an identified quantity is not subject to change during motion.
  - Here, we are obviously restricting ourselves to non-relativistic mechanics.
  - Otherwise, relativistic mass is $m\left(1-\left(\frac{v}{c}\right)^2\right)$

And the reduction becomes significant as velocity approaches that of light. Our interest lies where this second term is insignificant.

Definition: System

- **System:** A particular collection of matter in space that is of interest. The complement of this is the rest of matter – essentially, the rest of the universe.
- The **boundary** of the system is the surface that separates them. The kind of system depends on the nature of this surface – especially what are allowed to pass through.

Open & Closed System

* A system is **open** if matter or mass can pass through the boundary. Otherwise the system is **closed.** In a closed system therefore, we are dealing with the same quantity of matter throughout the motion as no new mass comes in and old matter are trapped within the boundary as defined.
* In addition to this, a system may also be closed to energy transfer. Such a system is said to be **isolated**. A thermally isolated system, closed to the transfer of heat energy across the boundary is said to be **insulated**. A system may also be only **mechanically isolated**.

Intensive & Extensive Properties

- An **intensive** or **bulk** property is a scale-invariant physical property of a system. Magnitude is independent of size.
  - Properties of subsystems do not add in consolidation.
  - **Examples:** Hardness, specific heat capacity, density, temperature, refractive index, etc.
  - It has been pointed out that this is not necessarily the "dictionary" meaning. It actually comes from **in**ex**tensive**

Intensive & Extensive Properties

- An **extensive** property of a system is directly dependent on the system size or the amount of material in the system.
  - Properties are additive from consolidated subsystems.
  - **Examples:** Momentum, energy, mass, etc.
- The ratio of two extensive properties of a system is an intensive property.
  - Density, an intensive property, is the ratio of Mass and Volume (two extensive properties) $\varrho = \frac{m}{V}$
  - Heat Capacity, an extensive property, divided by mass gives the specific heat capacity, an intensive property.
  - Body force (an extensive property) divided by volume or mass produces intensive properties of body force per unit mass/volume

Excursion: Elementary Calculus

* Let us first evaluate the simple expression in two ways:

$$\frac{d}{dt}\int_2^1 \left(2xt + t^2 \sin x\right) dx = \int_2^1 \frac{\partial}{\partial t}\left(2xt + t^2 \sin x\right) dx$$

* Did you expect the same answers? What did you get?
* In either case, we need to carry out an integration and a differentiation; in the one, we integrated first, in the other, we differentiated first.

Excursion: Elementary Calculus

- Notice the same integral now with a more generalized boundary varying with t.

$$\frac{d}{dt}\int_{1+t}^{t^2}\left(2xt + t^2 \sin x\right) dx = \int_{1+t}^{t^2} \frac{\partial}{\partial t}\left(2xt + t^2 \sin x\right) dx$$

- What can be wrong with it?
- In either case, we need to carry out an integration and a differentiation; in the one, we integrated first, in the other, we differentiated first.
- Are the results equal? If not, why not.
- Is it wrong to change the order of the two processes? When is it right to do so?

Important Results

* For us to have a meaningful discussion of the Balance Laws of Nature, there are three results, important enough to be called Theorems, that you will need to be well acquainted with:
  1. **Liouville**'s Theorem
  2. **Leibnitz-Reynolds** Transport Theorem
  3. The divergence Theorem of **Gauss**
* Number 1 is simple to understand but a bit tedious to prove. Number 2 is a familiar theorem that we will establish in a more general form.

Liouville's Theorem

* The differentiation of determinants has an important result credited to Joseph Liouville.
* For the determinant, $J \equiv |\mathbf{F}| = [\mathbf{Fe}_1, \mathbf{Fe}_2, \mathbf{Fe}_3]$, of the deformation gradient, $\mathbf{F}(\mathbf{X}, t)$, the simple form of Liouville's theorem that we shall apply here, states that,

$$\dot{J}(\mathbf{x}, t) \equiv \frac{D}{Dt} J(\mathbf{x}, t) = J \operatorname{div} \mathbf{v}$$

where $\mathbf{v}(\mathbf{x}, t)$ is the velocity.

* A general proof is supplied in Problem 139 of the Tensor problems on this webpage.

$$\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx$$

```
F[a_, b_] := 2 a b + b^2 Sin[a];
ϕ1[α_] := 1 + α; ϕ0[α_] := α^2;
J[α_] = Integrate[F[x, α], {x, ϕ0[α], ϕ1[α]}];
```

$$\frac{d}{dt}\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx$$

```
Φ1[t] = D[J[t], t]
```

$1+2t+t^2-t^4+t\,\mathrm{Cos}[t^2]-t\,\mathrm{Cos}[1+t]+t\,(2+2t-4t^3+\mathrm{Cos}[t^2]-\mathrm{Cos}[1+t]-2t^2\,\mathrm{Sin}[t^2]+t\,\mathrm{Sin}[1+t])$

```
Simplify[%]
```

$1+4t+3t^2-5t^4+2t\,\mathrm{Cos}[t^2]-2t\,\mathrm{Cos}[1+t]-2t^3\,\mathrm{Sin}[t^2]+t^2\,\mathrm{Sin}[1+t]$

$$\frac{\partial}{\partial t}f(x,t)$$

```
G[x_, β_] := D[F[x, β], β];
```

$$\int_{\phi_0}^{\phi_1}\frac{\partial}{\partial t}f(x,t)dx$$

```
Φ2[t] = Integrate[G[x, t], {x, ϕ0[t], ϕ1[t]}]
```

$1+2t+t^2-t^4+2t\,\mathrm{Cos}[t^2]-2t\,\mathrm{Cos}[1+t]$

$1+2t+t^2-t^4+2t\,\mathrm{Cos}[t^2]-2t\,\mathrm{Cos}[1+t]$

```
Rem[t_] := F[ϕ1[t], t] D[ϕ1[t], t] - F[ϕ0[t], t] D[ϕ0[t], t]
Simplify[Φ1[t] - Rem[t]]
```

$1+2t+t^2-t^4+2t\,\mathrm{Cos}[t^2]-2t\,\mathrm{Cos}[1+t]$

Leibniz Theorem

- Leibniz Integral Rule, well illustrated here states that,

$$\frac{d}{dt}\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx$$
$$= \int_{\phi_0(t)}^{\phi_1(t)} \frac{\partial}{\partial t} f(x,t)dx + f(\phi_1(t),t)\frac{d\phi_1(t)}{dt} - f(\phi_0(t),t)\frac{d\phi_0(t)}{dt}$$

- Answers our puzzle by the remainder terms seen above.
- Notice that when ϕ_0 and ϕ_1, the limits of the integration are constants, the remainder terms vanish and it is ok to change the order of integration and the original equation is correct.

Leibniz-Reynolds Transport Theorem

* A generalization of the above rule is known as the **Reynold's Transport Theorem** as follows:
* *The rate of change of an extensive property Φ per unit volume, for the system is equal to the time rate of change of Φ within the volume Ω and the net rate of flux of the property Φ through the surface $\partial\Omega$, or*

$$\frac{D}{Dt}\int_{\Omega}\Phi(\mathbf{x},t)dv = \int_{\Omega}\frac{\partial\Phi(\mathbf{x},t)}{\partial t}dv + \int_{\partial\Omega}\Phi(\mathbf{x},t)\mathbf{v}\cdot\mathbf{n}\,ds$$

Above Equation: 1-D

The first integral is easy to see:

$$\int_{x_0(t)}^{x_1(t)} \frac{\partial \Phi(x,t)}{\partial t} dx$$

$\mathbf{v} = \frac{dx(t)}{dt}\mathbf{i}$ the boundary Ω becomes the interval $[x_0(t), x_1(t)]$, at the beginning of the interval, $\mathbf{n}$, the outward drawn normal becomes $-\mathbf{i}$, at the end of the interval, it is $\mathbf{i}$. The line integral occurs only at two points which are now just the beginning points and end points, hence that sum of the evaluations at those two points:

$$\Phi(x_0(t), t)\left(\frac{dx_0(t)}{dt}\mathbf{i}\right) \cdot (-\mathbf{i}) + \Phi(x_1(t), t)\left(\frac{dx_1(t)}{dt}\mathbf{i}\right) \cdot (\mathbf{i})$$

and this recovers the original Leibniz rule.

Divergence of a Product

* On subsequent pages, we will need to evaluate the divergence of the product of a scalar and a vector. A simple elucidation of this matter is appropriate at this point:

$$
\begin{aligned}
\operatorname{div}(\mathbf{v}\phi) &= \operatorname{tr}[\operatorname{grad}(\mathbf{v}\phi)] \\
&= \operatorname{tr}\left[(v_i\phi)_{,j}\, \mathbf{e}_i \otimes \mathbf{e}_j\right] \\
&= \operatorname{tr}\left[\left(v_{i,j}\,\phi\right)\mathbf{e}_i \otimes \mathbf{e}_j + v_i\phi_{,j}\, \mathbf{e}_i \otimes \mathbf{e}_j\right] \\
&= \left(v_{i,j}\,\phi\right)\mathbf{e}_i \cdot \mathbf{e}_j + v_i\phi_{,j}\, \mathbf{e}_i \cdot \mathbf{e}_j \\
&= \left(v_{i,j}\,\phi\right)\delta_{ij} + v_i\phi_{,j}\, \delta_{ij} \\
&= v_{i,i}\,\phi + v_i\phi_{,i} \\
&= \phi \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \phi
\end{aligned}
$$

Symbols and Time Derivatives

Given a scalar, vector or tensor function $\Xi(\mathbf{x}, t)$ which is the same as $\Xi_R(\mathbf{X})$, the operation of differentiating with respect to time, is an important quantity.

- We are differentiating w.r.t time and keeping the other variable fixed. This is a fundamental cause of confusion in the textbooks you may read.
- If you keep $\mathbf{X}$ fixed, you have the material or substantial derivative and we shall represent this with the dot or $\frac{D}{Dt}$ (This derivative has at least ten names)
- If you keep $\mathbf{x}$ fixed, you have the local time derivative and we shall represent this with the prime or $\frac{\partial}{\partial t}$

Transport Theorem

* The fact that the volume is variable with time that is, $\Omega = \Omega(t)$ means that the derivative does not commute with the integral in spatial coordinates. A transformation to material coordinates simplifies the situation. Use the fact that in material coordinates, a derivative under the integral sign is the same as the derivative of the integral itself. If $I(t) = \int_{\Omega} \Xi(\mathbf{x}, t) dv$ then

Transport Theorem

$$
\begin{aligned}
\dot{I}(t) &= \frac{D}{Dt}\int_{\Omega} \Xi(\mathbf{x},t)\frac{dv}{dV}dV = \frac{D}{Dt}\int_{\Omega_0} \Xi(\mathbf{x},t)J\,dV \\
&= \int_{\Omega_0} \frac{D}{Dt}[\Xi(\mathbf{x},t)J]dV \\
&= \int_{\Omega_0} \left[\dot{\Xi}(\mathbf{x},t)J + \dot{J}\Xi(\mathbf{x},t)\right]dV \\
&= \int_{\Omega_0} \left[\dot{\Xi}(\mathbf{x},t) + \frac{\dot{J}}{J}\Xi(\mathbf{x},t)\right]J\,dV \\
&= \int_{\Omega} \left[\dot{\Xi}(\mathbf{x},t) + (\operatorname{div}\mathbf{v})\Xi(\mathbf{x},t)\right]dv
\end{aligned}
$$

For any spatial field, $\phi(\mathbf{x}, t)$,

$$d\phi = \frac{\partial\phi}{\partial\mathbf{x}} d\mathbf{x} + \frac{\partial\phi}{\partial t} dt$$

Whether this is a dot, or tensor product depends on the type of object ϕ is. For a scalar it is the dot product.

So that the material (substantive) time derivative,

$$\frac{D\phi(\chi(\mathbf{X}, t), t)}{Dt} = \frac{\partial\phi(\mathbf{x}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial\phi(\mathbf{x}, t)}{\partial\mathbf{x}}$$

Consequently, if $\Xi(\mathbf{x}, t)$ is a scalar function,

$$\dot{I}(t) = \int_\Omega \left[\frac{\partial\Xi(\mathbf{x}, t)}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \Xi(\mathbf{x}, t) + (\operatorname{div} \mathbf{v})\Xi(\mathbf{x}, t)\right] dv$$

$$= \int_\Omega \left[\frac{\partial\Xi(\mathbf{x}, t)}{\partial t} + \operatorname{div}(\mathbf{v}\,\Xi)\right] dv$$

which after applying the divergence theorem of Gauss, we find to be,

$$\dot{I}(t) \equiv \int_\Omega \left[\frac{\partial\Xi}{\partial t} + \operatorname{div}(\mathbf{v}\Xi)\right] dv = \int_\Omega \frac{\partial\Xi}{\partial t} dv + \int_{\partial\Omega} \Xi\mathbf{v} \cdot \mathbf{n}\, ds$$

as required.

Two Cases

1. Spatial Scalar Field $\phi(\mathbf{x}, t)$:

$$d\phi = \left(\frac{\partial \phi}{\partial \mathbf{x}}\right) \cdot d\mathbf{x} + \frac{\partial \phi}{\partial t} dt = (\text{grad } \phi) \cdot d\mathbf{x} + \frac{\partial \phi}{\partial t} dt$$

because grad of a scalar is a vector, hence the only consistent product here is the scalar.

2. Spatial Vector Field $\boldsymbol{\phi}(\mathbf{x}, t)$:

$$d\boldsymbol{\phi} = \left(\frac{\partial \boldsymbol{\phi}}{\partial \mathbf{x}}\right) d\mathbf{x} + \frac{\partial \boldsymbol{\phi}}{\partial t} dt = (\text{grad } \boldsymbol{\phi}) d\mathbf{x} + \frac{\partial \boldsymbol{\phi}}{\partial t} dt$$

The grad of a vector is a tensor. It operates on the vector to produce a vector.

Particular Case

In particular, $\boldsymbol{\phi} = \mathbf{v} \Rightarrow$

$$d\mathbf{v} = (\text{grad } \mathbf{v})d\mathbf{x} + \frac{\partial \mathbf{v}}{\partial t}dt$$

so that the substantial acceleration,

$$\frac{D\mathbf{v}}{Dt} = \mathbf{L}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

Substantial Acceleration

Convective Acceleration

Local Acceleration

Where $\mathbf{L}$ is the velocity gradient. For example, in steady fluid flow, the local acceleration, $\frac{\partial \mathbf{v}}{\partial t}$, is zero, but the material acceleration is not zero. That explains why a particle can move from the velocity of one location to a different velocity in another location. The velocity gradient term, called convective acceleration, is what subsists here. These two RHS terms occur only in the spatial description of motion.

Example: Material Velocity, Acceleration

We can look at the spatial configuration as a field defined over the reference configuration: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The velocity of this particle

1. $\mathbf{V}(\mathbf{X}, t) = \frac{D}{Dt}\boldsymbol{\chi}(\mathbf{X}, t)$which is the velocity of the material point that occupied the position $\mathbf{X}$ in the reference configuration. But this is the material occupying the position $\mathbf{x}$ in the spatial!
2. $\mathbf{v}(\mathbf{x}, t) = \mathbf{V}\left(\boldsymbol{\chi}^{-1}(\mathbf{x}), t\right)$ using the reference map

Example: Material Velocity, Acceleration

The acceleration of this particle

1. $\mathbf{A}(\mathbf{X}, t) = \frac{D^2}{Dt^2}\boldsymbol{\chi}(\mathbf{X}, t)$which is the velocity of the material point that occupied the position $\mathbf{X}$ in the reference configuration. But this is the material ocuupying the position $\mathbf{x}$ in the spatial!
2. $\mathbf{a}(\mathbf{x}, t) = \mathbf{A}\left(\boldsymbol{\chi}^{-\mathbf{1}}(\mathbf{x}), t\right)$ using the reference map

All works fine provided we have the material description and the explicit form of the motion. When this is not the case, we have to refer to $\frac{D\mathbf{v}}{Dt} = \mathbf{L}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$ as the following example shows:

Consider a motion defined by

$$\mathbf{x} = (1+t)X_1\mathbf{e}_1 + (1+t)^2X_2\mathbf{e}_2 + \left(1+t^2\right)X_3\mathbf{e}_3$$

Let us find the velocity and acceleration. Clearly, in Material terms,

$$\mathbf{V}(\mathbf{X},t) = X_1\mathbf{e}_1 + 2(1+t)X_2\mathbf{e}_2 + 2tX_3\mathbf{e}_3$$

and acceleration,

$$\mathbf{A}(\mathbf{X},t) = 2X_2\mathbf{e}_2 + 2X_3\mathbf{e}_3$$

And if we observe that the reference map here is,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = \frac{x_1}{1+t}\mathbf{e}_1 + \frac{x_2}{(1+t)^2}\mathbf{e}_2 + \frac{x_3}{1+t^2}\mathbf{e}_3$$

We can substitute here and obtain the spatial description of the velocity and acceleration:

$$\mathbf{v}(\mathbf{x},t) = \frac{x_1\mathbf{e}_1}{1+t} + \frac{2x_2\mathbf{e}_2}{1+t} + \frac{2tx_3\mathbf{e}_3}{1+t^2}$$

$$\mathbf{a}(\mathbf{x},t) = \frac{2x_2}{(1+t)^2}\mathbf{e}_2 + \frac{2x_3}{1+t^2}\mathbf{e}_3$$

Now try to evaluate this substantial acceleration from the spatial velocity! Be careful that you are not getting just the local acceleration!

```
v[x1_, x2_, x3_, t_] := {x1/(1+t), 2 x2/(1+t), 2 t x3/(1+t^2)}
```

```
vGrad = Grad[v[x1, x2, x3, t], {x1, x2, x3}]
```

$$\left\{\left\{\frac{1}{1+t}, 0, 0\right\}, \left\{0, \frac{2}{1+t}, 0\right\}, \left\{0, 0, \frac{2\,t}{1+t^2}\right\}\right\}$$

```
aLocal = D[v[x1, x2, x3, t], t]
```

$$\left\{-\frac{x_1}{(1+t)^2}, -\frac{2\,x_2}{(1+t)^2}, -\frac{4\,t^2\,x_3}{(1+t^2)^2}+\frac{2\,x_3}{1+t^2}\right\}$$

```
vGrad.v[x1, x2, x3, t]
```

$$\left\{\frac{x_1}{(1+t)^2}, \frac{4\,x_2}{(1+t)^2}, \frac{4\,t^2\,x_3}{(1+t^2)^2}\right\}$$

```
aTot = aLocal + vGrad.v[x1, x2, x3, t]
```

$$\left\{0, \frac{2\,x_2}{(1+t)^2}, \frac{2\,x_3}{1+t^2}\right\}$$

Example 2

Consider the motion,

$$\mathbf{x} = \left((1+t)X_2 - tX_1\right)\mathbf{e}_1 + \left((1+t)^2 X_1 + tX_2\right)\mathbf{e}_2 + \left(1+t^2\right)X_3\mathbf{e}_3$$

Find the *Reference Map*, *Spatial Velocity* and *Substantial Acceleration*. Show that the latter can be found either by directly differentiating the material velocity or adding the local acceleration to the velocity gradient tensor operation on the spatial velocity.

Ans: The link here is the Mathematica code that does this computation. See that the velocity gradient here, unlike the previous example, is NOT a diagonal tensor.

Conservation of Mass

Based upon the above theorem, we can express the balance of mass compactly, considering the fact that,

$$\int_{\Omega} \varrho(\mathbf{x}, t) dv = \int_{\Omega_0} \varrho_0(\mathbf{X}) dV$$

The right hand of the above equation is independent of time t. Hence a time derivative,

$$\frac{D}{Dt} \int_{\Omega} \varrho(\mathbf{x}, t) dv = \frac{D}{Dt} \int_{\Omega_0} \varrho_0(\mathbf{X}) dV = 0$$

Invoking the Leibniz-Reynolds theorem, we conclude that,

$$\int_{\Omega} \left(\frac{\partial \varrho}{\partial t} + \operatorname{div}(\mathbf{v}\varrho) \right) dv = \int_{\Omega} \frac{\partial \varrho}{\partial t} dv + \int_{\partial\Omega} \varrho \mathbf{v} \cdot \mathbf{n}\, ds = 0$$

Conservation of Mass

The mass generation inside the system plus the net mass transport across the boundary sum up to zero.

$$\int_{\Omega} \frac{\partial \varrho}{\partial t} dv + \int_{\partial\Omega} \varrho \mathbf{v} \cdot \mathbf{n} \, ds = 0$$

Equivalently, in differential form, The time rate of change of spatial density ϱ plus the divergence of mass flow rate equals zero.

local derivative

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\mathbf{v}\varrho) = 0$$

Or equivalently, $$\frac{D\varrho}{Dt} + \varrho \operatorname{div} \mathbf{v} = 0$$

substantive derivative

Isochoric Motion

* A motion that evolves in such a way that the density does not change is called Isochoric Motion. It is also called "incompressible" motion.
* From the above equations, we can see that for Isochoric Motion, $\text{div}\,\mathbf{v} = 0$ because, conservation of mass imposes the condition,

$$\frac{D\varrho}{Dt} + \varrho\,\text{div}\,\mathbf{v} = 0$$

and the result follows if the substantial derivative of density is zero. Furthermore, under Isochoric motion, Liouville's theorem also assures that the third invariant of the deformation Gradient, $\dot{J}(\mathbf{x}, t) = J\,\text{div}\,\mathbf{v} = 0$.