Balance Laws II

Conservation of Mass, Momentum & Energy

Conservation of Mass

We can express the balance of mass compactly, considering the fact that,

$$\int_{\Omega} \varrho(\mathbf{x}, t)\, dv = \int_{\Omega_0} \varrho_0(\mathbf{X})\, dV$$

The right hand of the above equation is independent of time t.

A change of variables ⇒

$$\int_{\Omega} \varrho(\mathbf{x}(\mathbf{X}, t), t) \frac{dv}{dV}\, dV = \int_{\Omega_0} \varrho_0(\mathbf{X})\, dV$$

Or,

$$0 = \int_{\Omega_0} \left(\varrho_0(\mathbf{X}) - \varrho(\mathbf{x}(\mathbf{X}, t), t) J\right) dV$$

Conservation of Mass

We can see that at all points,

$$\varrho_0(\mathbf{X}) - \varrho(\mathbf{x}(\mathbf{X}, t), t)J = 0$$

Taking a material time derivative of the mass integrals, we have,

$$\frac{D}{Dt}\int_{\Omega} \varrho(\mathbf{x}, t)dv = \frac{D}{Dt}\int_{\Omega_0} \varrho_0(\mathbf{X})dV = 0$$

Invoking the Leibniz-Reynolds theorem, we conclude that,

$$\int_{\Omega}\left(\frac{\partial \varrho}{\partial t} + \mathrm{div}(\mathbf{v}\varrho)\right)dv = \int_{\Omega}\frac{\partial \varrho}{\partial t}dv + \int_{\partial\Omega} \varrho\mathbf{v}\cdot\mathbf{n}\, ds = 0$$

Conservation of Mass

The mass generation inside the system plus the net mass transport across the boundary sum up to zero.

$$\int_{\Omega} \frac{\partial \varrho}{\partial t} dv + \int_{\partial \Omega} \varrho \mathbf{v} \cdot \mathbf{n} \, ds = 0$$

Equivalently, in differential form, The time rate of change of spatial density ϱ plus the divergence of mass flow rate equals zero.

local derivative

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\mathbf{v}\varrho) = 0$$

Or equivalently, $\dfrac{D\varrho}{Dt} + \varrho \operatorname{div} \mathbf{v} = 0$

substantive derivative

Steady & Isochoric Motions

- We do well to observe the significance and differences between the two time derivatives in the above equations. In the first,

$$\frac{\partial \varrho}{\partial t} + \mathrm{div}(\mathbf{v}\varrho) = 0$$

We are looking at a local derivative. It vanishes in **steady motion**. Here, mass balance states that the divergence of the product $\mathbf{v}(\mathbf{x}, t)\varrho(\mathbf{x}, t)$ vanishes.

In the second,

$$\frac{D\varrho}{Dt} + \varrho \, \mathrm{div} \, \mathbf{v} = 0$$

we are looking at the material time derivative and therefore following a particle. It vanishes in **isochoric motion**. Here, the divergence of the spatial velocity, $\mathbf{v}(\mathbf{x}, t)$ is zero.

Isochoric Motion

* A motion that evolves in such a way that the density does not change is called Isochoric Motion.
* From the above equations, we can see that for Isochoric Motion, $\text{div}\,\mathbf{v} = 0$ because, conservation of mass imposes the condition,

$$\frac{D\varrho}{Dt} + \varrho\,\text{div}\,\mathbf{v} = 0$$

and the result follows if the substantial derivative of density is zero. Furthermore, under Isochoric motion, Liouville's theorem also assures that the material time derivative of the third invariant of the deformation Gradient,

$$\dot{J}(\mathbf{x}, t) = J\,\text{div}\,\mathbf{v} = 0.$$

Incompressible Bodies

- A body is said to be incompressible if the material time derivative of its density,

$$\frac{D}{Dt}\varrho(\mathbf{x}(\mathbf{X},t)) = 0$$

for all particles $\mathbf{X}$, and at all times t.

- A compressible body can have isochoric motion. All the possible motions of an incompressible body are isochoric. For a compressible fluid, isochoric motion may exist for a short time and could be localized.

Cauchy's Laws of Motion

The momentum balance principles in this section are generalizations of Newton's second law of motion in the context of a continuously distributed body instead of a particle. These principles lead to the Cauchy's Laws of motion. We begin this section with the linear momentum balance. Continuing from the last section, we can express the linear momentum of a body in the spatial frame as,

$$\boldsymbol{P}(t) = \int_{\Omega} \varrho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)dv = \int_{\Omega_0} \varrho_0(\mathbf{X})\mathbf{V}(\mathbf{X}, t)dV$$

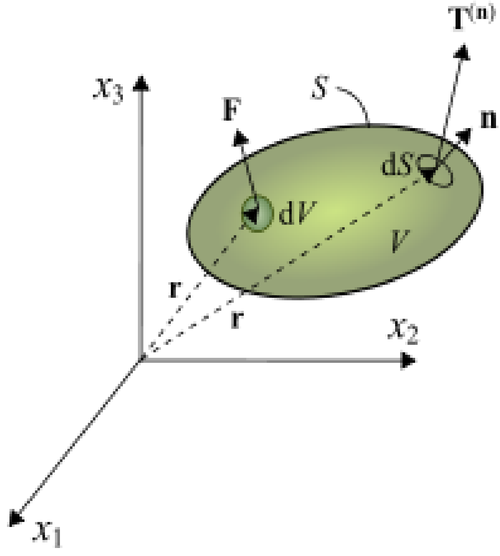
where Ω is the spatial configuration volume and Ω_0 the reference configuration.

Linear Momentum

The balance of linear momentum, according to the second law of Newton is that,

$$\frac{D\mathbf{P}(t)}{Dt} = \mathbf{F}(t)$$

where $\mathbf{F}(t)$ is the resultant force on the system. Hence by the conservation of linear momentum, we may write,



Cauchy' Law

$$\dot{\mathbf{P}}(t) = \frac{D}{Dt}\int_{\Omega} \varrho(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)dv$$
$$= \frac{D}{Dt}\int_{\Omega_0} \varrho_0(\mathbf{X})\mathbf{V}(\mathbf{X},t)dV = \mathbf{F}(t)$$

We now look at the forces acting on the body from the categorization of surface and body forces. The surface forces are measured by the tractions $\mathbf{t}^{(\mathbf{n})}$ per unit area of the surface while the body forces are in terms of the specific body force $\mathbf{b}$ per unit volume. Clearly,

$$\mathbf{F}(t) = \int_{\partial\Omega} \mathbf{t}^{(\mathbf{n})}(\mathbf{x},t)ds + \int_{\Omega} \mathbf{b}(\mathbf{x},t)dv$$

Cauchy law of Motion

By Cauchy's stress law, $\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$. Consequently, we may write,

$$\frac{D}{Dt}\int_{\Omega} \varrho \mathbf{v} dv = \int_{\partial\Omega} \mathbf{t}^{(\mathbf{n})}(\mathbf{x}, t) ds + \int_{\Omega} \mathbf{b}(\mathbf{x}, t) dv$$

$$= \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n} ds + \int_{\Omega} \mathbf{b} dv$$

$$= \int_{\Omega} (\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}) dv$$

where we have invoked the divergence theorem of Gauss

Cauchy's Law

The law of conservation of mass allows us to take the above substantial derivative under the integral, we are allowed to treat the mass measure as a constant, hence we can write that,

$$\frac{D}{Dt}\int_{\Omega}\varrho\mathbf{v}dv = \int_{\Omega}\varrho\frac{D}{Dt}(\mathbf{v})dv$$
$$= \int_{\Omega}\varrho\frac{D\mathbf{v}(\mathbf{x},t)}{Dt}dv$$
$$= \int_{\Omega}(\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{b})dv$$

We can write the above equation in differential form as,

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \varrho\frac{D\mathbf{v}(\mathbf{x},t)}{Dt}$$

Formal Statement

Euler-Cauchy First Law of Motion:

The divergence of the stress tensor $\boldsymbol{\sigma}$ *plus the body force* $\mathbf{b}$ *per unit volume equals material time rate of change of linear momentum.*

$$\operatorname{div} \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t) = \varrho(\mathbf{x}, t) \frac{D\mathbf{v}(\mathbf{x}, t)}{Dt}$$

Cauchy Second Law of Motion

The rate of change of angular momentum about a point, $\dot{\mathbf{L}}(t)$ *is equal to the sum of the external moments about that point. In a continuously distributed medium, the* ***Cauchy Stress Tensor field is a symmetric tensor***

$$\boldsymbol{\sigma}^{\mathrm{T}}(\mathbf{x}, t) = \boldsymbol{\sigma}(\mathbf{x}, t)$$

This is a consequence of applying the same balance law to the moments of momentum as follows

Angular Momentum Balance

- Given a spatial point $\mathbf{x}$, we can write its position vector as $\mathbf{r} = \mathbf{x} - \mathbf{o}$. Consequently,

$$\mathbf{L}(t) \equiv \int_{\Omega} \mathbf{r} \times \big(\varrho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)\big)\, dv$$

Is the total angular momentum of the spatial body. We can express this in terms of the two sources of forces and take the material time derivative of angular momentum:

$$\frac{D}{Dt}\int_{\Omega} \mathbf{r} \times (\varrho \mathbf{v}) dv = \int_{\partial\Omega} \mathbf{r} \times \mathbf{t}^{(\mathbf{n})} ds + \int_{\Omega} \mathbf{r} \times \mathbf{b} dv$$

omitting the dependencies while expressing the fact that the change in angular momentum equals the total moments acting on the body.

Angular Momentum Balance

$$\frac{D}{Dt}\int_{\Omega} \mathbf{r} \times (\varrho \mathbf{v}) dv = \int_{\partial\Omega} \mathbf{r} \times \boldsymbol{\sigma}\mathbf{n} ds + \int_{\Omega} \mathbf{r} \times \mathbf{b} dv$$
$$= \int_{\Omega} \operatorname{div}(\mathbf{r} \times \boldsymbol{\sigma}) dv + \int_{\Omega} \mathbf{r} \times \mathbf{b} dv$$
$$= \int_{\Omega} (\mathbf{r} \times \operatorname{div} \boldsymbol{\sigma} - \mathbf{E} : \boldsymbol{\sigma}) dv + \int_{\Omega} \mathbf{r} \times \mathbf{b} dv$$
$$= \int_{\Omega} \mathbf{r} \times (\operatorname{div} \boldsymbol{\sigma} + \mathbf{b}) dv - \int_{\Omega} \mathbf{E} : \boldsymbol{\sigma} \, dv$$

$\mathbf{E}$ is the Levi-Civita Tensor. LHS, $\mathbf{v} \times (\varrho \mathbf{v})$ vanishes first term on RHS vanishes as a result of the balance of linear momentum, hence the inner product,

$$\mathbf{E} : \boldsymbol{\sigma} = 0$$

Inner Product of Two Tensors

Given two tensors, $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{B} = B_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ the product

$$\begin{aligned}\mathbf{A}\mathbf{B}^{\mathrm{T}} &= \left(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(B_{\alpha\beta}\mathbf{e}_\beta \otimes \mathbf{e}_\alpha\right)\\ &= A_{ij}B_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\alpha\delta_{j\beta} = A_{ij}B_{\alpha j}\mathbf{e}_i \otimes \mathbf{e}_\alpha\end{aligned}$$

The inner product of $\mathbf{A}$ and $\mathbf{B}$ is defined as,

$$\mathbf{A}:\mathbf{B} \equiv \mathrm{tr}\left(\mathbf{A}\mathbf{B}^{\mathrm{T}}\right) = \mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{B}\right) = A_{ij}B_{\alpha j}\delta_{i\alpha} = A_{ij}B_{ij}$$

For a third-order tensor $\mathbf{E} = E_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$,

$$\begin{aligned}\mathbf{E}:\mathbf{B} \equiv \mathrm{tr}\left(\mathbf{E}\mathbf{B}^{\mathrm{T}}\right) &= E_{ijk}B_{\alpha\beta}\mathbf{e}_i\left(\mathbf{e}_k \otimes \mathbf{e}_\beta\right)\left(\mathbf{e}_j \otimes \mathbf{e}_\alpha\right)\\ &= E_{ijk}B_{\alpha\beta}\mathbf{e}_i\left(\mathbf{e}_k \cdot \mathbf{e}_\beta\right)\left(\mathbf{e}_j \cdot \mathbf{e}_\alpha\right)\\ &= E_{ijk}\,B_{jk}\mathbf{e}_i\end{aligned}$$

Cauchy Second Law

In concluding this section, observe that the ***natural tensor characterizations*** of strain such as **deformation gradient**, displacement gradient are not symmetric.

- Various strain tensors are defined. These were done essentially to separate the rigid body displacements from actual deformations.
- The symmetry of Lagrangian and Eulerian strains are definitions. On the other hand, the **Cauchy true stress** is symmetric as a natural principle.

Divergence Theorem

1. For a tensor field $\boldsymbol{\Xi}$ of any order, the volume integral in the region $\Omega \subset \mathcal{E}$,

$$\int_{\Omega} (\text{grad}\, \boldsymbol{\Xi})\, dv = \int_{\partial\Omega} \boldsymbol{\Xi} \otimes \mathbf{n}\, ds$$

where $\mathbf{n}$ is the outward drawn normal to $\partial\Omega$ – the boundary of Ω.

For a second-order tensor, taking the traces, we have

$$\int_{\Omega} (\text{div}\, \boldsymbol{\Xi})\, dv = \int_{\partial\Omega} \boldsymbol{\Xi}\mathbf{n}\, ds$$

For a vector field $\mathbf{f}$

$$\int_{\Omega} (\text{div}\, \mathbf{f})\, dv = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n}\, ds$$

Divergence Theorem for Vector Fields

To show that this is true simply replace Ξ by the vector $\mathbf{f}$,

$$\int_{\Omega} (\text{grad } \mathbf{f})\, dv = \int_{\partial\Omega} \mathbf{f} \otimes \mathbf{n}\, ds$$

Take the trace of both sides,

$$\int_{\Omega} (\text{div } \mathbf{f})\, dv = \int_{\partial\Omega} \text{tr}(\mathbf{f} \otimes \mathbf{n})\, ds = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n}\, ds$$

Divergence Theorem for Scalar Fields

For a scalar field ϕ consider a **constant** vector $\mathbf{a}$. Clearly, $\mathbf{f} \equiv \phi\mathbf{a}$ is a vector field. Hence the above result applies:

$$\int_{\Omega} \left(\text{div}\,(\phi\mathbf{a})\right) dv = \int_{\partial\Omega} (\phi\mathbf{a}) \cdot \mathbf{n}\, ds$$

But observe that we can express $\phi\mathbf{a} = \phi a_i \mathbf{e}_i$ so that

$$\text{grad}\,(\phi\mathbf{a}) = \phi_{,j}\, a_i \mathbf{e}_i \otimes \mathbf{e}_j$$

And therefore,

$$\text{div}\,(\phi\mathbf{a}) = \text{tr}\left(\phi_{,j}\, a_i \mathbf{e}_i \otimes \mathbf{e}_j\right) = \phi_{,i}\, a_i = \mathbf{a} \cdot \text{grad}\,\phi$$

Hence, $\mathbf{a} \cdot \int_{\Omega} \text{grad}\,\phi\, dv = \mathbf{a} \cdot \int_{\partial\Omega} \phi\mathbf{n}\, ds \Rightarrow$

$$\int_{\Omega} \text{grad}\,\phi\, dv = \int_{\partial\Omega} \phi\mathbf{n}\, ds$$

Balance Equations in Cartesian Coordinates

* In their Simplest Forms, the three balance equations we have are:

1. Mass balance $\text{div}\,(\varrho \mathbf{v}) = 0$
2. Linear momentum (equilibrium) $\text{div}\,\boldsymbol{\sigma} + \mathbf{b} = 0$
3. Angular momentum $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$

Class exercise: Express these in Cartesian Coordinates. We will use Mathematica to obtain these in other systems.

Given two tensors, $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ and $\mathbf{B} = B_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ the product

$$\begin{aligned}\mathbf{AB} &= \left(A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(B_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\\ &= A_{ij}B_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\delta_{j\alpha} = A_{ij}B_{j\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\end{aligned}$$

Gradient of this product is,

$$\begin{aligned}\operatorname{grad}(\mathbf{AB}) &= \left(A_{ij}B_{j\beta}\right)_{,k}\mathbf{e}_i \otimes \mathbf{e}_\beta \otimes \mathbf{e}_k\\ &= \left(A_{ij,k}B_{j\beta} + A_{ij}B_{j\beta,k}\right)\mathbf{e}_i \otimes \mathbf{e}_\beta \otimes \mathbf{e}_k\end{aligned}$$

Taking traces on both sides,

$$\begin{aligned}\operatorname{div}(\mathbf{AB}) &= \left(A_{ij,k}B_{j\beta} + A_{ij}B_{j\beta,k}\right)\mathbf{e}_i\delta_{\beta k}\\ &= A_{ij,k}B_{jk}\mathbf{e}_i + A_{ij}B_{jk,k}\mathbf{e}_i\\ &= (\operatorname{grad}\mathbf{A}){:}\,\mathbf{B}^{\mathrm{T}} + \mathbf{A}\operatorname{div}\mathbf{B}\end{aligned}$$

To evaluate the expression, $\operatorname{div}(\mathbf{r} \times \boldsymbol{\sigma})$ simply substitute $\mathbf{A} = \mathbf{r} \times$ and let $\mathbf{B} = \boldsymbol{\sigma}$.

Observe that $\mathbf{A} = (\mathbf{r}\,\times) = e_{i\alpha j}x_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$. Note also that

$$\text{grad}(\mathbf{r}\,\times) = e_{i\alpha j}x_{\alpha,k}\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = e_{ikj}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

on account of the fact that $x_{\alpha,k} = \delta_{\alpha k}$. This is the second transpose of the Levi-Civita Tensor $\mathbf{E} = e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$

$$\begin{aligned} \text{div}(\mathbf{r} \times \boldsymbol{\sigma}) &= \left(e_{i\alpha j}x_\alpha\right)_{,k} \sigma_{jk}\mathbf{e}_i + \mathbf{r}\,\text{div}\,\boldsymbol{\sigma} \\ &= \mathbf{E}^{\mathrm{T}}\colon \boldsymbol{\sigma} + \mathbf{r}\,\text{div}\,\boldsymbol{\sigma} \\ &= -\mathbf{E}\colon \boldsymbol{\sigma} + \mathbf{r}\,\text{div}\,\boldsymbol{\sigma} \end{aligned}$$

For zero change in angular momentum, Cauchy second law states that,

$$-\mathbf{E}\colon \boldsymbol{\sigma} = \mathbf{o}$$

- the zero vector.

$$\mathbf{E}:\boldsymbol{\sigma} = \mathbf{o}$$

As we showed earlier,

$$\mathbf{E}:\boldsymbol{\sigma} = e_{ijk}\sigma_{jk}\mathbf{e}_i = \mathbf{o}$$

Lead to the three equations,

$$\sigma_{23} = \sigma_{32}$$
$$\sigma_{31} = \sigma_{13}$$
$$\sigma_{12} = \sigma_{21}$$

That the Cauchy stress tensor is necessarily symmetrical is a direct result of the balance of angular momentum.

1. **The components of Cauchy stress in Cartesian coordinates are**

$$\begin{pmatrix} x_1x_2 & x_1^2 & -x_2 \\ x_1^2 & 0 & 0 \\ -x_2 & 0 & x_1^2+x_2^2 \end{pmatrix}$$

Find the body forces in the system to keep the system in equilibrium.

2. **The components of Cauchy stress in Cartesian coordinates are**

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & x_2+\alpha x_3 & \Phi(x_2,x_3) \\ 0 & \Phi(x_2,x_3) & x_2+\beta x_3 \end{pmatrix}$$ ***(a)*** **Find $\Phi(x_2, x_3)$ so that the equilibrium equations are satisfied assuming body forces are zero.**

(b) **Use the value of $\Phi\left(x^2, x^3\right)$ found in *(a)* to compute the Cauchy Traction vector on the plane $\psi = x_1 + x_2 + x_3$.**

3. **Given the stress components,**

$$\begin{pmatrix} x_1^2x_2 & \alpha - x_2^2 & -\beta^2x_2 \\ \beta - x_2^2 & \alpha\beta - x_2^2 & 0 \\ -\alpha^2x_2 & 0 & x_1^2+x_2^2 \end{pmatrix}$$

Find the values α and β that make this an admissible Cauchy stress tensor and the body forces that keeps the body in equilibrium.

4. By evaluating the divergence operation on a tensor show that the equilibrium equations can be expressed in Cylindrical Polar coordinates as,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{zr}}{\partial z} + b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + b_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0$$

5. By evaluating the divergence operation on a tensor show that the equilibrium equations can be expressed in Spherical Polar coordinates as,

$$\frac{1}{\rho}\left(\sigma_{\rho\theta}\cot\theta + \frac{\partial \sigma_{\rho\theta}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \sigma_{\rho\phi}}{\partial \phi} + \rho\frac{\partial \sigma_{\rho\rho}}{\partial \rho} - \sigma_{\theta\theta} + 2\sigma_{\rho\rho} - \sigma_{\phi\phi}\right) + b_\rho = 0$$

$$\frac{1}{\rho}\left((\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + \rho\frac{\partial \sigma_{\rho\theta}}{\partial \rho} + 3\sigma_{\rho\theta} + \frac{1}{\sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right) + b_\theta = 0$$

$$\frac{1}{\rho}\left(\frac{1}{\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \rho\frac{\partial \sigma_{\rho\phi}}{\partial \rho} + 3\sigma_{\rho\phi} + 2\sigma_{\theta\phi}\cot\theta + \frac{\partial \sigma_{\theta\phi}}{\partial \theta}\right) + b_\phi = 0$$

6. The law of conservation of mass is expressed in the vector form, $\frac{\partial \varrho}{\partial t} + \mathrm{div}(\mathbf{v}\varrho) = 0$. Express this law in tensor components and find the equivalent in physical components for Cartesian, Cylindrical polar and Spherical polar coordinate systems.

```
SS[x1_, x2_, x3_] := {{σρ[x1, x2, x3], τρθ[x1, x2, x3], τρϕ[x1, x2, x3]},
   {τ1ρθ[x1, x2, x3], σθ[x1, x2, x3], τθϕ[x1, x2, x3]},
   {τρϕ[x1, x2, x3], τθϕ[x1, x2, x3], σϕ[x1, x2, x3]}};
bb[x1_, x2_, x3_] := {bρ[x1, x2, x3], bθ[x1, x2, x3], bϕ[x1, x2, x3]};
```

```
Div[SS[ρ, θ, ϕ], {ρ, θ, ϕ}, "Spherical"] + bb[ρ, θ, ϕ] // MatrixForm
```

ιForm=

$$b_{\rho}[\rho,\theta,\phi]+\frac{\mathrm{Csc}[\theta]\left(\mathrm{Sin}[\theta]\,\sigma_{\rho}[\rho,\theta,\phi]-\mathrm{Sin}[\theta]\,\sigma_{\phi}[\rho,\theta,\phi]+\mathrm{Cos}[\theta]\,\tau_{\rho\theta}[\rho,\theta,\phi]+\tau_{\rho\phi}^{(0,0,1)}[\rho,\theta,\phi]\right)}{\rho}+\frac{-\sigma_{\theta}[\rho,}{}$$

$$b_{\theta}[\rho,\theta,\phi]+\frac{\mathrm{Csc}[\theta]\left(\mathrm{Cos}[\theta]\,\sigma_{\theta}[\rho,\theta,\phi]-\mathrm{Cos}[\theta]\,\sigma_{\phi}[\rho,\theta,\phi]+\mathrm{Sin}[\theta]\,\tau_{\rho\theta}[\rho,\theta,\phi]+\tau_{\theta\phi}^{(0,0,1)}[\rho,\theta,\phi]\right.}{\rho}$$

$$b_{\phi}[\rho,\theta,\phi]+\frac{\tau_{\rho\phi}[\rho,\theta,\phi]+\tau_{\theta\phi}^{(0,1,0)}[\rho,\theta,\phi]}{\rho}+\tau_{\rho\phi}^{(1,0,0)}[\rho,\theta,\phi]+\frac{\mathrm{Csc}[\theta]\left(2\,\mathrm{Cos}\right.}{}$$

rt | reverse | all subsets | permutations | more...

```
Div[SS[r, θ, z], {r, θ, z}, "Cylindrical"] + bb[r, θ, z] // MatrixForm
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ixForm=