SSG 516
Mechanics of Continua

Kinematics 05
More Deformation Examples

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Natural or Covariant Basis

- From previous class, it is clear that the position vector takes non-linear forms in curvilinear systems. We also noted that there are two kinds of related bases for such systems.
- Today, we will look at the logic of these bases in the way they help correct the dimensionality problem that arises when the variables you use for the three numbers representing a point in the Euclidean point space are not of the same units.
- Perhaps the easiest way to do this is to attempt to write a change in the position vector. We begin with the Cartesian system:

Changes in Position Vectors

Cartesian Position Vector:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$
$$d\mathbf{r} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3$$

That is easy! Now, attempt to write the change in the Cylindrical Polar position vector:

$$\mathbf{r} = r\mathbf{e}_r(\theta) + z\mathbf{e}_z$$

Is it correct to write,

$$d\mathbf{r} = dr\mathbf{e}_r(\theta) + dz\mathbf{e}_z?$$

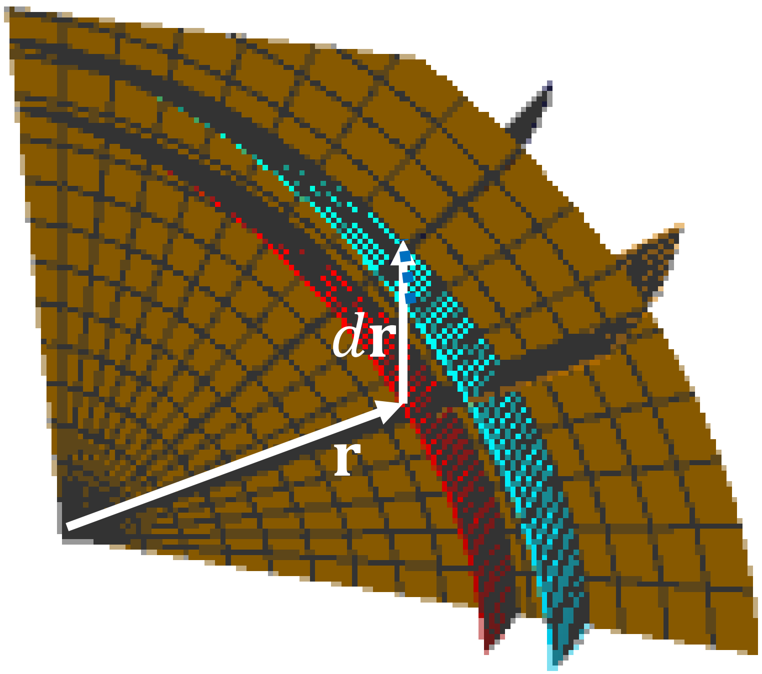
Or in the Spherical Polar system, $\mathbf{r} = \rho\mathbf{e}_\rho(\theta, \phi)$. What is the change in the position vector? Is it correct to write,

$$d\mathbf{r} = d\rho\mathbf{e}_\rho(\theta, \phi)$$

Spherical Coordinates

```
s1=ParametricPlot3D[{ Sin[φ]Cos [θ], Sin[φ]Sin[θ],Cos[φ]},{θ,π/5,Pi/3},{φ,0,Pi/2},PlotStyle->Red];
s2=ParametricPlot3D[{ ρ Sin[φ]Cos [π/4],ρ Sin[φ]Sin[π/4],ρ Cos[φ]},{ρ,0,1.5},{φ,0,Pi/2},PlotStyle->Green];
s3=ParametricPlot3D[{ ρ Sin[π/6]Cos [θ],ρ Sin[π/6]Sin[θ],ρ Cos[π/6]},{ρ,0,1.5},{θ,π/5,Pi/3},PlotStyle->Yellow];
s4=ParametricPlot3D[{ ρ Sin[π/8]Cos [θ],ρ Sin[π/8]Sin[θ],ρ Cos[π/8]},{ρ,0,1.5},{θ,π/5,Pi/3},PlotStyle->Blue];
s5=ParametricPlot3D[{ 1.15Sin[φ]Cos [θ],1.15 Sin[φ]Sin[θ],1.15Cos[φ]},{θ,π/5,Pi/3},{φ,0,Pi/2},PlotStyle->Cyan];
Show[s1,s2,s3,s4,s5,PlotRange->{{-1.4,1.4},{-1.4,1.5},{0,1.2}}, Ticks->None]
```

In the spherical polar diagram we have shown the points $\mathbf{r}$ and a location $d\mathbf{r}$ away from it. You will observe that, in general, they are not pointing in the same direction as we might have been led to believe from the formula above! It is clear from here that the Cartesian strategy of computing the changes in the position vector fails in the curvilinear system. The solution? Partial differentiation!



Cylindrical Polar Changes

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z$$
$$= r\mathbf{e}_r(\theta) + z\mathbf{e}_z \rightarrow \mathbf{r}(r, \theta, z)$$

Because we earlier defined $\mathbf{e}_r(\theta) \equiv \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2$

Hence,

$$d\mathbf{r} = \frac{\partial\mathbf{r}}{\partial r}dr + \frac{\partial\mathbf{r}}{\partial\theta}d\theta + \frac{\partial\mathbf{r}}{\partial z}dz$$
$$= \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta + \mathbf{e}_z dz$$

Notice what has happened! There is automatic adjustment of the basis to account for unit consistency. Hence, even though it is not orthonormal, the natural basis for the cylindrical is made up of the three vectors,

$$\mathbf{e}_r, r\mathbf{e}_\theta, \text{and } \mathbf{e}_z$$

Spherical Coordinates representation of Position Vector change

$$\mathbf{r} = \rho \sin\phi \cos\theta \, \mathbf{i} + \rho \sin\theta \sin\theta \, \mathbf{j} + \rho \cos\phi \, \mathbf{k}$$
$$= \rho \mathbf{e}_\rho(\theta, \phi) \to \mathbf{r}(\rho, \theta, \phi)$$

Showing that the position vector is a non-linear function of the three coordinate variables, $\rho, \theta,$ and ϕ. The full differential,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \rho} d\rho + \frac{\partial \mathbf{r}}{\partial \theta} d\theta + \frac{\partial \mathbf{r}}{\partial \phi} d\phi$$
$$= \mathbf{e}_\rho d\rho + \rho \mathbf{e}_\theta d\theta + \rho \sin\theta \mathbf{e}_\phi d\phi$$

Notice what has happened! There is automatic adjustment of the basis to account for unit consistency. Hence, even though it is not orthonormal, the natural basis for the spherical polar coordinate system is made up of the three vectors, not the trio of $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi$ but $\mathbf{e}_\rho, \rho\mathbf{e}_\theta, \rho \sin\theta \, \mathbf{e}_\phi$.

Reciprocal or Dual Basis

- The fact that a curvilinear (or non-Cartesian) coordinate system has another set of basis vectors in addition to the natural basis we have defined may look strange.
- Stranger still may be the fact that the Cartesian system we have been used to all along, has two sets of bases also.
- This is how anyone would feel when introduced to the fact that a circle is actually a degenerate ellipse whose two foci have merged into one!
- The concept of dual basis is a fundamental concept that eases the way we deal with curvilinear systems and must be introduced at this stage:

Reciprocity

* For simplicity, we restrict ourselves to general coordinates that are orthogonal. All the coordinate systems of interest to us have this property and hence we do not lose too much by avoiding the details of reciprocal basis at this stage.
* Let $\mathbf{g}_i, i = 1,2,3$ be the natural bases is a curvilinear system of coordinates. The dual (or reciprocal or contravariant) base vectors are $\mathbf{g}^i$such that,

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

Dual Bases Examples

* For Cylindrical Polar, the natural basis set is:

$$\mathbf{e}_r, r\mathbf{e}_\theta, \text{and } \mathbf{e}_z$$

The dual set is,

$$\mathbf{e}_r, \frac{\mathbf{e}_\theta}{r}, \text{and } \mathbf{e}_z$$

For the spherical, the natural basis set is, $\mathbf{e}_\rho, \rho\mathbf{e}_\theta$, $\rho \sin\theta \, \mathbf{e}_\phi$ and the dual is

$$\mathbf{e}_\rho, \frac{\mathbf{e}_\theta}{\rho}, \frac{\mathbf{e}_\phi}{\rho \sin\theta}$$

Application Of Dual Bases

Given a curvilinear system (for example, the spherical coordinates, ρ, θ, ϕ), the fact that the coordinate variables are in different units makes it impossible for us to write the differential in a position vector, $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$

$$d\mathbf{r} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3$$

The Cartesian scheme fails because, $\mathbf{e}_\rho d\rho + \mathbf{e}_\theta d\theta + \mathbf{e}_\phi d\phi$ is dimensionally faulty as we are adding lengths to angles! The use of the natural bases remedied the situation as in

$$d\mathbf{r} = \mathbf{e}_\rho d\rho + \rho\mathbf{e}_\theta d\theta + \rho \sin\theta \mathbf{e}_\phi d\phi$$

Which automatically adjusts the angular measures by an appropriate multiplication factor to give the correct value.

Computing Vector Bases

- The code below computes the natural bases of Spherical Polar Coordinates by differentiating the position vector expressed in the coordinates with respect to the the coordinate variables:
- `x[ρ_,θ_,φ_]:=ρ Cos[φ]Sin[θ];y[ρ_,θ_,φ_]:=ρ Sin[φ]Sin[θ];z[ρ_,θ_,φ_]:=ρ Cos[θ];`
- `NatBasSph:=D[{x[ρ,θ,φ],y[ρ,θ,φ],z[ρ,θ,φ]},{{ρ,θ,φ}}]`
- The result of the computation is packed into the columns of the matrix below when we ask that the result be put in MatrixForm:

$$\begin{pmatrix} \text{Cos}[\phi]\,\text{Sin}[\theta] & \rho\,\text{Cos}[\theta]\,\text{Cos}[\phi] & -\rho\,\text{Sin}[\theta]\,\text{Sin}[\phi] \\ \text{Sin}[\theta]\,\text{Sin}[\phi] & \rho\,\text{Cos}[\theta]\,\text{Sin}[\phi] & \rho\,\text{Cos}[\phi]\,\text{Sin}[\theta] \\ \text{Cos}[\theta] & -\rho\,\text{Sin}[\theta] & 0 \end{pmatrix}$$

Computing the Dual Basis Vectors

- Inverting the above matrix produces the Dual basis vectors, DualBasSph=Inverse[NatBasSph]; packed into the rows of the matrix below:

$$\begin{pmatrix} \cos[\phi]\sin[\theta] & \sin[\theta]\sin[\phi] & \cos[\theta] \\ \frac{\cos[\theta]\cos[\phi]}{\rho} & \frac{\cos[\theta]\sin[\phi]}{\rho} & -\frac{\sin[\theta]}{\rho} \\ -\frac{\csc[\theta]\sin[\phi]}{\rho} & \frac{\cos[\phi]\csc[\theta]}{\rho} & 0 \end{pmatrix}$$

- Take note that the basis vectors are in the rows in this matrix. A quick proof of the duality can be seen by multiplying the dual with the natural basis vectors in,
- DualBasSph. NatBasSph//MatrixForm which obviously gives us the identity matrix.

Application Of Dual Bases

* Similarly, we know that the gradient of a scalar function $\Phi(x, y, z)$ in Cartesian coordinates can be written as,

$$\text{grad }\Phi = \frac{\partial \Phi}{\partial x}\mathbf{i} + \frac{\partial \Phi}{\partial y}\mathbf{j} + \frac{\partial \Phi}{\partial z}\mathbf{k}$$

This simple form cannot be extended to a curvilinear system so, it is wrong to expect that, for spherical coordinates,

$$\text{grad }\Phi \neq \frac{\partial \Phi}{\partial \rho}\mathbf{e}_\rho + \frac{\partial \Phi}{\partial \theta}\mathbf{e}_\theta + \frac{\partial \Phi}{\partial \phi}\mathbf{e}_\phi$$

Same problem of dimensional inconsistency again! The fact that the coordinate variables appear here in quotient form means that the dual basis is needed to correct this problem:

Gradient in Orthogonal Curvilinear Systems

* If we use the dual basis in the Cylindrical Polar and Spherical Polar, we can write the gradient in these systems following the Cartesian method and obtain:

$$\text{grad}\,\Phi = \frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi$$

For the spherical, and

$$\text{grad}\,\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\Phi}{\partial z}\mathbf{e}_z$$

For the cylindrical to correct the dimensional inconsistency.

* We will subsequently use this to obtain correct deformation gradients in non-Cartesian Systems.

When to use the dual bases: Simple Rule:

* If the coordinate variables are in the numerator, use the natural bases.
* When the coordinate variables appear in the denominator, the dual basis will be used.
* You are, for example, already in a position to obtain the grad of a scalar function in other orthogonal curvilinear systems once you can use the transformation equations to find the position vector as we previously demonstrated.

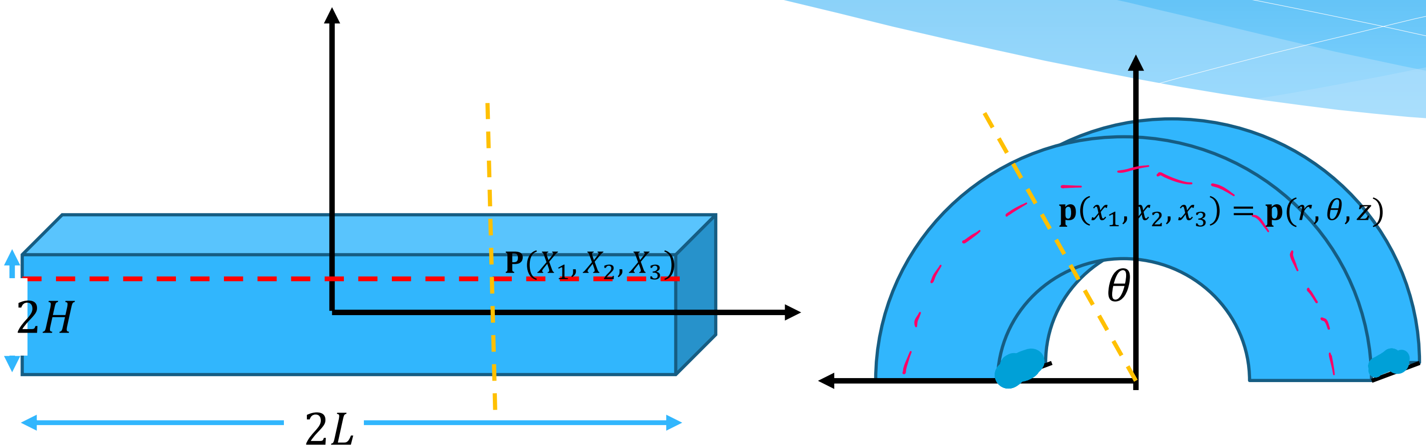
Bending into a circular arc

- If we deform a straight bar into a circular bar as shown below, the transformation function can be found by the following consideration:
- Note that each horizontal filament in the original bar becomes a circular filament in the spatial configuration. The vertical undeformed sections become radial sections in the spatial state. For the moment, we assume nothing happens in the axial or z direction in each case.
- Consequently, we can write,

$$r = r(X_2), \theta = \theta(X_1) \text{ and } z = z(X_3)$$

As a general set of functions transforming X_1, X_2, X_3 to r, θ, z

Bar bending to a semi-circle



1. Let the centerline be a semicircle at a distance R and let the thickness contract uniformly with a factor α

$$\Rightarrow r = R + \alpha X_2, \text{and}$$

$$\theta = \frac{\pi X_1}{2L}$$

2. If the bar contracts uniformly in X_3 direction, $z = \beta X_3$

Natural Basis Adjust

* The (spatial) coordinate system in the quotient is cylindrical hence we will use the natural basis in this situation.
* The referential coordinate system in the quotient is Cartesian, there is no difference between the natural and dual bases. Hence, we make no adjustments.
* The computations follow the same pattern as previously after the simple adjustment above:

Bending

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \alpha \mathbf{e}_\theta \otimes \mathbf{E}_1 + \frac{\pi r}{2L}\mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3$$

Clearly,

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

and the Right Stretch Tensor,

$$\mathbf{U} = \frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta \mathbf{E}_3 \otimes \mathbf{E}_3$$

Bending into a circular arc; Revisited

- In previous slides, we looked at the bending of a circular bar where the transformation equations are given as follows:

$$r = r(X_2), \theta = \theta(X_1) \text{ and } z = z(X_3)$$

as a general set of functions transforming X_1, X_2, X_3 to r, θ, z

- We use Cylindrical Polar Coordinates for the spatial configuration and Cartesian for the Reference configuration.
- We now attempt the same problem again using Cartesian coordinates for both spatial and referential

Transformation of Coordinates

$$x_1 = r\cos\theta = r(X_2)\cos\theta(X_1) = x_1(X_1, X_2)$$
$$x_2 = r\sin\theta = r(X_2)\sin\theta(X_1) = x_2\,(X_1, X_2)$$
$$x_3 = \beta X_3$$

Here, we have used our knowledge of the coordinate system relationships to express the dependency in the spatial configuration on the reference with Cartesian coordinates. Our deformation gradient can be calculated directly using the more elementary approach of simply looking at the components with no reference to the tensor bases.

Deformation Gradient in Cartesian Coordinates

$$\frac{\partial x_1}{\partial X_1} = -r\sin\theta\frac{d\theta}{dX_1};\ \frac{\partial x_1}{\partial X_2} = \cos\theta\frac{dr}{dX_2};\ \ \frac{\partial x_1}{\partial X_3} = 0$$

$$\frac{\partial x_2}{\partial X_1} = r\cos\theta\frac{d\theta}{dX_1};\ \frac{\partial x_2}{\partial X_2} = \cos\theta\frac{dr}{dX_2};\frac{\partial x_2}{\partial X_3} = 0$$

$$\frac{\partial x_3}{\partial X_1} = \frac{\partial x_3}{\partial X_2} = 0;\frac{\partial x_3}{\partial X_3} = \beta$$

The deformation gradient can now be put in the matrix form,

$$\mathbf{F} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]\begin{bmatrix} -r\sin\theta\frac{d\theta}{dX_1} & \cos\theta\frac{dr}{dX_2} & 0 \\ r\cos\theta\frac{d\theta}{dX_1} & \cos\theta\frac{dr}{dX_2} & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

Right Cauchy-Green Tensor

$$\mathbf{F}^{\mathrm{T}} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3] \begin{bmatrix} -r\sin\theta \dfrac{d\theta}{dX_1} & r\cos\theta \dfrac{d\theta}{dX_1} & 0 \\ \cos\theta \dfrac{dr}{dX_2} & \cos\theta \dfrac{dr}{dX_2} & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

So that the Right Cauchy-Green Tensor, $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is,

$$[\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3] \begin{bmatrix} -r\sin\theta \dfrac{d\theta}{dX_1} & r\cos\theta \dfrac{d\theta}{dX_1} & 0 \\ \cos\theta \dfrac{dr}{dX_2} & \cos\theta \dfrac{dr}{dX_2} & 0 \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} -r\sin\theta \dfrac{d\theta}{dX_1} & \cos\theta \dfrac{dr}{dX_2} & 0 \\ r\cos\theta \dfrac{d\theta}{dX_1} & \cos\theta \dfrac{dr}{dX_2} & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

Right Cauchy-Green

$$[\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]\begin{bmatrix} r^2(\sin^2\theta+\cos^2\theta)\left(\frac{d\theta}{dX_1}\right)^2 & r(\cos\theta\sin\theta-\cos\theta\sin\theta)\frac{d\theta}{dX_1}\frac{dr}{dX_2} & 0 \\ r(\cos\theta\sin\theta-\cos\theta\sin\theta)\frac{d\theta}{dX_1}\frac{dr}{dX_2} & (\sin^2\theta+\cos^2\theta)\left(\frac{dr}{dX_2}\right)^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}\otimes\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}$$

$$= [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]\begin{bmatrix} r^2\left(\frac{d\theta}{dX_1}\right)^2 & 0 & 0 \\ 0 & \left(\frac{dr}{dX_2}\right)^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}\otimes\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]\begin{bmatrix} r^2\left(\frac{\pi}{2L}\right)^2 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}\otimes\begin{bmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{bmatrix}$$

As we previously obtained, and shown again on the next page.

Bending

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \alpha \mathbf{e}_\theta \otimes \mathbf{E}_1 + \frac{\pi r}{2L}\mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3$$

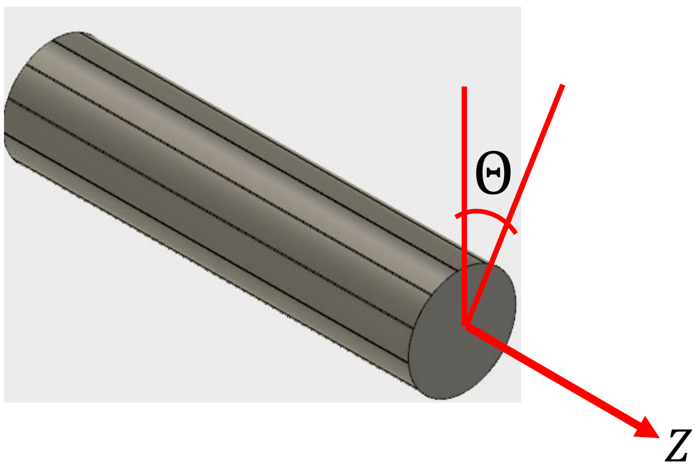
Clearly,

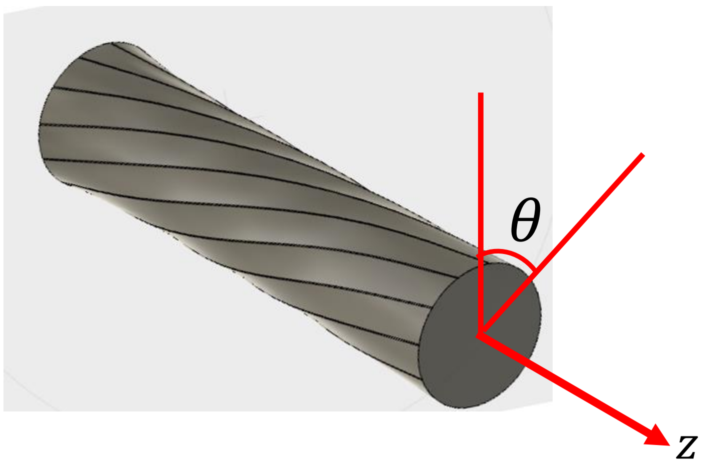
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

and the Right Stretch Tensor,

$$\mathbf{U} = \frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta \mathbf{E}_3 \otimes \mathbf{E}_3$$

Torsion of a Circular Bar Revisited





Circular Bar

* It is convenient to refer the torsion problem to cylindrical coordinates. In consistency with our practice so far, we select R, Θ and Z for the undeformed body and r, θ and z for the typical point in the spatial configuration.
* For a cylindrical bar, it is reasonable to assume that each there are no changes to the radial and axial components in any element;
* Only the angular coordinates are altered by an amount depending on the undeformed value and the axial component Z. Hence,

$$r = R, \theta = \Theta + f(Z), Z = Z$$

are the transformation equations of the deformation.

Natural/Dual Basis Adjust

- The (spatial) coordinate system in the quotient is cylindrical hence we will use the natural basis in this situation.
- The referential coordinate system in the quotient is also Cylindrical Polar, We therefor use the dual basis for the referential system as shown in the second coordinate basis, $\mathbf{E}_{\Theta}/R$
- The computations follow the same pattern as previously after the simple adjustment above:

Cauchy & Strain Tensors

* $$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta / R \\ \mathbf{E}_Z \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & r\frac{\partial f}{\partial Z} \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

```
F := {{1, 0, 0}, {0, r / R, r f[Z]}, {0, 0, 1}}

CC = Transpose[F].F
```

$$\left\{\{1, 0, 0\}, \left\{0, \frac{r^2}{R^2}, \frac{r^2 f[Z]}{R}\right\}, \left\{0, \frac{r^2 f[Z]}{R}, 1 + r^2 f[Z]^2\right\}\right\}$$

```
EE = (1 / 2) (CC - IdentityMatrix[3]) // MatrixForm
```

atrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(-1 + \frac{r^2}{R^2}\right) & \frac{r^2 f[Z]}{2R} \\ 0 & \frac{r^2 f[Z]}{2R} & \frac{1}{2} r^2 f[Z]^2 \end{pmatrix}$$

Torsional Strains

* From the above computations, we find that the Green Lagrange strains are:

$$\mathbf{E} = \frac{1}{2}\left[\left(\frac{r}{R}\right)^2 - 1\right]\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \frac{1}{2}r^2 f^2(Z)\mathbf{E}_\mathrm{Z} \otimes \mathbf{E}_\mathrm{Z} + \frac{1}{2R}\left(r^2 f(Z)\right)(\mathbf{E}_\Theta \otimes \mathbf{E}_\mathrm{Z} + \mathbf{E}_\mathrm{Z} \otimes \mathbf{E}_\Theta)$$

And the right Cauchy-Green Tensor for the deformation is:

$$\mathbf{C} = \mathbf{E}_R \otimes \mathbf{E}_R + \left(\frac{r}{R}\right)^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + [1 + r^2 f^2(Z)]\mathbf{E}_\mathrm{Z} \otimes \mathbf{E}_\mathrm{Z} + \frac{r^2 f(Z)}{\mathrm{R}}(\mathbf{E}_\Theta \otimes \mathbf{E}_\mathrm{Z} + \mathbf{E}_\mathrm{Z} \otimes \mathbf{E}_\Theta)$$

Explain the meaning of the components

Blood Flow

- **When a blood vessel is under pressure, the following deformation transformations were observed,** $r = r(R), \phi = \Phi + \psi Z\ , z = \lambda Z$ **Compute the deformation gradient, Cauchy-Green Tensor, Lagrangian. and Eulerian strain tensors for this deformation.**
- To solve this problem, first observe that both reference and spatial systems are in Cylindrical Polar Coordinates. It is easiest to arrange our deformation gradient with the bases in the way we have been doing:

Deformation Gradient

$$\mathbf{F} = [\mathbf{e}_r \quad r\mathbf{e}_\phi \quad \mathbf{e}_z] \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Phi} & \frac{\partial r}{\partial Z} \\ \frac{\partial \phi}{\partial R} & \frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Phi} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Phi}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

$$= [\mathbf{e}_r \quad \mathbf{e}_\phi \quad \mathbf{e}_z] \begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Phi} & \frac{\partial r}{\partial Z} \\ \frac{\partial \phi}{\partial R} r & \frac{r}{R}\frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial Z} r \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Phi} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Phi \\ \mathbf{E}_Z \end{bmatrix}$$

Blood Vessel

$$\mathbf{F} = [\mathbf{e}_r \quad \mathbf{e}_\phi \quad \mathbf{e}_z] \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & \frac{r}{R} & \psi r \\ 0 & 0 & \lambda \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Phi \\ \mathbf{E}_Z \end{bmatrix}$$

$$\mathbf{F}^\mathrm{T}\mathbf{F} = [\mathbf{E}_R \quad \mathbf{E}_\Phi \quad \mathbf{E}_Z] \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & \frac{r}{R} & \psi r \\ 0 & \psi r & \lambda \end{pmatrix} \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Phi \\ \mathbf{E}_Z \end{bmatrix}$$

$$\mathbf{F}^\mathrm{T}\mathbf{F} = [\mathbf{E}_R \quad \mathbf{E}_\Phi \quad \mathbf{E}_Z] \begin{pmatrix} (r')^2 & 0 & 0 \\ 0 & \frac{r^2}{R^2} & \frac{r^2\psi}{R} \\ 0 & \frac{r^2\psi}{R} & \lambda^2 + r^2\psi^2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Phi \\ \mathbf{E}_Z \end{bmatrix}$$

Lagrange Strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

$$= \begin{pmatrix} \frac{1}{2}\left(\left(\frac{dr}{dR}\right)^2 - 1\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(\frac{r^2}{R^2} - 1\right) & \frac{r^2\psi}{2R} \\ 0 & \frac{r^2\psi}{2R} & \frac{1}{2}(\lambda^2 + r^2\psi^2 - 1) \end{pmatrix}$$

Finger & Piola Tensors

* Finger Tensor, $\boldsymbol{F}\,\boldsymbol{F}^T$,

$$\boldsymbol{F}\,\boldsymbol{F}^T = \begin{pmatrix} (r')^2 & 0 & 0 \\ 0 & \psi^2 r^2 + \dfrac{r^2}{R^2} & r\lambda\psi \\ 0 & r\lambda\psi & \lambda^2 \end{pmatrix}$$

* The inverse of this also called the Piola Tensor is,

$$\boldsymbol{B} = \boldsymbol{F}^{-T}\boldsymbol{F}^{-1} = \begin{pmatrix} \dfrac{1}{(r')^2} & 0 & 0 \\ 0 & \dfrac{R^2}{r^2} & -\dfrac{R^2\psi}{r\lambda} \\ 0 & -\dfrac{R^2\psi}{r\lambda} & \dfrac{R^2\left(r^2\psi^2(r')^2 + \dfrac{r^2(r')^2}{R^2}\right)}{r^2\lambda^2(r')^2} \end{pmatrix}$$

Eulerian Strain

Eulerian strain, $\frac{1}{2}(\mathbf{I}-\mathbf{B}) =$

$$\begin{pmatrix} \frac{1}{2}\left(1-\frac{1}{(r')^2}\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(1-\frac{R^2}{r^2}\right) & \frac{R^2\psi}{2r\lambda} \\ 0 & \frac{R^2\psi}{2r\lambda} & \frac{1}{2}\left[1-\frac{R^2\left(r^2\psi^2(r')^2+\frac{r^2(r')^2}{R^2}\right)}{r^2\lambda^2(r')^2}\right] \end{pmatrix}$$

$$\boldsymbol{e}=\frac{1}{2}(\boldsymbol{I}-\boldsymbol{B})=\begin{pmatrix} \frac{1}{2}\left(1-\frac{1}{(r')^2}\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(1-\frac{R^2}{r^2}\right) & \frac{R^2\psi}{2r\lambda} \\ 0 & \frac{R^2\psi}{2r\lambda} & \frac{1}{2}\left[1-\frac{R^2\left(r^2\psi^2(r')^2+\frac{r^2(r')^2}{R^2}\right)}{r^2\lambda^2(r')^2}\right] \end{pmatrix}$$

Spherical Deformation Gradient

Find, by direct computation, the physical components of the Deformation gradient if the material (ρ, θ, ϕ) and spatial $(\varrho, \vartheta, \varphi)$ frames are referred to spherical polar coordinates

$$\mathbf{F} = \frac{\partial x^i}{\partial X^j}\left(\mathbf{g}_i \otimes \mathbf{G}^j\right)$$

$$= [\mathbf{e}_\varrho \quad \varrho \mathbf{e}_\vartheta \quad \varrho \sin\vartheta\, \mathbf{e}_\varphi] \begin{bmatrix} \frac{\partial \varrho}{\partial \rho} & \frac{\partial \varrho}{\partial \theta} & \frac{\partial \varrho}{\partial \phi} \\ \frac{\partial \vartheta}{\partial \rho} & \frac{\partial \vartheta}{\partial \theta} & \frac{\partial \vartheta}{\partial \phi} \\ \frac{\partial \varphi}{\partial \rho} & \frac{\partial \varphi}{\partial \theta} & \frac{\partial \varphi}{\partial \phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \frac{\mathbf{e}_\theta}{\rho} \\ \frac{\mathbf{e}_\phi}{\rho \sin\theta} \end{bmatrix}$$

$$= \frac{\partial \varrho}{\partial \rho}\mathbf{e}_\varrho \otimes \mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial \varrho}{\partial \theta}\mathbf{e}_\varrho \otimes \mathbf{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial \varrho}{\partial \phi}\mathbf{e}_\varrho \otimes \mathbf{e}_\phi$$
$$+ \varrho \frac{\partial \vartheta}{\partial \rho}\mathbf{e}_\vartheta \otimes \mathbf{e}_\rho + \frac{\varrho}{\rho}\frac{\partial \vartheta}{\partial \theta}\mathbf{e}_\vartheta \otimes \mathbf{e}_\theta$$
$$+ \frac{\varrho}{\rho \sin\theta}\frac{\partial \vartheta}{\partial \phi}\mathbf{e}_\vartheta \otimes \mathbf{e}_\phi + \varrho \frac{\partial \varphi}{\partial \rho}\mathbf{e}_\varphi \otimes \mathbf{e}_\rho$$
$$+ \frac{\varrho \sin\vartheta}{\rho}\frac{\partial \varphi}{\partial \theta}\mathbf{e}_\varphi \otimes \mathbf{e}_\theta + \frac{\varrho \sin\vartheta}{\rho \sin\theta}\frac{\partial \varphi}{\partial \phi}\mathbf{e}_\varphi \otimes \mathbf{e}_\phi$$

Spherical Deformation Gradient

And whose matrix representation of components is,

$$\begin{bmatrix} \frac{\partial \varrho}{\partial \rho} & \frac{1}{\rho}\frac{\partial \varrho}{\partial \theta} & \frac{1}{\rho \sin\theta}\frac{\partial \varrho}{\partial \phi} \\ \varrho\frac{\partial \vartheta}{\partial \rho} & \frac{\varrho}{\rho}\frac{\partial \vartheta}{\partial \theta} & \frac{\varrho}{\rho \sin\theta}\frac{\partial \vartheta}{\partial \phi} \\ \varrho\frac{\partial \varphi}{\partial \rho} & \frac{\varrho \sin\vartheta}{\rho}\frac{\partial \varphi}{\partial \theta} & \frac{\varrho \sin\vartheta}{\rho \sin\theta}\frac{\partial \varphi}{\partial \phi} \end{bmatrix}$$