SSG 516
Mechanics of Continua

Kinematics 04
Curvilinear Coordinates

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Summary

* The Cartesian Coordinate System and why it is easy to use.
* Partial Differentiation of position vectors to obtain basis vectors.
* Generalizations to Orthogonal Curvilinear Systems
* Deformation Gradients in Curvilinear systems.

Euclidean Point Space

The 3D Euclidean Point Space we live in is where all engineering objects of interest to us reside. This space contains point locations that can be occupied by a location in an object at a particular time. It is often of interest to be able to do several things:

1. Locate the point in an unambiguous way,
2. Relate the point to one or more other points in its vicinity, and
3. Define quantities that take up values of interest at that point.

* Temperature map of this classroom (one thousand thermometers)
* Temperature distribution, Temperature field.
* Tensor Fields

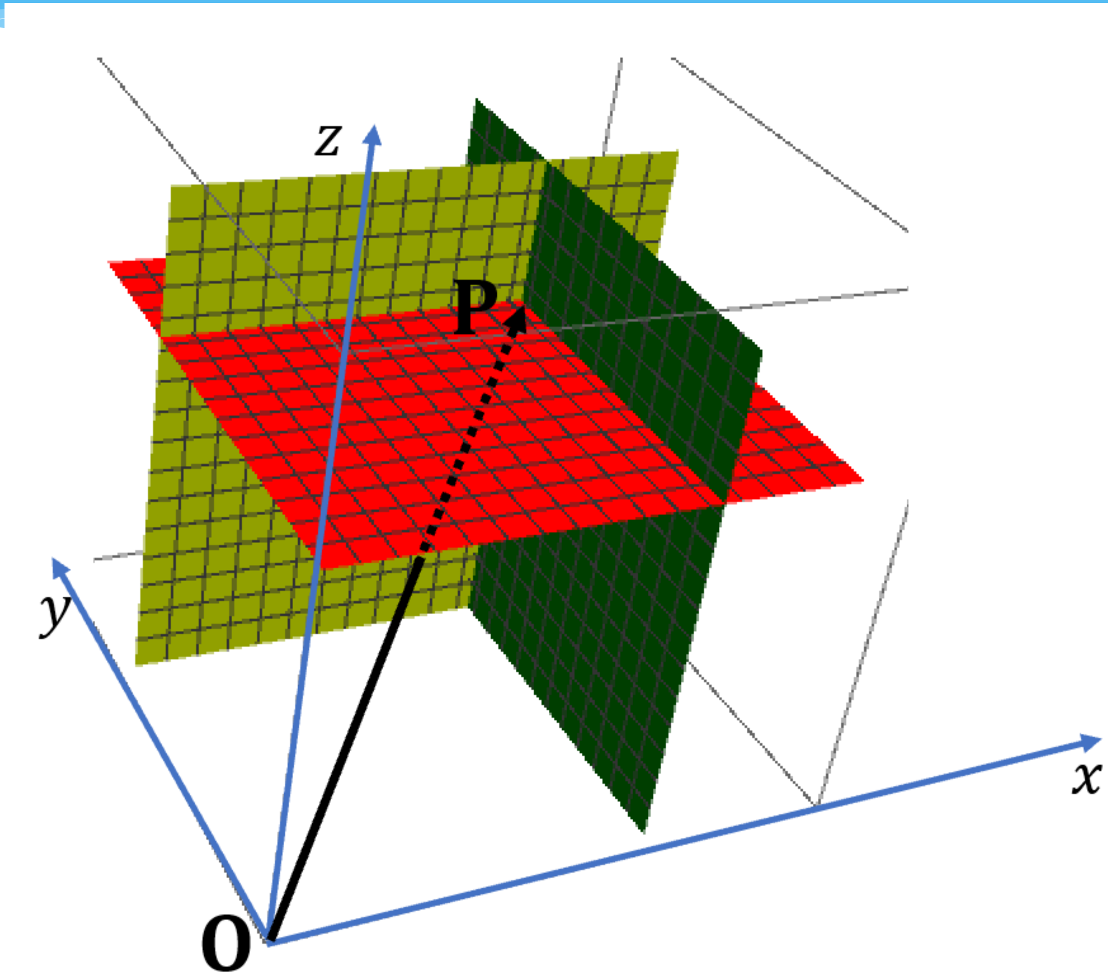
Cartesian Coordinate System

- We can refer the room to a set of Cartesian coordinates (x, y, z).
- In this system, each location is represented by three ordered numbers. The first represents the x coordinate, the second the y coordinate, and the third, the z coordinate respectively.
- Following code implements this idea

```
Cart1=ParametricPlot3D[{ 1,y,z},{y,0,1.4},{z,0,1.4},PlotStyle->Red];
Cart2=ParametricPlot3D[{ x,1,z},{x,0,1.4},{z,0,1.4},PlotStyle->Green];
Cart3=ParametricPlot3D[{ x,y,1},{x,0,1.4},{y,0,1.4},PlotStyle->Yellow];
Show[Cart1,Cart2,Cart3,PlotRange->{{0,1.5},{0,1.5},{0,1.5}}, Ticks->None]
```

Cartesian Coordinates

- In locating point $\mathbf{P}(x_1, y_1, z_1)$ above, we constructed three coordinate planes:
    - A dark coloured plane perpendicular to the x −axis,
    - A yellow plane perpendicular to the y −axis, and
    - A red plane perpendicular to the z −axis.



Position Vector

* Furthermore, we can define a vector for the point location $\mathbf{P}(x_1, y_1, z_1)$ .Such a vector is defined by joining the point $\mathbf{P}$ to the origin to form the vector **OP** represented by the line shown.
* The vector whose magnitude is defined by the length of **OP**, and whose direction is indicated by the direction of OP, a ***Position Vector.***
* We defined a vector (a member of the Euclidean Vector Space, that is now embedded in the Euclidean point space of our daily experience.
    * The latter contains just points, the former is a collection of objects that obey certain rules that make us label them "vectors".
    * This particular one is not just a vector, it is a position vector because it is the point $\mathbf{P}(x_1, y_1, z_1)$ that gave birth to it. At any other point we define by three numbers, we can also get a position vector in this simple way.

Cartesian Coordinates Properties

* Notice several things that are attractive in the Cartesian system we have described.
  * Each coordinate surface is a plane. The three defined at a particular point are respectively parallel to the three you can define at any other point.
  * Each coordinate lines: the intersection of these planes that are parallel to the axes are similarly parallel straight lines at all points in the system.
  * The basis vectors – usually defined as unit vectors along the axes, are always the same at any point in the Cartesian system. It does not matter where the point P is located, the basis vectors are the same unit vectors we define as $(\mathbf{i}, \mathbf{j}$ and $\mathbf{k})$ or $(\mathbf{e}_1, \mathbf{e}_2,$ and $\mathbf{e}_3)$ along the coordinate lines at the origin.
* These properties combine to make the Cartesian coordinate system very simple and easy to use. It is no wonder that it is the first coordinate system you get introduced to – for most people, as early as secondary school!

Cartesian Position Vector

* The position vector OP can be written simply as,

$$\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$$

* Or, more conveniently as,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

* Where we have replaced (x_1, y_1, z_1) by (x_1, x_2, x_3) so we may benefit from the compactness of the Einstein's summation convention.
* There are other hidden reasons why this coordinate system is so simple and easy to use. It may not be obvious that the simple expression of the position vector we have here is possible only in the Cartesian system.

General Position Vector

- In other coordinate systems, the position vector is usually a much more complicated function of the coordinate variables and the basis vectors. In general, if we do not assume that we are using the Cartesian system,

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2, \alpha_3, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$$

- If $\alpha_i, i = 1,2,3$ are the coordinate variables and $\mathbf{g}_i, i = 1,2,3$ are the basis vectors. The simple linear form we have for the Cartesian case, as we shall see is a rare exception and a special case. The functional for of the position vectors can be complicated.

Constancy of Basis Vectors

* A second reason that the Cartesian system is so easy, useful and pervasive is the related fact of the constancy of the basis unit vectors. To illustrate this, imagine we continue with our thought experiment to get a temperature map for the room, then we have a scalar field $T(x_1, x_2, x_3)$. If we have a vector function defined at each point, then we get a vector field $\mathbf{v}(x_1, x_2, x_3)$ .We can easily write the vector field in terms of three scalar fields that we call its components; hence, we may write,

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

Where $v_i(x_1, x_2, x_3), i = 1,2,3$ are the components of the velocity vector. The fact that the basis vectors $\mathbf{e}_i, i = 1,2,3$ neither varies temporally nor spatially means that differential and integral calculus with the Cartesian system take a particularly easy form. Differentiating the above equations, whether with respect to time or to space, we simply focus on the functions, $v_i(x_1, x_2, x_3)$ and ignore the constants $\mathbf{e}_i, i = 1,2,3$!

Varied Bases & Dimensions

- A third reason for the simplicity of the Cartesian system is in the fact that the three numbers representing the coordinates are of the same dimensionality.
- The numbers, $x_1, x_2,$ and x_3 for the coordinates of P are all lengths. They are all the same dimension.
    - There is nothing compelling you to use lengths for your coordinate variables in a coordinate system.
    - In fact, the two next most popular systems – the Spherical and Cylindrical systems use a combination of lengths and angles!
    - If you are not careful, and you use these coordinate systems just the way you do the Cartesian, your first error might be that you are adding quantities of different dimensions and units in the same expression and will be guaranteed to obtain wrong results.

Cylindrical Polar Coordinates

- In the cylindrical system, we select the three numbers that we shall use to represent a typical point P using a different strategy. We select two lengths and an angle. Since we already are quite used to the Cartesian system, let us first note that the third coordinate in the Cylindrical Polar System is shared with the Cartesian. Even if we represent it with a different symbol, note that the z-coordinate as well as the $\mathbf{k}$, $\boldsymbol{e}_3$ or $\boldsymbol{e}_z$ essentially remain the same in both Cartesian and the Cylindrical Polar system'
- Begin with our familiar Cartesian system of coordinates. We can represent the position of a point (position vector) with three coordinates x_1, x_2, x_3 $(\in R)$ such that,

Cylindrical Polar Coordinates

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

- That is, the choice of any three scalars can be used to locate a point. We now introduce a transformation (called a polar transformation) of $\{x_1, x_2\} \rightarrow \{r, \phi\}$ such that, $x_1 = r\cos\phi$, and $x_2 = r\sin\phi$ .Note also that this transformation is invertible: $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$ With such a transformation, we can locate any point in the 3-D space with three scalars $\{r, \phi, z\}$ instead of our previous set $\{x_1, x_2, x_3\}$. Our position vector is now,

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$

Cylindrical Polar Coordinates

where we define $\mathbf{e}_r \equiv \cos\phi\, \mathbf{e}_1 + \sin\phi\, \mathbf{e}_2$,$\mathbf{e}_z$ is no different from $\mathbf{e}_3$ or $\mathbf{k}$. In order to complete our triad of basis vectors, we need a third vector, $\mathbf{e}_\phi$ .In selecting $\mathbf{e}_\phi$,we want it to be such that $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ can form an orthonormal (pairwise orthogonal and individually normalized) basis. Let

$$\mathbf{e}_\phi = \xi \mathbf{e}_1 + \eta \mathbf{e}_2$$

* To satisfy our conditions, $\mathbf{e}_\phi \cdot \mathbf{e}_r = 0$,$\mathbf{e}_\phi \cdot \mathbf{e}_z = 0$ (automatically satisfied by not choosing a third coordinate) and $\sqrt{\xi^2 + \eta^2} = 1$.

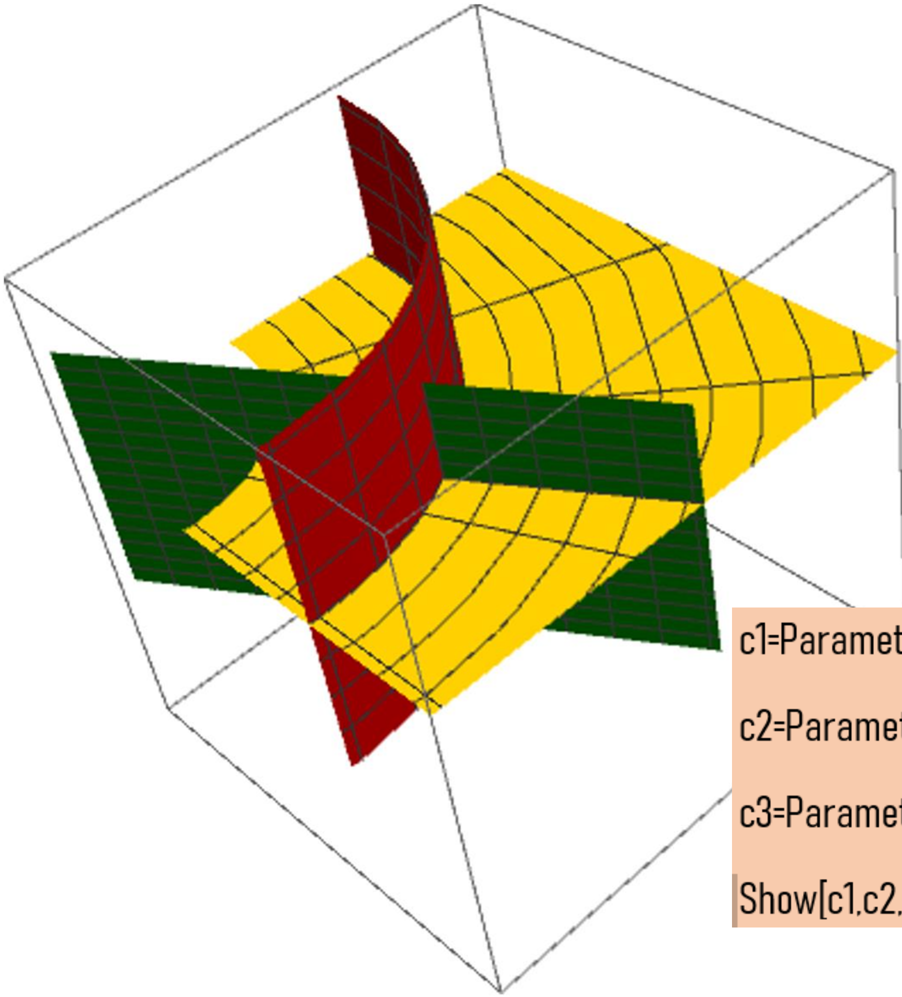
Cylindrical Polar Coordinates

- It is easy to see that $\boldsymbol{e}_\phi \equiv -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$ satisfies these requirements. $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ forms an orthonormal (that is, *each member has unit magnitude and they are pairwise orthogonal*) triad just like $\mathbf{e}_i, i = 1,2,3$.
- The transformation we have just described can be given a geometric interpretation. In either case, it is the definition of the ***Cylindrical Polar coordinate system***.
- Unlike our Cartesian system, we note that $\{\mathbf{e}_r(\phi), \mathbf{e}_\phi(\phi), \mathbf{e}_z\}$ as the first two of these are not constants but vary with angular orientation. $\mathbf{e}_z$ remains a constant vector as in the Cartesian case.

Geometrical Interpretation

* The coordinate system just described requires us, as before, to select three ordered numbers to uniquely represent a point in the Euclidean point space. The first is a length, r, the second, an angle ϕ, and the third, a length, z. These are the coordinate variables.
* Recall that in the Cartesian case, the coordinate planes have equations, $x_1 = const$, $x_2 = const$, and $x_3 = const$ giving us three planes that intersect at the point defined by those three values of the constants used.
* In a similar way, the coordinate planes in the Cylindrical Polar are: $r = const$ describing a cylinder with the z-axis as its axis, $\phi = const$ describing a plane through the axis and another plane, $z = const$ describing a plane that is perpendicular to the cylinder axis. This is as shown in the figure below:

Cylindrical Polar Coordinates



```
c1=ParametricPlot3D[{ Sin[ϕ], Cos[ϕ],z},{ ϕ,0,Pi},{z,1.5,3.5},PlotStyle->Red];

c2=ParametricPlot3D[{r Sin[Pi/3],r Cos[Pi/3],z},{r,0,2},{z,1.5,2.3},PlotStyle->Green];

c3=ParametricPlot3D[{r Sin[θ],r Cos[ϕ],2},{ ϕ,0,2 Pi},{r,0.5,2.5},PlotStyle->Yellow];

Show[c1,c2,c3, PlotRange->{{0,1.4},{0,1.5},{1,2.5}},Ticks->None]
```

Differentiate Position Vectors

$$\mathbf{r} = x_i \boldsymbol{e}_i \; => \; \frac{\partial \mathbf{r}}{\partial x_i} = \boldsymbol{e}_i$$

And for the Cylindrical,

$$\mathbf{r} = r\boldsymbol{e}_r(\phi) + z\boldsymbol{e}_z \; =>$$
$$\frac{\partial \mathbf{r}}{\partial r} = \boldsymbol{e}_r(\phi);$$
$$\frac{\partial \mathbf{r}}{\partial \phi} = r\frac{\partial}{\partial \phi}(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) = r\mathbf{e}_\phi(\phi)$$
$$\frac{\partial \mathbf{r}}{\partial z} = \boldsymbol{e}_z$$

Mistakes to avoid

* Two easy mistakes that can be made are :
  * That the Cylindrical position vector is $r\boldsymbol{e}_r(\phi) + \phi\mathbf{e}_\phi + z\mathbf{e}_z$ which is a simplistic copy of the Cartesian formula. This is wrong in at least two ways. For one thing, it is dimensionally incorrect because the unit of the middle basis component is an angle while the other components are measuring lengths. Secondly, we cannot obtain the Cartesian result from this via a coordinate transformation.
  * That the basis vectors are constants. They are NOT all constants. $\boldsymbol{e}_r(\phi)$ and $\mathbf{e}_\phi(\phi)$ are both functions of ϕ unlike in the Cartesian case, but $\mathbf{e}_z$ is a constant like the Cartesian case!

Spherical Coordinates.

* The spherical Polar coordinate system selects its three ordered triplets with yet another strategy. This can be explained by the same transformation route we started. Continuing further with our transformation, we may again introduce two new scalars such that $\{r, z\} \to \{\rho, \theta\}$ in such a way that the position vector,

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = \rho \sin\theta\, \mathbf{e}_r + \rho \cos\theta\, \mathbf{e}_z \equiv \rho\mathbf{e}_\rho$$

* Here, $r = \rho \sin\theta$, $z = \rho \cos\theta$. As before, we can use three scalars, $\{\rho, \theta, \phi\}$ instead of $\{r, \phi, z\}$. In comparison to the original Cartesian system we began with, we have that,
* $$\begin{aligned} \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} &= \rho \sin\theta\, \mathbf{e}_r + \rho \cos\theta\, \mathbf{e}_z \\ &= \rho \sin\theta\, (\cos\phi\, \mathbf{i} + \sin\phi\, \mathbf{j}) + \rho \cos\theta\, \mathbf{k} \\ &= \rho \sin\theta \cos\phi\, \mathbf{i} + \rho \sin\theta \sin\phi\, \mathbf{j} + \rho \cos\theta\, \mathbf{k} \\ &\equiv \rho\mathbf{e}_\rho \end{aligned}$$

Spherical Coordinates

* it is clear that the unit vector

$$\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

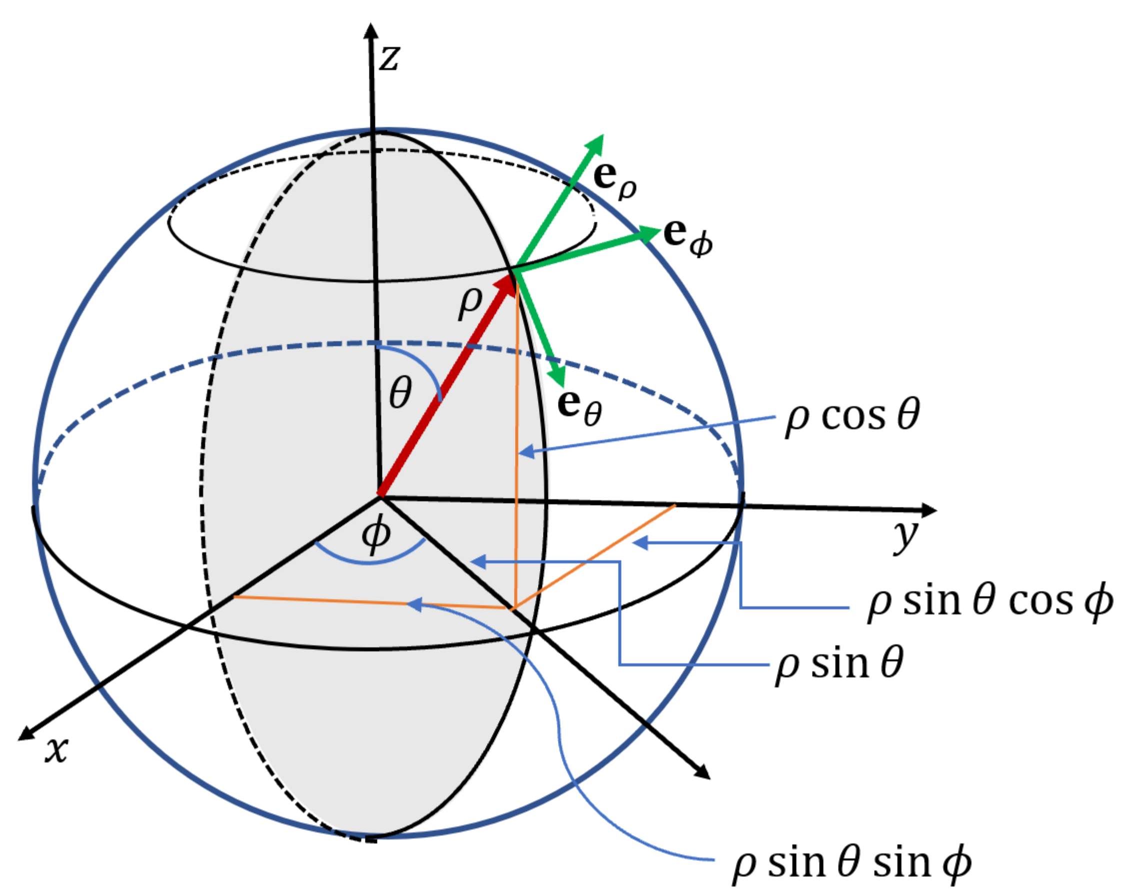
.Again, we introduce the unit vector, $\mathbf{e}_\theta \equiv \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$ and retain $\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$ as before**.** It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho, \phi, \theta\}$ directly by the transformation equations, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\theta$ .The orthonormal set of basis for the $\{\rho, \theta, \phi\}$ system is $\left\{\boldsymbol{e}_\rho(\theta,\phi), \boldsymbol{e}_\theta\,(\theta,\phi), \boldsymbol{e}_\phi\,(\phi)\right\}$

$$\mathbf{r}(\rho,\theta,\phi) \equiv \rho\mathbf{e}_\rho(\theta,\phi)$$

Showing that the position vector depends on the three coordinate variables representing the radial distance, ρ, from the origin on the azimuthal (great circle, longitudinal) plane inclined at an angle θ to the meridian plane $(x - z)$, with a polar angle ϕ as shown below:

Spherical Polar

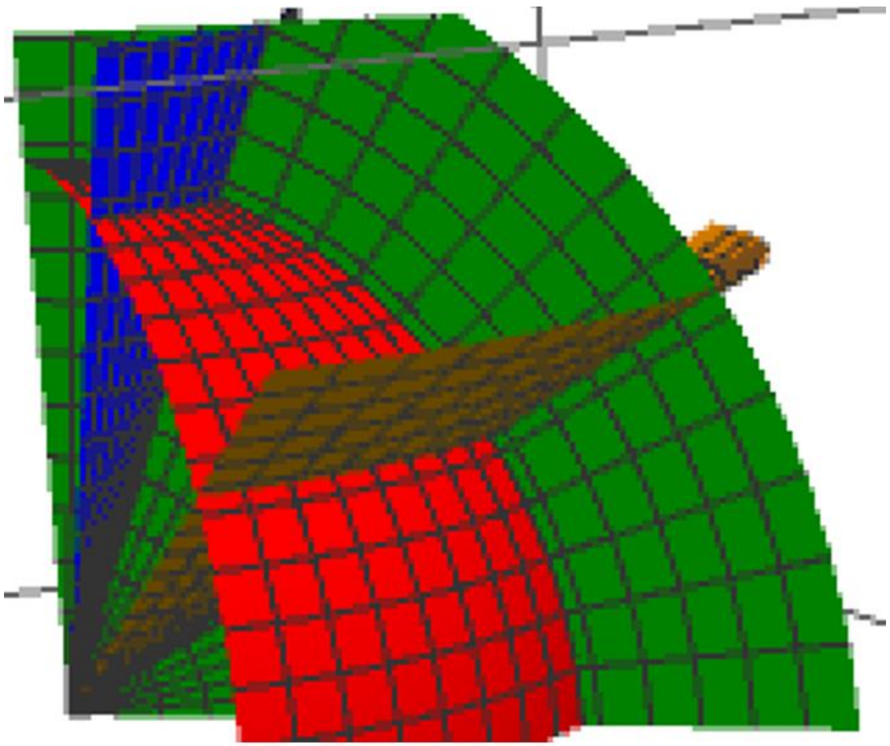
* The orthonormal basis vectors are shown at the point of interest. The projection of the radial distance to the "equatorial" plane is also shown.



Spherical Coordinates

- Coordinate surfaces

Here we have two points on the same sphere (equal radii) and the same θ but with two values of θ. The coordinate surfaces are spheres for $\rho = const$; planes through the vertical axis for $\phi = const$ and cones through the origin for $\theta = const$. As before, the coordinate surfaces are orthogonal as well as the tangents to the coordinate lines that are at the intersections of the coordinate planes



Other Coodinate Systems

* There are many other ways of selecting three ordered scalars to create a coordinate system. The ones we have seen so far are all orthogonal coordinate systems because the coordinate planes meet at all points at right angles. Other orthogonal coordinate systems that have engineering significance include:
    1. Parabolic and Parabolic Cylindric
    2. Elliptic Cylinder, Elliptic, Bipolar,
    3. Confocal,
    4. Prolate and Oblate spheroidal, Toroidal

  The strategy of definition is similar in each case. A few:

Parabolic Cylinder Coordinates

Parabolic Cylinder Coordinates (ξ, η, z). Here, the first two are square roots of length while the third scalar is length. $x = \xi\eta, y = \frac{1}{2}(\xi^2 - \eta^2)$ *and* $z = z$, and the position vector is,

$$\xi\eta\mathbf{e}_1 + \frac{\mathbf{e}_2}{2}(\xi^2 - \eta^2) + z\mathbf{e}_3$$

The basis vectors, $\eta\mathbf{e}_1 + \xi\mathbf{e}_2, \xi\mathbf{e}_1 - \eta\mathbf{e}_2$, and $\mathbf{e}_3$ obtained by differentiation with respect to the coordinate variables can be seen to be orthogonal, though not orthonormal. These can easily be normalized to become orthonormal.

Parabolic Coordinates

Parabolic Coordinates. (ξ, η, ϕ) Here, the first two are square roots of length while the third scalar is an angle. $x = \xi\eta\cos\phi\,, y = \xi\eta\sin\phi\,, z = \frac{1}{2}(\xi^2 - \eta^2)$ and the position vector is,

$$\xi\eta\cos\phi\,\mathbf{e}_1 + \xi\eta\sin\phi\,\mathbf{e}_2 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_3$$

The basis vectors are orthogonal and they are: $\eta\cos\phi\mathbf{e}_1 + \eta\sin\phi\,\mathbf{e}_2 + \xi\mathbf{e}_3$, $\xi\cos\phi\mathbf{e}_1 + \xi\sin\phi\,\mathbf{e}_2 - \eta\mathbf{e}_3, -\xi\eta\sin\phi\,\mathbf{e}_1 + \xi\eta\cos\phi\,\mathbf{e}_2$

Natural & Dual Bases

Non orthogonal systems exist but are not very commonly used in Engineering. Apart from basis vectors not necessarily normalized, they are also not orthogonal.

In general, a coordinate system where, unlike the Cartesian, the coordinate lines are curves rather than straight lines is said to be curvilinear. The explicit set of systems we have here are orthogonal and curvilinear systems.

Dual Bases

We can generalize and represent such system as having coordinate variables, $\alpha_i, i = 1,2,3$ are the coordinate variables and $\mathbf{g}_i, i = 1,2,3$ formed by differentiating the position vector

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2, \alpha_3, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$$

Are called covariant basis vectors. For OCC, we can always find another set of basis vectors, $\mathbf{g}^j, j = 1,2,3$ such that,

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

The set $\mathbf{g}^j, j = 1,2,3$ related in the reciprocity rule shown is said to be a reciprocal basis. Using these two basis instead of just one, it is possible to greatly reduce the computational complexity associated with the variability in the basis vectors with the coordinate variables.

Spherical Deformation Gradient

Find, by direct computation, the physical components of the Deformation gradient if the material (ρ, θ, ϕ) and spatial $(\varrho, \vartheta, \varphi)$ frames are referred to spherical polar coordinates

$$\mathbf{F} = \frac{\partial x^i}{\partial X^j}(\mathbf{g}_i \otimes \mathbf{G}^j)$$

$$= [\mathbf{e}_\varrho \quad \varrho\mathbf{e}_\vartheta \quad \varrho\sin\vartheta\,\mathbf{e}_\varphi]\begin{bmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{\partial\varrho}{\partial\theta} & \frac{\partial\varrho}{\partial\phi} \\ \frac{\partial\vartheta}{\partial\rho} & \frac{\partial\vartheta}{\partial\theta} & \frac{\partial\vartheta}{\partial\phi} \\ \frac{\partial\varphi}{\partial\rho} & \frac{\partial\varphi}{\partial\theta} & \frac{\partial\varphi}{\partial\phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \frac{\mathbf{e}_\theta}{\rho} \\ \frac{\mathbf{e}_\phi}{\rho\sin\theta} \end{bmatrix}$$

$$= \frac{\partial\varrho}{\partial\rho}\mathbf{e}_\varrho \otimes \mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\varrho}{\partial\theta}\mathbf{e}_\varrho \otimes \mathbf{e}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial\varrho}{\partial\phi}\mathbf{e}_\varrho \otimes \mathbf{e}_\phi + \varrho\frac{\partial\vartheta}{\partial\rho}\mathbf{e}_\vartheta \otimes \mathbf{e}_\rho + \frac{\varrho}{\rho}\frac{\partial\vartheta}{\partial\theta}\mathbf{e}_\vartheta \otimes \mathbf{e}_\theta$$
$$+ \frac{\varrho}{\rho\sin\theta}\frac{\partial\vartheta}{\partial\phi}\mathbf{e}_\vartheta \otimes \mathbf{e}_\phi + \varrho\sin\vartheta\frac{\partial\varphi}{\partial\rho}\mathbf{e}_\varphi \otimes \mathbf{e}_\rho + \frac{\varrho\sin\vartheta}{\rho}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\varphi \otimes \mathbf{e}_\theta$$
$$+ \frac{\varrho\sin\vartheta}{\rho\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\varphi \otimes \mathbf{e}_\phi$$

Spherical Deformation Gradient

And whose matrix representation of components is,

$$\begin{bmatrix} \frac{\partial \varrho}{\partial \rho} & \frac{1}{\rho}\frac{\partial \varrho}{\partial \theta} & \frac{1}{\rho \sin\theta}\frac{\partial \varrho}{\partial \phi} \\ \varrho\frac{\partial \vartheta}{\partial \rho} & \frac{\varrho}{\rho}\frac{\partial \vartheta}{\partial \theta} & \frac{\varrho}{\rho \sin\theta}\frac{\partial \vartheta}{\partial \phi} \\ \varrho \sin\vartheta\frac{\partial \varphi}{\partial \rho} & \frac{\varrho \sin\vartheta}{\rho}\frac{\partial \varphi}{\partial \theta} & \frac{\varrho \sin\vartheta}{\rho \sin\theta}\frac{\partial \varphi}{\partial \phi} \end{bmatrix}$$

Toroidal Coordinates

- The coordinate variables are: u, v, ϕ such that,

$$x_1 = \frac{a \sinh v \cos \phi}{\cosh v \ - \cos u}, 0 \leq u < 2\pi$$

$$x_2 = \frac{a \sinh v \sin \phi}{\cosh v \ - \cos u}, -\infty < v < \infty$$

$$x_3 = \frac{a \sin u}{\cosh v \ - \cos u}, 0 \leq \phi < 2\pi$$

- By setting the coordinate variables to constants, you will find that the coordinate planes are spheres, tores and planes.
- *Write Mathematica codes to make this happen*

Bipolar Coordinates

- The coordinate variables are: u, v, z such that,

$$x_1 = \frac{a \sinh v}{\cosh v - \cos u}, 0 \leq u < 2\pi$$

$$x_2 = \frac{a \sin u}{\cosh v - \cos u}, -\infty < v < \infty$$

$$x_3 = z, -\infty < z < \infty$$

- *By* setting the coordinate variables to constants, you will find that the coordinate planes are cylinders, cylinders and Planes.
- *Write Mathematica codes to make this happen*

Confocal Ellipsoidal Coordinates

- The coordinate variables are: u, v, w such that,

$$x_1^2 = \frac{(a^2 - u)(a^2 - v)(a^2 - w)}{(a^2 - b^2)(a^2 - c^2)}, u < c^2 < b^2 < a^2$$

$$x_2^2 = \frac{(b^2 - u)(b^2 - v)(b^2 - w)}{(b^2 - a^2)(b^2 - c^2)}, c^2 < v < b^2 < a^2$$

$$x_3^2 = \frac{(c^2 - u)(c^2 - v)(c^2 - w)}{(c^2 - a^2)(c^2 - b^2)}, c^2 < b^2 < w < a^2$$

Prolate Spheroidal Coordinates

* The coordinate variables are: u, v, ϕ such that,

$$x_1 = a \sinh u \sin v \cos \phi \,, u \geq 0$$
$$x_2 = a \sinh u \sin v \sin \phi \,, 0 \leq v \leq \pi$$
$$x_1 = a \cosh u \cos v \,, 0 \leq \phi < 2\pi$$

* Coordinate surfaces are Prolate ellipsoids, Two sheet hyperboloids, and Planes

Oblate Spheroidal Coordinates

* The coordinate variables are: ξ, η, ϕ such that,

$$x_1 = a\cosh\xi\cos\eta\cos\phi\,,\xi \geq 0$$

$$x_2 = a\cosh\xi\cos\eta\sin\phi\,,-\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2}$$

$$x_1 = a\sinh\xi\sin\eta\,,0 \leq \phi \leq 2\pi$$

Find the coordinate planes that are:

Oblate spheroids, One sheet hyperbolas and planes