SSG 516
Mechanics of Continua

Kinematics 03
Strain Tensors

Omotayo Fakinlede
oafak.com

Summary

* The meaning of strain. Separating Rigid Body motions from shape changes
* Deformation Gradient contains too much information: Needs to be decomposed.
* Lagrangian & Eulerian Strain functions. Seth-Hill functions generalizing strains
* Simple Deformation Examples.

Points, Areas & Volumes

* Deformation function $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ gives information on the location of a material point as a result of the deformation at a particular time t.
* The deformation gradient

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j = \text{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t)$$

Gives information on the transformation of small material vectors in the same process.

* Areas transform by the tensor $\mathbf{F}^{\mathbf{c}}(\mathbf{X}, t)$, the cofactor of $\mathbf{F}$ while volumes transform by $J = \det \mathbf{F}$

Strain

* Strain is our attempt to quantify relative displacements and changes in orientations of material line elements as a result of the deformation.
* A wholesale movement of the entire element itself is a transformation that does not qualify as strain. We call such transformations Rigid Body Motions. Examples are:
  * Rotation: of all material points in the element about an axis
  * Translation: of all the material element by the same amount in a given direction.

Removal of RBM

- A proper strain function must satisfy two conditions:
  1. Two deformations, differing only by rigid body motions represent the same strained system.
  2. When $\mathbf{F} = \mathbf{I}$ the identity tensor, the strain function must vanish everywhere. Why is this?
- Many strain functions can be defined in so far as they satisfy the above conditions.
- A number have been used successfully in certain situations.

Referential & Spatial Strain

* The most successful strain functions are
  1. Green-Lagrange Strain Tensor
  $$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$
  where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the right Cauchy Green Tensor
  2. Euler-Almansi Strain Tensor
  $$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$$
  where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ the left Cauchy-Green Tensor
* We have shown that $\mathbf{C} = \mathbf{U}^2$ is a material tensor while $\mathbf{B} = \mathbf{V}^2$ is spatial. Consequently, $\mathbf{E}$ is a material tensor field while $\mathbf{e}$ is spatial.

Generalized Strain

* It was shown by Seth and Hill that these strain systems are actually special cases of more generalized strain tensors. Hence we have the two sets of strain functions:

1. $\frac{1}{m}(\mathbf{U}^m - \mathbf{I})$ for $m \neq 0,\ \log_{\mathrm{e}} \mathbf{U}, m = 0$

2. $\frac{1}{m}(\mathbf{V}^m - \mathbf{I})$for $m \neq 0,\ \log_{\mathrm{e}} \mathbf{V}, m = 0$

Lagrange and Eulerian

* The two special cases can now be seen as,
  1. Lagrangian : $\frac{1}{m}(\mathbf{U}^m - \mathbf{I})$ for $m = 2$
  2. Eulerian : $\frac{1}{m}(\mathbf{V}^m - \mathbf{I})$ for $m = -2$

The Stretch Tensor

* The Polar decomposition Theorem immediately shows why the deformation gradient cannot be a proper measure of strain.
* Consider the expression, $\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}, \mathbf{F}_2 = \mathbf{R}_2\mathbf{U}$ so that the only difference between the two deformation gradients is the fact that the rotations are different but the stretch tensors are the same.
* The strains should be the same but they are represented by two different deformation gradients.

The Stretch Tensor

- Consider two infinitesimal material vectors, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ and subject the material in which they are placed to the deformation gradient $\mathbf{F}$. Clearly, the images of these two elements in the spatial state will be:

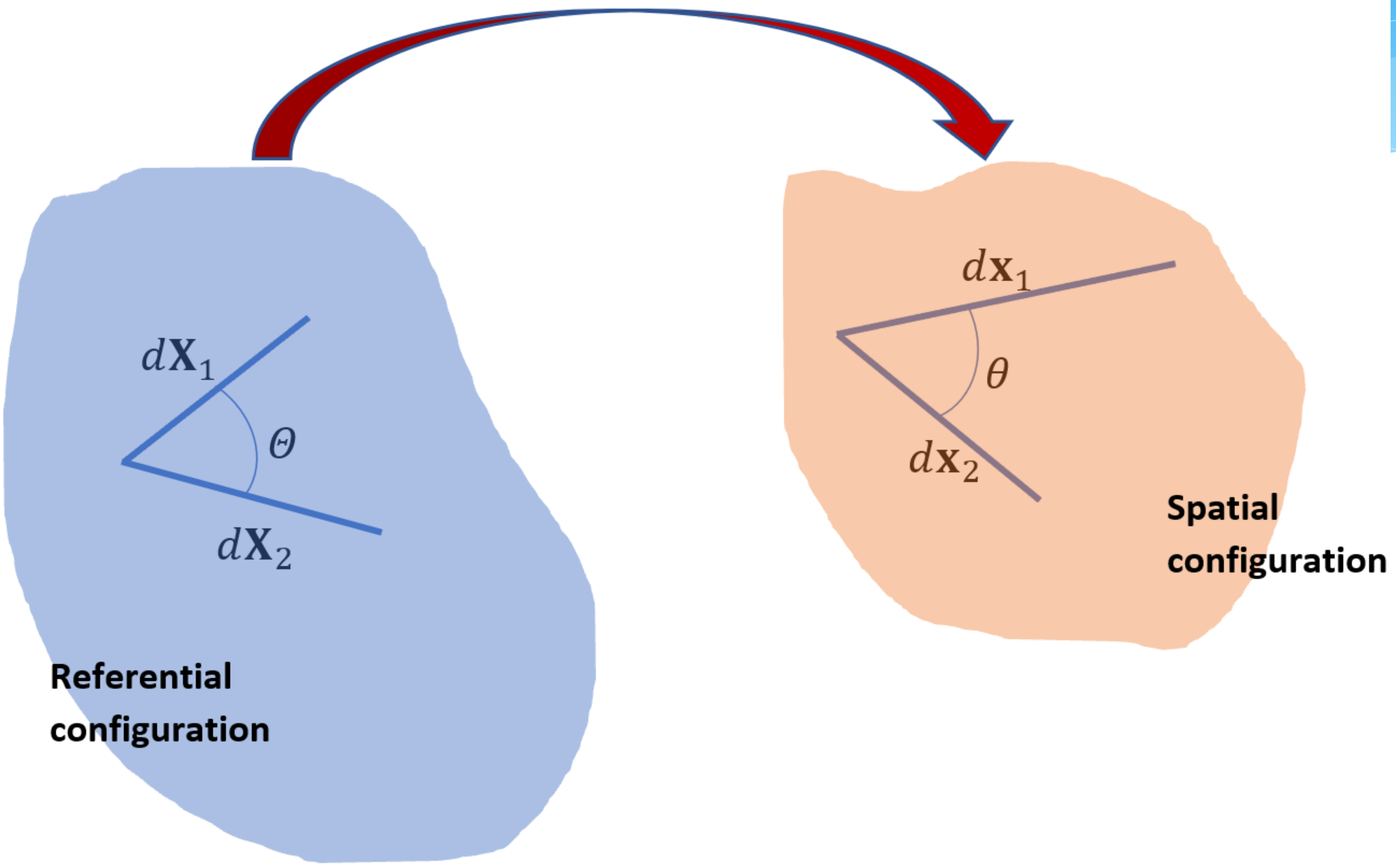
$$d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1 \text{and } d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$$

- We now proceed to find the magnitude of the image vectors by taking the scalar products as follows:

$$\begin{aligned} d\mathbf{x}_1 \cdot d\mathbf{x}_2 &= \mathbf{F}d\mathbf{X}_1 \cdot \mathbf{F}d\mathbf{X}_2 \\ &= \mathbf{RU}d\mathbf{X}_1 \cdot \mathbf{RU}d\mathbf{X}_2 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{R}^{\mathrm{T}}\mathbf{RU}d\mathbf{X}_2 \\ &= \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2 \end{aligned}$$

Upon recalling that the transpose of a rotation is its inverse.

Deformation of Lines & Angles



Stretch

$$d\mathbf{x}_1 \cdot d\mathbf{x}_2 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2$$

* So that, if both vectors are the same, we have that,

$$d\mathbf{x}_1 \cdot d\mathbf{x}_1 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_1$$

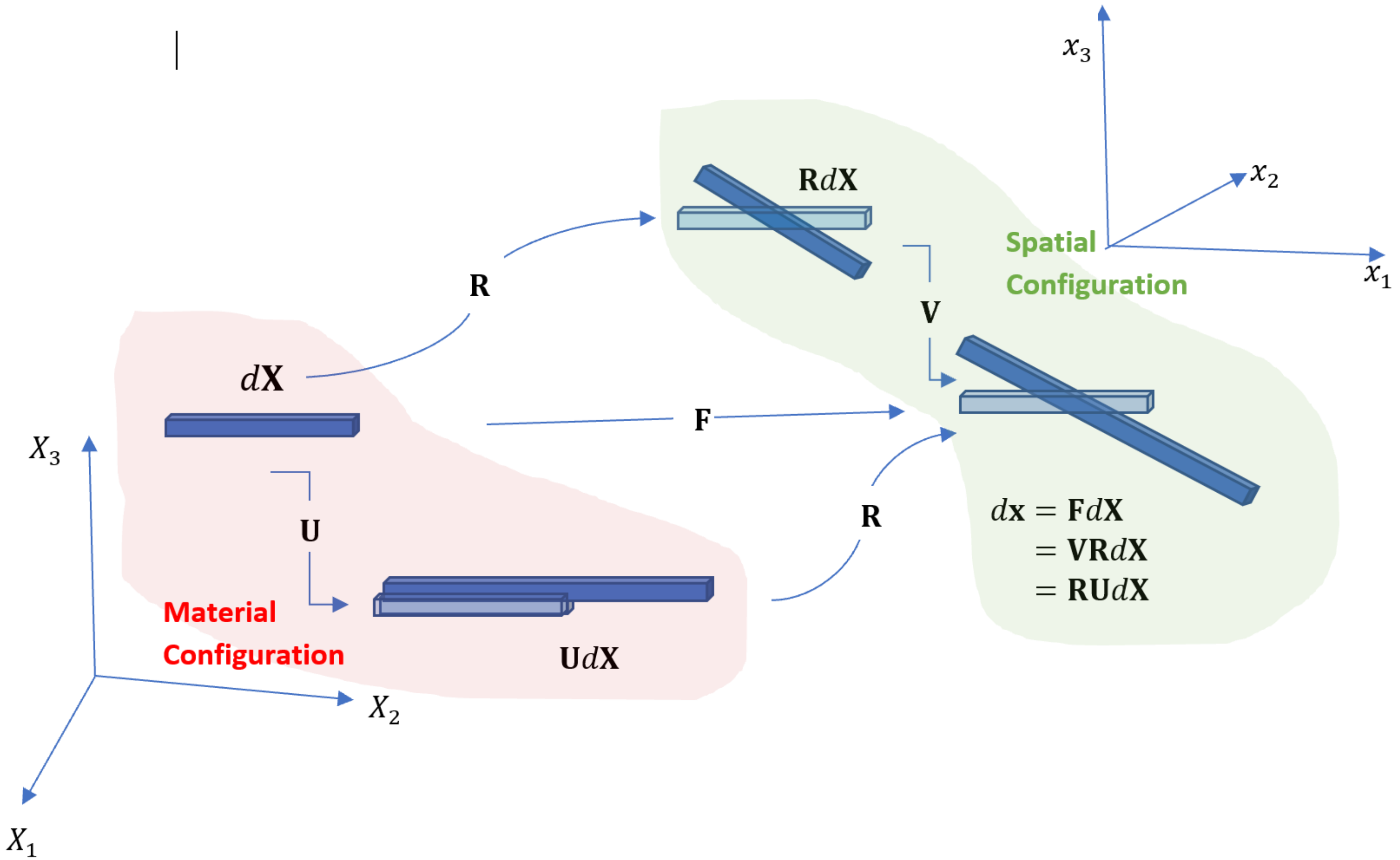
* And after taking square roots, we see that,

$$\|d\mathbf{x}\| = \|\mathbf{U}d\mathbf{X}\|$$

* Which tells us that the ***magnitude*** of the spatial vector is governed by a transformation of the material vector, not by the deformation gradient, but by the **right stretch tensor**.
* It is left as an exercise for you to make the argument that,

$$\|d\mathbf{X}\| = \left\|\mathbf{V}^{-1}d\mathbf{x}\right\|$$

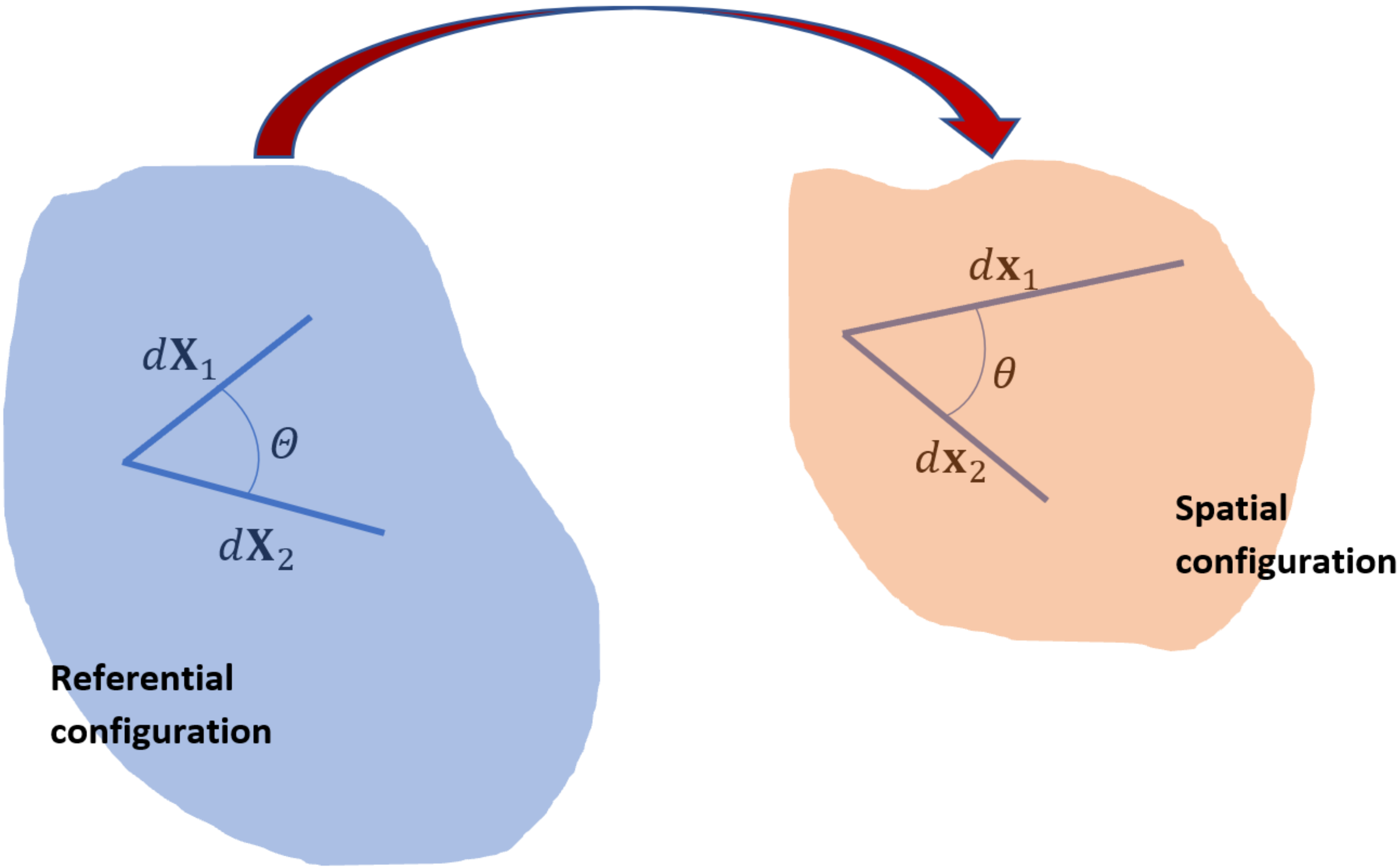
Polar Decomposition of Deformation Gradient



Normal and Shear Strains

* The above arguments helps us clarify issues with normal strains on infinitesimal elements. Once we know the Right stretch tensor, we can find the new length of any fibre.
* In shear strain, we are interested in the changes in the angles between infinitesimal elements.

Deformation of Lines & Angles



Shear Strain

* In the referential configuration, the angle between the line elements, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ is,

$$\Theta = \cos^{-1}\left(\frac{d\mathbf{X}_1 \cdot d\mathbf{X}_2}{\|d\mathbf{X}_1\|\|d\mathbf{X}_2\|}\right)$$

To find the angle between any two elements in the spatial configuration we simply recall that the angle we seek is

$$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\|d\mathbf{x}_2\|}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\|\mathbf{U}d\mathbf{X}_1\|\|\mathbf{U}d\mathbf{X}_2\|}\right)$$

Shear Strain

* To find shear strain, we look at two elements in the referential configuration that are at right angles.
* Shear strain is DEFINED as the change in the right angle between these two elements: We subtract the new angle θ in radians from $\frac{\pi}{2}$.

Right Stretch Tensor

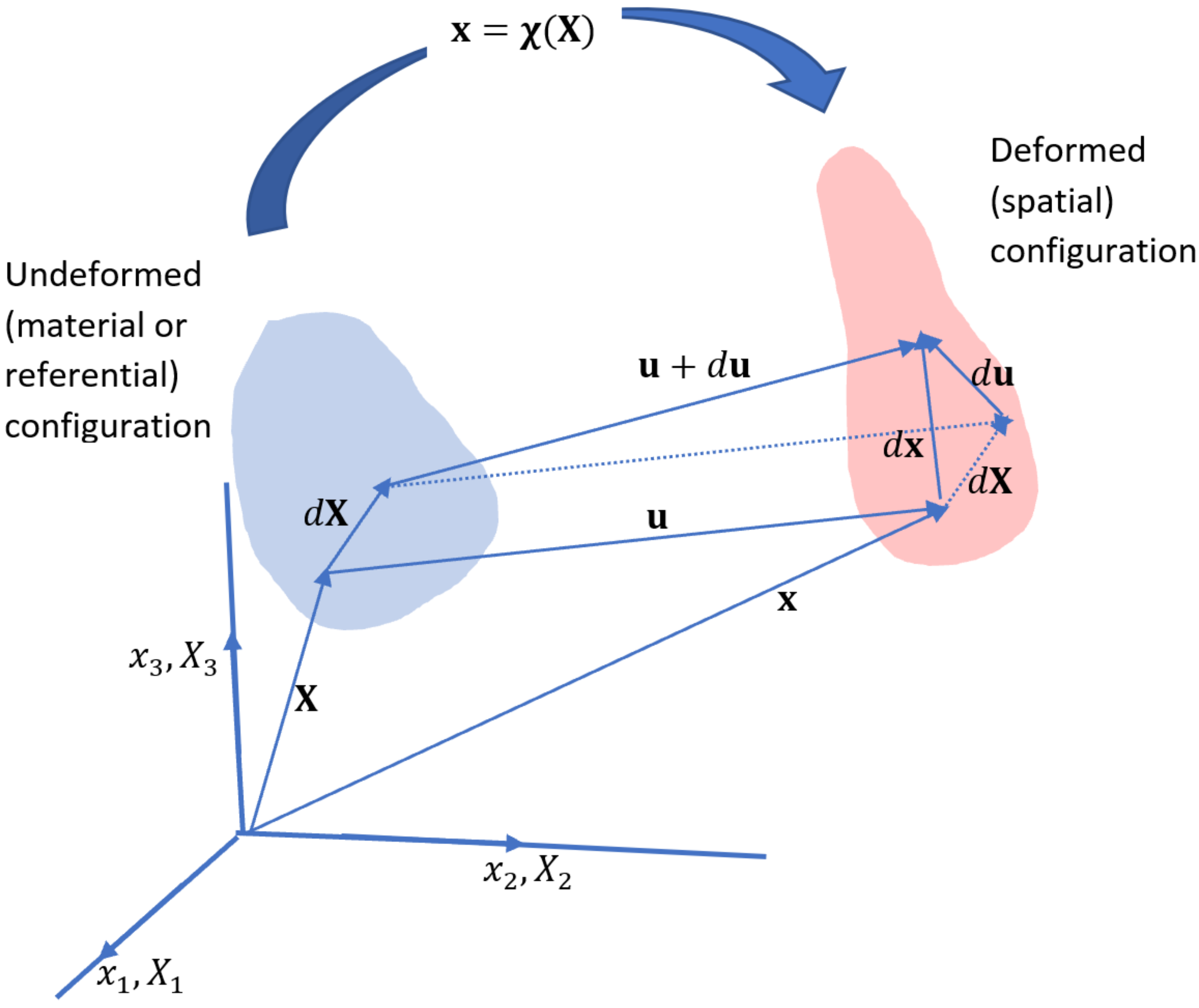
- The above expressions show that the right stretch tensor governs, not only the transformation of lengths, but also the transformation of angles.
- When you add this knowledge to the fact that the strain must vanish once the deformation gradient is either a rotation or an identity, it becomes clear why successful material strain functions take the Seth-Hill forms of:

$$\frac{1}{m}(\mathbf{U}^m - \mathbf{I}) \text{ for } m \neq 0, \log_e \mathbf{U}\,, m = 0$$

Strain the Displacement Function

* Consider a material that has been subjected to a deformation as shown below. Here, for simplicity, we refer both configurations to the same Cartesian origin and let the two coordinate systems coincide.
* Let a point $\mathbf{P}$ be located at the point $\mathbf{X}$ in the material configuration be such that it transforms to the point $\mathbf{p}$ located at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the spatial.
* Consider the vector $\mathbf{u} = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X}$ Let us take the material gradient of this equation and write,

$$\mathbf{H} \equiv \mathbf{Grad\,u} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}) - \mathbf{I} = \mathbf{F} - \mathbf{I}$$


$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$
Deformed (spatial) configuration
Undeformed (material or referential) configuration
$\mathbf{u} + d\mathbf{u}$
$d\mathbf{u}$
$d\mathbf{x}$
$d\mathbf{X}$
$d\mathbf{X}$
$\mathbf{u}$
$\mathbf{x}$
$\mathbf{X}$
x_3, X_3
x_2, X_2
x_1, X_1

Small Strains

In component form, we can write,

$$H_{ij} = \frac{\partial u_i}{\partial X_j}$$

Upon noting that the identity tensor, in Cartesian coordinates has the Kronecker delta as its coefficients, we can therefore write,

$$F_{ij} = \delta_{ij} + H_{ij}$$

Again, in Cartesian, the transpose is simply the reversal of the indices. Hence we can write,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left((\mathbf{H} + \mathbf{I})^{\mathrm{T}}(\mathbf{H} + \mathbf{I}) - \mathbf{I}\right)$$

Component Form

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left((\mathbf{H}+\mathbf{I})^{\mathrm{T}}(\mathbf{H}+\mathbf{I}) - \mathbf{I}\right)$$
$$= \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right)$$

In component form as,

$$E_{ij} = \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_i}{\partial X_j}\right)$$

If we can neglect second-order terms, and realizing that the spatial is indistinguishable from the material, then,

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

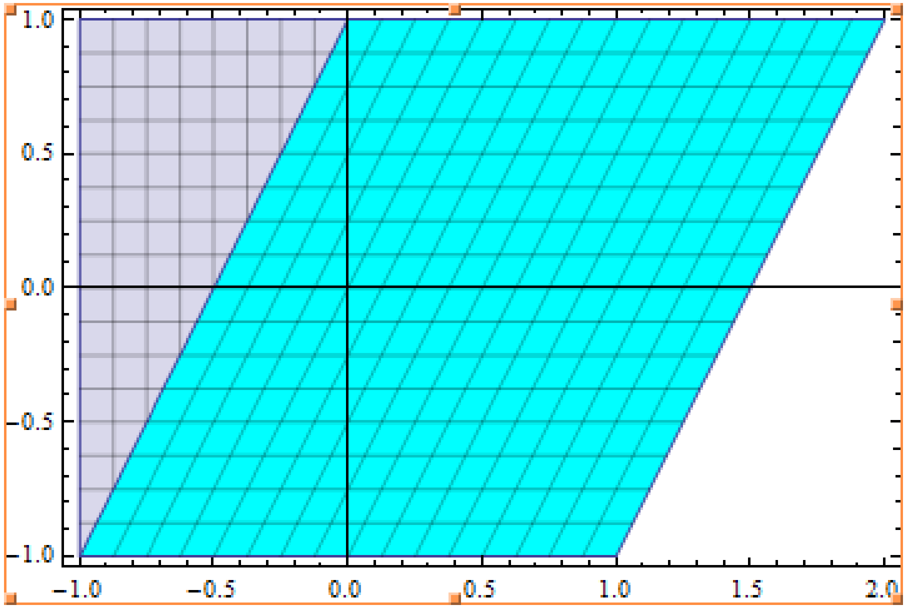
Deformation Examples: Uniform Shear

```
myMap[X1_, X2_] := {0.5 + X1 + 0.5 X2 , X2} // Flatten

initialConfig = ParametricPlot[{X1, X2}, {X1, -1, 1}, {X2, -1, 1}];

deformedConfig = ParametricPlot[myMap[X1, X2], {X1, -1, 1},
  {X2, -1, 1} , MeshShading → {{Cyan, Cyan}}];

Show[initialConfig, deformedConfig, PlotRange → All]
```



Deformation Gradient

- The Deformation gradient here is easily calculated by hand. Do this to ensure you don't get lost in the mechanical computation and lose the context:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \mathbf{x} = (0.5 + X_1 + 0.5X_2)\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$$

for the element occupying $X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$ initially.

Clearly, $\frac{\partial x_1}{\partial X_1} = 1, \frac{\partial x_1}{\partial X_2} = 0.5, \frac{\partial x_1}{\partial X_3} = 0$ and $\frac{\partial x_2}{\partial X_2} = \frac{\partial x_3}{\partial X_3}$ with all other components of the deformation gradient vanishing.

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the matrix of the deformation gradient components

Use Mathematica to find $\mathbf{F}, \mathbf{C}, \mathbf{E}, \mathbf{U}$ and $\mathbf{R}$. Look at my code for the homework example.

Home Work

In Cartesian Coordinates, the deformation of a rectangular sheet is given by: $= (\boldsymbol{\lambda_1 X_1} + \boldsymbol{k_1 X_2})\mathbf{g}_1 + (\boldsymbol{k_2 X_1} + \boldsymbol{\lambda_2 X_2})\mathbf{g}_2 + \boldsymbol{\lambda_3 X_3}\mathbf{g}_3$ **Compute the tensors $\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{E}, \boldsymbol{U}$ and $\boldsymbol{R}$. Show that $\boldsymbol{R}^T\boldsymbol{R} = \mathbf{1}$. For** $\lambda_1 = 1.1,\ \lambda_2 = 1.25, k_1 = 0.15, k_2 = -0.2$, **determine the principal values and directions of $\boldsymbol{E}$. Verify that the principal directions are mutually orthogonal. Compute the strain invariants and show that they are consistent with the characteristic equation.**

$$F = \begin{bmatrix} \lambda_1 & k_1 & 0 \\ k_2 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$C = \begin{pmatrix} k_2^2 + \lambda_1^2 & k_1\lambda_1 + k_2\lambda_2 & 0 \\ k_1\lambda_1 + k_2\lambda_2 & k_1^2 + \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$$

Work through taber02.nb posted where you got your .pdfs

Triaxial extension

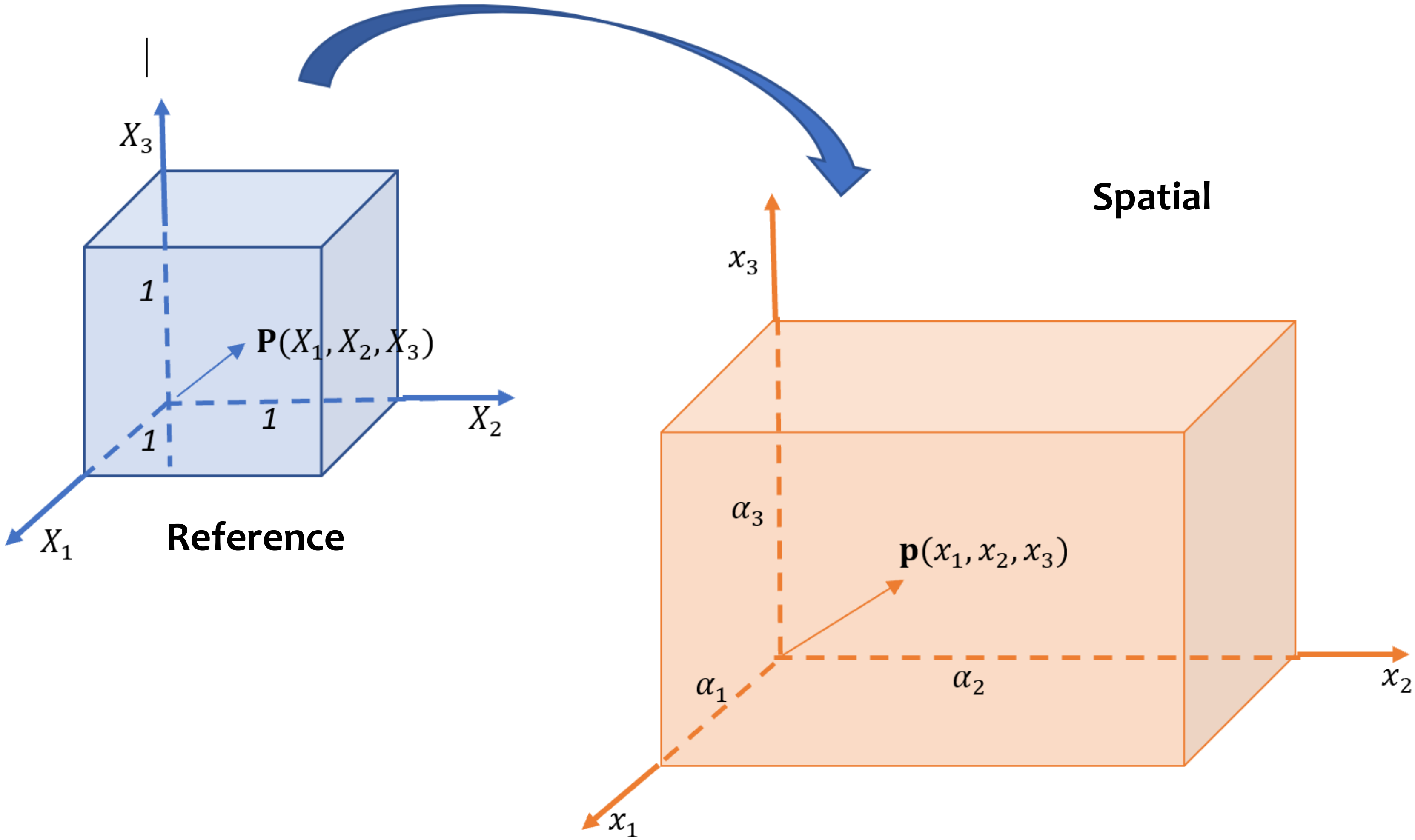
* Consider the unit cube shown below in a triaxial extension so that a typical point $\mathbf{P}$ located at (X_1, X_2, X_3)in the undeformed state, moves to (x_1, x_2, x_3) in such a way that,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$$

Note that uniaxial extension can be obtained by allowing two of the constants to be unity while biaxial will be ensured by one of the constants becoming one as follows:

Uniaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$

Biaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$



X_3
1
$\mathbf{P}(X_1, X_2, X_3)$
1
X_2
1
X_1
Reference
Spatial
x_3
α_3
$\mathbf{p}(x_1, x_2, x_3)$
α_1
α_2
x_2
x_1

Deformation Gradient for Triaxial Extension

$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \alpha_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \alpha_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + \alpha_3 \mathbf{e}_3 \otimes \mathbf{E}_3$$

Strain Function for Extension

* The following Mathematica code shows that,

```
F := {{α1, 0, 0}, {0, α2, 0}, {0, 0, α3}}

CC = Transpose[F].F
```

$\{\{\alpha_1^2, 0, 0\}, \{0, \alpha_2^2, 0\}, \{0, 0, \alpha_3^2\}\}$

```
EE = (1/2) (CC - IdentityMatrix[3]) // MatrixForm
```

atrixForm=

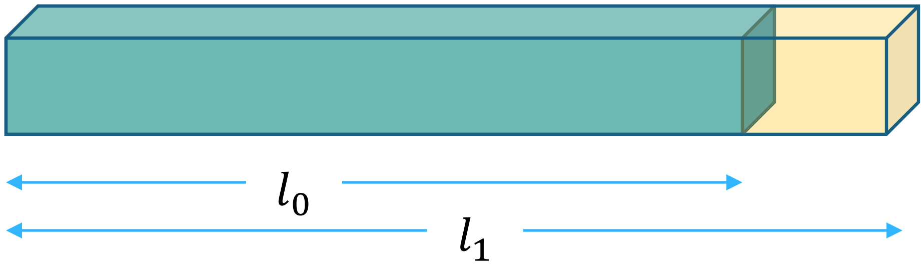
$$\begin{pmatrix} \frac{1}{2}\left(-1+\alpha_1^2\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(-1+\alpha_2^2\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(-1+\alpha_3^2\right) \end{pmatrix}$$

The Green Lagrange strain tensor is,

$$\mathbf{E} = -\frac{1}{2}\left(1-\alpha_1^2\right)\mathbf{E}_1 \otimes \mathbf{E}_1 - \frac{1}{2}\left(1-\alpha_2^2\right)\mathbf{E}_2 \otimes \mathbf{E}_2 - \frac{1}{2}\left(1-\alpha_3^2\right)\mathbf{E}_3 \otimes \mathbf{E}_3$$

Uniaxial Extension

We noted earlier that Uniaxial extension transformation function is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$. Let us write $\alpha_1 = l_1/l_0$ and examine the implications.



What is the value of $\alpha_1 X_1$ when $\alpha_1 = l_1/l_0$? Of course it is zero when $X_1 = 0$, and it is equal to l_1 when $X_1 = l_0$. In one word, it properly defines the spatial configuration for the uniaxial extension we are so used to!

Reddy's Example

* You may notice that my function here differs from Reddy's. For uniaxial extension, you will obtain a similar function if you write, $l_1 = l_0 + \epsilon$ and $\alpha_1 = 1 + \epsilon/l_0$. From where we could have written, $\alpha_1 = 1 + \alpha$.
* If you work with that function, you will obtain a strain α as defined here which is still extension divided by original length for small strains.

Small Uniaxial Extension

- Recall that, in this case, $\alpha_2 = \alpha_3 = 1$. Consequently, the Lagrangian Strain becomes,

The Green Lagrange strain tensor is,

$$\mathbf{E} = -\frac{1}{2}\left(1 - \left(\frac{l_1}{l_0}\right)^2\right)\mathbf{E}_1 \otimes \mathbf{E}_1 \approx \frac{l_1 - l_0}{l_0}\mathbf{E}_1 \otimes \mathbf{E}_1$$

To see that this is true, consider that,

$$\frac{l_1^2 - l_0^2}{2l_0^2} = \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0}$$

Meaning of Strain Tensor

Now, observe that,

$$\lim_{l_0 \to l_1} \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0} = \frac{l_1 - l_0}{l_0}$$

When strains are small, in uniaxial extension, it is correct to state that change in length divided by original length is equal to strain!

* What do the components of the strain tensor mean?
* Begin with the meaning of the deformation gradient
* The strain tensor components deal with the fibres along the coordinate axes. Explain

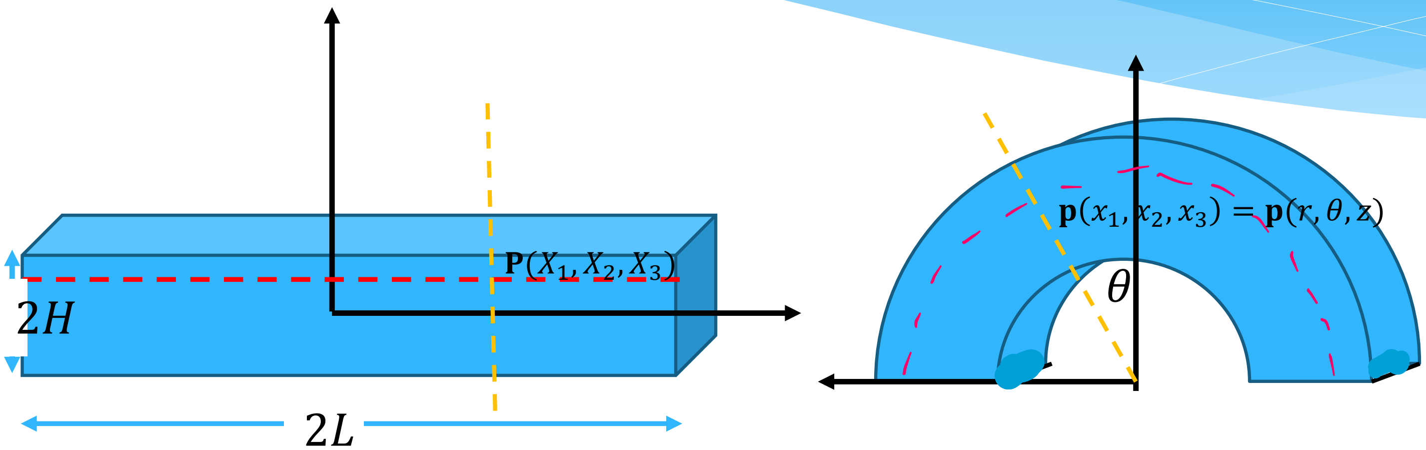
Bending into a circular arc

- If we deform a straight bar into a circular bar as shown below, the transformation function can be found by the following consideration:
- Note that each horizontal filament in the original bar becomes a circular filament in the spatial configuration. The vertical undeformed sections become radial sections in the spatial state. For the moment, we assume nothing happens in the axial or z direction in each case.
- Consequently, we can write,

$$r = r(X_2), \theta = \theta(X_1) \text{ and } z = z(X_3)$$

As a general set of functions transforming X_1, X_2, X_3 to r, θ, z

Bar bending to a semi-circle



1. Let the centerline be a semicircle at a distance R and let the thickness contract uniformly with a factor α

$$\Rightarrow r = R + \alpha X_2, \text{and}$$

$$\theta = \frac{\pi X_1}{2L}$$

2. If the bar contracts uniformly in X_3 direction, $z = \beta X_3$

Bending

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \alpha \mathbf{e}_\theta \otimes \mathbf{E}_1 + \frac{\pi r}{2L}\mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3$$

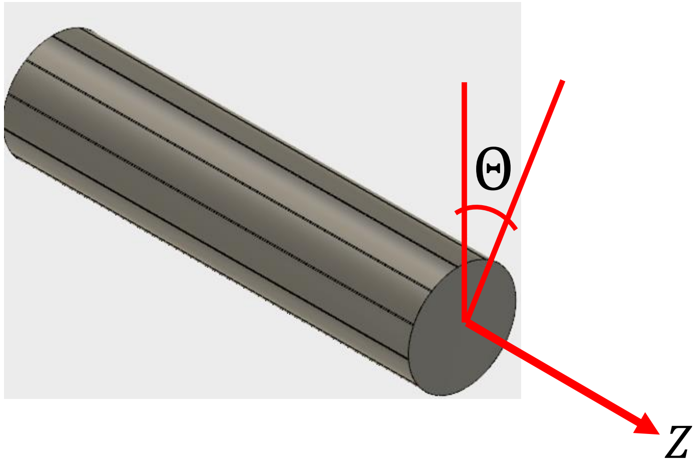
Clearly,

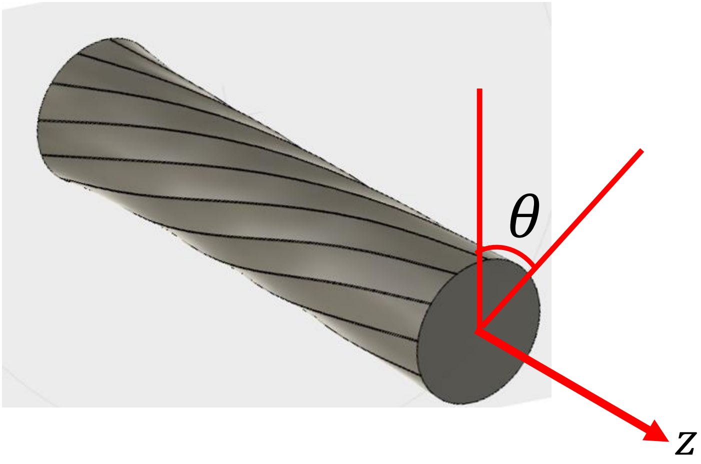
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \alpha^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

and the Right Stretch Tensor,

$$\mathbf{U} = \alpha \mathbf{E}_1 \otimes \mathbf{E}_1 + \frac{\pi r}{2L}\mathbf{E}_2 \otimes \mathbf{E}_2 + \beta \mathbf{E}_3 \otimes \mathbf{E}_3$$

Torsion of a Circular Bar





Circular Bar

* It is convenient to refer the torsion problem to cylindrical coordinates. In consistency with our practice so far, we select R, Θ and Z for the undeformed body and r, θ and z for the typical point in the spatial configuration.
* For a cylindrical bar, it is reasonable to assume that each there are no changes to the radial and axial components in any element;
* Only the angular coordinates are altered by an amount depending on the undeformed value and the axial component Z. Hence,

$$r = R, \theta = \Theta + f(Z), Z = Z$$

are the transformation equations of the deformation.

Cauchy & Strain Tensors

* $$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta / R \\ \mathbf{E}_Z \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & r\frac{\partial f}{\partial Z} \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

```
F := {{1, 0, 0}, {0, r / R, r f[Z]}, {0, 0, 1}}

CC = Transpose[F].F
```

$$\left\{\{1, 0, 0\}, \left\{0, \frac{r^2}{R^2}, \frac{r^2 f[Z]}{R}\right\}, \left\{0, \frac{r^2 f[Z]}{R}, 1 + r^2 f[Z]^2\right\}\right\}$$

```
EE = (1 / 2) (CC - IdentityMatrix[3]) // MatrixForm
```

atrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(-1 + \frac{r^2}{R^2}\right) & \frac{r^2 f[Z]}{2R} \\ 0 & \frac{r^2 f[Z]}{2R} & \frac{1}{2} r^2 f[Z]^2 \end{pmatrix}$$

Torsional Strains

* From the above computations, we find that the Green Lagrange strains are:

$$\mathbf{E} = \frac{1}{2}\left[\left(\frac{r}{R}\right)^2 - 1\right]\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \frac{1}{2}r^2 f^2(Z)\mathbf{E}_Z \otimes \mathbf{E}_Z + \frac{1}{2R}\left(r^2 f(Z)\right)(\mathbf{E}_\Theta \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_\Theta)$$

And the right Cauchy-Green Tensor for the deformation is:

$$\mathbf{C} = \mathbf{E}_R \otimes \mathbf{E}_R + \left(\frac{r}{R}\right)^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + [1 + r^2 f^2(Z)]\mathbf{E}_Z \otimes \mathbf{E}_Z + \frac{r^2 f(Z)}{\mathrm{R}}(\mathbf{E}_\Theta \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_\Theta)$$

Explain the meaning of the components

Homework

1. Follow the same logic and evaluate the Deformation Gradient and the Right Stretch tensor for the bending to an arc with arc angle of Θ, all other conditions remaining the same as in the above example.
2. Find the inverse of the stretch tensor and show that $\mathbf{e}_\theta \otimes \mathbf{E}_1 + \mathbf{e}_r \otimes \mathbf{E}_2 + \mathbf{e}_z \otimes \mathbf{E}_3$ is a rotation and in particular, it is the rotation tensor in bending of the straight bar shown previously to a semicircle.