# SSG 516
# Mechanics of Continua

## Kinematics 02
## The Deformation Gradient and other tensors

Omotayo Fakinlede
oafak.com

# Summary

* Deformation Gradient tensor. A result of the differential transformation of the deformation function.
* Decomposition of the Deformation Gradient into a Rotation and a left or right Stretch Tensor.
* Show that the Deformation Gradient and its Rotation Tensor are both mixed tensors with basis from both referential and spatial configurations.
* The Right Stretch Tensor is a material tensor field while the Left Stretch Tensor is a spatial tensor field.

# Deformation Function

- From last lecture, we can see that the deformation function in any material can be written as

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$$

Here, $\mathbf{X}$ is a typical position vector on the initial state or configuration, and $\mathbf{x}$ is the position vector in the deformed configuration.

- The vector function

$$\boldsymbol{\chi}(\cdot)$$

is the *deformation function*, or simply *the deformation* in that, when we supply a position vector in the initial configuration as an argument to $\boldsymbol{\chi}(\cdot)$, we immediately obtain the deformation position vector $\mathbf{x}$ that tells where the point element that was located at $\mathbf{X}$ is presently occupying in the deformed state. Of course, we can be careless, and write, $\mathbf{x} = \mathbf{x}(\mathbf{X})$ in which case we are not distinguishing between the function and its value.

# Component Form

* The vector equation here can also be written in terms of components as,

$$x_i = \chi_i(X_1, X_2, X_3), \qquad i = 1,2,3$$

for each component because each vector equation is actually three scalar equations

* Note also that for something to be dependent on the position vector $\mathbf{X}$ means exactly the same thing as to be dependent on its three scalar components.

# Differential Form

- Clearly, from multivariable calculus, we know that the differential,

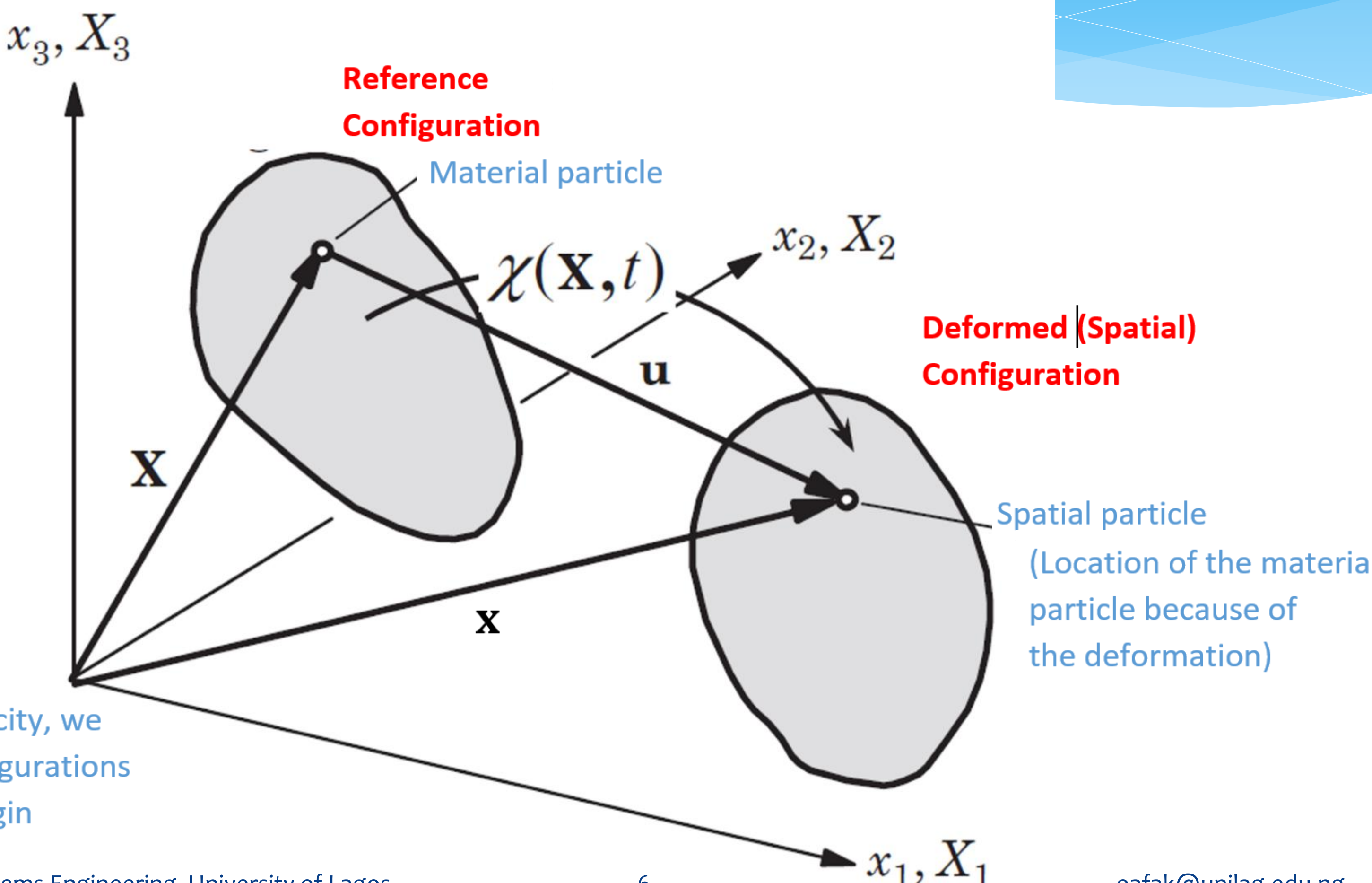
$$dx_i = \frac{\partial \chi_i(X_1, X_2, X_3)}{\partial X_1} dX_1 + \frac{\partial \chi_i(X_1, X_2, X_3)}{\partial X_2} dX_2 + \frac{\partial \chi_i(X_1, X_2, X_3)}{\partial X_3} dX_3$$

- Or, re writing these three equations more compactly,

$$dx_i = \frac{\partial \chi_i}{\partial X_1} dX_1 + \frac{\partial \chi_i}{\partial X_2} dX_2 + \frac{\partial \chi_i}{\partial X_3} dX_3 = \frac{\partial \chi_i}{\partial X_j} dX_j$$

In which case we have not made the dependency of the $\boldsymbol{\chi}$ components on $X_1, X_2$ $and$ $X_3$ explicit.

# Reference and Spatial Configurations




Here, for simplicity, we refer both configurations to the same origin

# Differential in Invariant Form

$$dx_i = \frac{\partial \chi_i}{\partial X_j} dX_j = \frac{\partial x_i}{\partial X_j} dX_j$$

* In which case we have not made the dependency of the $\boldsymbol{\chi}$ components on $X_1, X_2 \; and \; X_3$ explicit. And if we decide to write this equation in its invariant form, we have,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}$$

* Where $\mathbf{F}$, the tensor whose components are $\frac{\partial \chi_i}{\partial X_j}$ is the deformation gradient of the new configuration.

# Equivalency

- In Cartesian coordinates, let $\mathbf{e}_\alpha, \alpha = 1,2,3$ and $\mathbf{E}_k, k = 1,2,3$ be the basis vectors in the spatial and reference coordinates respectively
- For the differential vectors and the tensor, we can write, in component form,

$$d\mathbf{x} = dx_\alpha \mathbf{e}_\alpha$$

$$\mathbf{F} = \frac{\partial \chi_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j \ and$$

$$d\mathbf{X} = dX_k \mathbf{E}_k$$

- Deformation gradient, $\mathbf{F}$ is called a two-toe tensor because it belongs to both configurations (one toe on each). It is a mixed tensor, having basis dyads from two configurations. It transforms the infinitesimal tensor from reference to spatial configuration.
- We noted above that the function $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is the deformation.

# The Reference Map

- We noted earlier that the deformation gradient cannot vanish. It is clear therefore that we can invert this equation and obtain,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x})$$

- Which, given a deformation and a typical position vector, $\mathbf{x}$, we can easily find the corresponding location in the undeformed configuration.
- This is called the reference map because it maps us from the present configuration into the undeformed configuration which is our reference configuration.

# Deformation & Reference Map Example

* Consider the general deformation,

$$\mathbf{x} = \chi(\mathbf{X}, t) = (tX_1 + k_1X_2)\mathbf{e}_1 + (k_2X_1 + tX_2)\mathbf{e}_2 + t\mathbf{e}_3$$

* We can invert this function and obtain,

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t) = \frac{tx_1 - k_1x_2}{t^2 - k_1k_2}\mathbf{E}_1 + \frac{tx_2 - k_2x_1}{t^2 - k_1k_2}\mathbf{E}_2 + \frac{x_3}{t}\mathbf{E}_3$$

Mathematica code for this inversion is,

```
Solve[{x1 == t X1 + X2 k1 , x2 == k2 X1 + X2 t, x3 == t X3}, {X1, X2, X3}]
```

$$\left\{\left\{X_1 \to -\frac{-t\,x_1 + k_1\,x_2}{t^2 - k_1\,k_2},\ X_2 \to -\frac{k_2\,x_1 - t\,x_2}{t^2 - k_1\,k_2},\ X_3 \to \frac{x_3}{t}\right\}\right\}$$

# Deformation Example

* For a more specific deformation at a given time, say $t = 1$,
  $$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = (X_1 + k_1 X_2)\mathbf{e}_1 + (k_2 X_1 + X_2)\mathbf{e}_2 + t\mathbf{e}_3$$
* We can invert this function and obtain,
  $$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = \frac{x_1 - k_1 x_2}{1 - k_1 k_2}\mathbf{E}_1 + \frac{x_2 - k_2 x_1}{1 - k_1 k_2}\mathbf{E}_2 + x_3\mathbf{E}_3$$

Mathematica for specific values for $k_1, k_2$,

```
Solve[{x1 == t X1 + X2 k1 , x2 == k2 X1 + X2 t, x3 == t X3}, {X1, X2, X3}] /.
 {t → 1, k1 -> 0.15, k2 -> -.2}

{{X1 → -0.970874 (-x1 + 0.15 x2), X2 → -0.970874 (-0.2 x1 - x2), X3 → x3}}
```

# Time & Motion

* The Mathematica Example of last week and the homework you submitted show that a sequence of deformations over time is motion and is represented by the function,
$$\mathbf{x} = \chi(\mathbf{X}, t)$$
* And its reference map for any particular time $t$ is,
$$\mathbf{X} = \chi^{-\mathbf{1}}(\mathbf{x}, t)$$

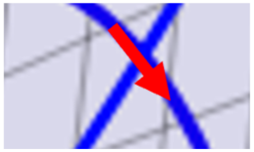
which tells us that the material that is occupying the spatial location $\mathbf{x}$ is the same one that was in $\mathbf{X}$ in the reference configuration; that given $\mathbf{x}$ and time $t$, the reference map function can be used to calculate $\mathbf{X}$ as in the above example.

# Stretch and Rotation

* Consider again the typical elemental vector transformed from the referential to the spatial last week slides:

Look at the referential vector $d\mathbf{X}$,

Which, as a result of the transformation now turns to the spatial vector $d\mathbf{x}$



* It is easy to see that the spatial vector has a different length and a different angular orientation from the referential vector. In other words, there was a stretch (or contraction) as well as a rotation.

# Polar Decomposition Theorem

For a given deformation gradient $\mathbf{F}$, there is a unique rotation tensor $\mathbf{R}$, and unique, positive definite symmetric tensors $\mathbf{U}$ and $\mathbf{V}$ for which,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

This is a fundamental theorem in continuum mechanics called the Polar decomposition theorem.

# Polar Decomposition

By the results of this theorem,

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$$

$\mathbf{R}$ is a rotation tensor while $\mathbf{U}$ and $\mathbf{V}$ are the right (or material) stretch tensor and the left (spatial) stretch tensors respectively. Being a rotation tensor, $\mathbf{R}$ must be proper orthogonal. In addition to its components being an orthogonal matrix, the matrix representation of $\mathbf{R}$ must have a determinant that is positive:

$$\det \mathbf{R} = +1.$$

Note that

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\,\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{I}\,\mathbf{U} = \mathbf{U}^{2}.$$

Definition: **Positive Definite.** A tensor $\mathbf{T}$ is positive definite if for every real vector $\mathbf{u}$, the quadratic form $\mathbf{u} \cdot \mathbf{Tu} > \mathbf{0}$. If $\mathbf{u} \cdot \mathbf{Tu} \geq \mathbf{0}$ Then $\mathbf{T}$ is said to be positive semi-definite.

Now every positive definite tensor $\mathbf{T}$ has a square root $\mathbf{U}$ such that,

$$\mathbf{U}^{2} \equiv \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{T}$$

# Proof

To prove this theorem, we must first show that $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is symmetric and positive definite. Take its transpose; symmetry becomes obvious.

To show positive definiteness, For an arbitrary real vector $\mathbf{u}$ consider the expression, $\mathbf{u}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{u}$**.** Let the vector $\mathbf{b}=\mathbf{F}\mathbf{u}$. Then we can write,

$$\mathbf{u}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{u}=\mathbf{u}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{b}=\mathbf{b}\cdot\mathbf{F}\mathbf{u}=\mathbf{b}\cdot\mathbf{b}=|\mathbf{b}|^2>0$$

as the magnitude of any real vector must be positive. Hence $\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is positive definite.

# Polar Decomposition

- Since every positive definite tensor has a positive definite square root. Let that square root be $\mathbf{U}$

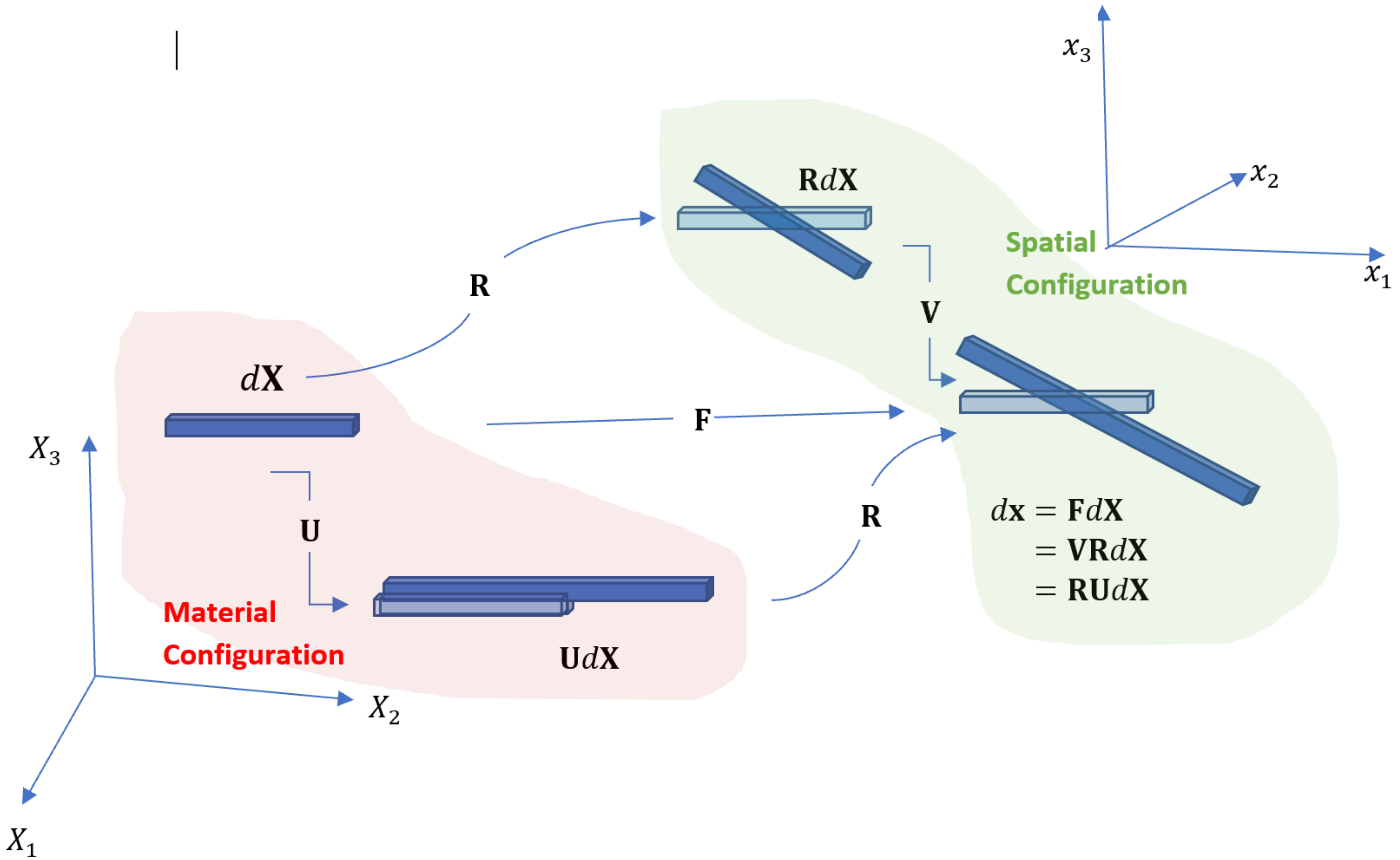
$$\begin{aligned}\mathbf{F}^{\mathbf{T}}\mathbf{F} &= \mathbf{UU} \\ &= \mathbf{U}^{\mathbf{T}}\mathbf{U} = \mathbf{U}^{\mathbf{T}}\mathbf{IU} = \mathbf{U}^{\mathbf{T}}\mathbf{R}^{\mathbf{T}}\mathbf{RU} \\ &= (\mathbf{RU})^{\mathbf{T}}\mathbf{RU}\end{aligned}$$

- Which shows that $\mathbf{F} = \mathbf{RU}$
- We can also find a positive definite tensor $\mathbf{V}$ such that $\mathbf{F} = \mathbf{VR}$

Write $\mathbf{F} = \mathbf{RU} = \mathbf{VR} \Rightarrow \mathbf{V} = \mathbf{RUR}^{-1}$

The fact that $\mathbf{V}$ is positive definite can also be established from the fact that $\mathbf{V}^2 = \mathbf{RUR}^{-1}\mathbf{RUR}^{-1} = \mathbf{RUUR}^{-1} = \mathbf{RUUR}^{\mathbf{T}} = \mathbf{RU}(\mathbf{RU})^{\mathbf{T}} = \mathbf{FF}^{\mathbf{T}}$ which is obviously positive definite.

# Polar Decomposition of Deformation Gradient



# Referential & Spatial Transformations

* In the above figure, we have an exaggerated infinitesimal material vector $d\mathbf{X}$ transforming in three different ways to the spatial vector $d\mathbf{x}$.
  1. *Direct application of the deformation gradient,* $\mathbf{F}$
  2. *A rotation followed by a stretch* $\mathbf{VR}$*, and*
  3. *A stretch followed by a rotation* $\mathbf{RU}$*.*
* The fact that the first is a direct transformation from material to spatial is obvious. How do we explain that the rotation tensor is also a mixed tensor from material to spatial while the left stretch is a spatial tensor and the right stretch a material tensor?

# Transpose and Inverses of the deformation gradient

* Consider a spatial vector $\mathbf{s}$. The dot product $\mathbf{s} \cdot d\mathbf{x}$ has physical significance while $\mathbf{s} \cdot d\mathbf{X}$ does not as the two operands do not exist at the same time so an operation between them makes no physical sense.
* Clearly, $\mathbf{s} \cdot d\mathbf{x} = \mathbf{s} \cdot \mathbf{F} d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{s}$ meaning that $\mathbf{F}^{\mathrm{T}}\mathbf{s}$ is a material vector so that $\mathbf{F}^{\mathrm{T}}$ transforms spatial vectors to material.
* Beginning with a material vector $\mathbf{t}$. The physically meaningful product,

$$\mathbf{t} \cdot d\mathbf{X} = \mathbf{t} \cdot \mathbf{F}^{-1} d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-\mathbf{T}}\mathbf{t}$$

  Showing that $\mathbf{F}^{-\mathbf{T}}$ transforms material to spatial while $\mathbf{F}^{-1}$transforms spatial vectors to material.

# Stretch and Rotation tensors

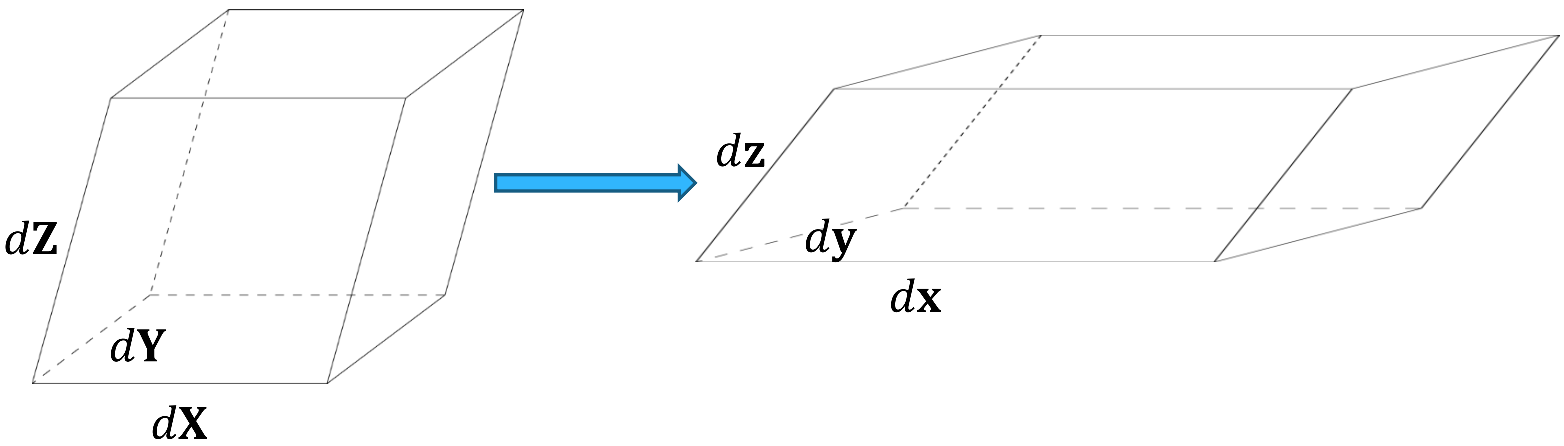
- Above arguments immediately lead to the conclusion that $\mathbf{F}^\mathrm{T}\mathbf{F}$ is a material tensor field while $\mathbf{F}\mathbf{F}^\mathrm{T}$ is a spatial tensor field. Their square roots, $\mathbf{U}, \mathbf{V}$ have their respective properties.
- $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ and the nature of $\mathbf{U}, \mathbf{V}$ as material and spatial fields respectively conclude the proof that $\mathbf{R}$ transforms a material vector to spatial. Is a mixed tensor field like the deformation gradient itself.
- These facts are demonstrated in the above figure.

# Volume & Area Changes

Consider an elemental volume in the reference state in the form of a parallelepiped with dimensions $d\mathbf{X}$, $d\mathbf{Y}$ and $d\mathbf{Z}$. Let this deform to the parallelepiped bounded by $d\mathbf{x}$, $d\mathbf{y}$ and $d\mathbf{z}$ in the current placement caused by a deformation gradient $\mathbf{F}$.

* We require that this parallelepiped be of a non-trivial size, i.e. $[d\mathbf{X}, d\mathbf{Y}, d\mathbf{Z}] \neq 0$
* This means the material vectors $d\mathbf{X}$, $d\mathbf{Y}$ and $d\mathbf{Z}$ are linearly independent.
* Clearly, we must have that $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$, $d\mathbf{y} = \mathbf{F}\, d\mathbf{Y}$ and $d\mathbf{z} = \mathbf{F}\, d\mathbf{Z}$.

# Deformation of Volume



# The Volume change

The scalar, undeformed, volume is given by,

$$dV = [d\mathbf{X}, d\mathbf{Y}, d\mathbf{Z}]$$

and the deformed volume

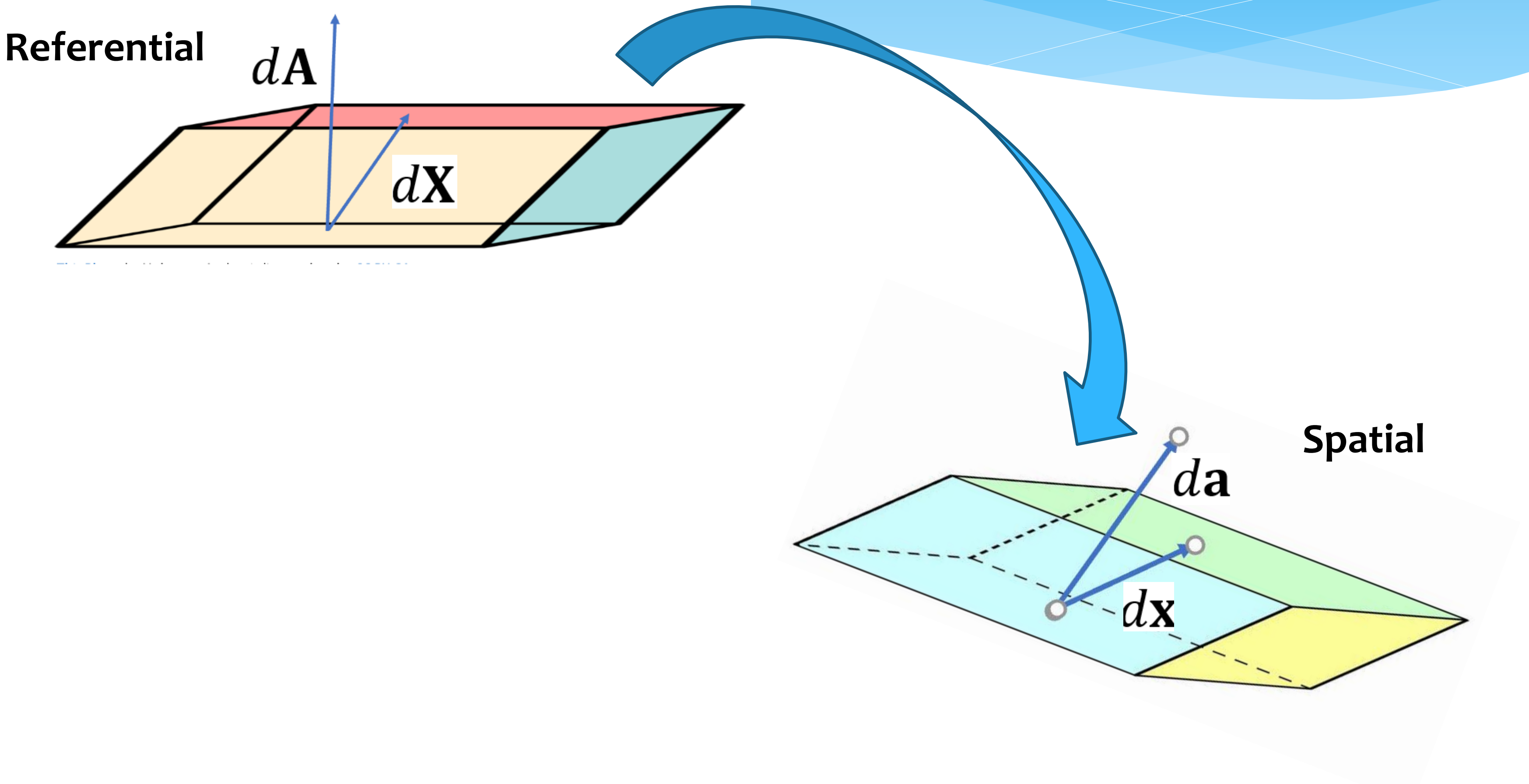
$$dv = [d\mathbf{x}, d\mathbf{y}, d\mathbf{z}] = [\mathbf{F}d\mathbf{X}, \mathbf{F}d\mathbf{Y}, \mathbf{F}d\mathbf{Z}]$$

Clearly seeing that $d\mathbf{X}, d\mathbf{Y}$ and $d\mathbf{Z}$ are independent vectors,

$$\frac{dv}{dV} = \frac{[\mathbf{F}d\mathbf{X}, \mathbf{F}d\mathbf{Y}, \mathbf{F}d\mathbf{Z}]}{[d\mathbf{X}, d\mathbf{Y}, d\mathbf{Z}]} = I_3(\mathbf{F}) = \det \mathbf{F} \equiv J > 0$$

We can also write, $dv = J dV$

# Area Transformation



# Area Changes

For an element of area $d\mathbf{a}$ in the deformed body with a vector $d\mathbf{x}$ projecting out of its plane (does not have to be normal to it). For the elemental volume, we have the following relationship:

$$d\mathbf{v} = J d\mathbf{V} = d\mathbf{a} \cdot d\mathbf{x} = J d\mathbf{A} \cdot d\mathbf{X}$$

where $d\mathbf{A}$ is the element of area that transformed to $d\mathbf{a}$ and $d\mathbf{X}$ is the image of $d\mathbf{x}$ in the undeformed material. Noting that, $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ we have,

$$\begin{aligned} d\mathbf{a} \cdot \mathbf{F} d\mathbf{X} - J d\mathbf{A} \cdot d\mathbf{X} &= 0 \\ &= \left(\mathbf{F}^{\mathrm{T}} d\mathbf{a} - J d\mathbf{A}\right) \cdot d\mathbf{X} \end{aligned}$$

# Nanson Formula

For an arbitrary vector $d\mathbf{X}$, we have:

$$\mathbf{F}^{\mathrm{T}} d\mathbf{a} - J d\mathbf{A} = \mathbf{o}$$

so that,

$$d\mathbf{a} = J\mathbf{F}^{-\mathrm{T}} d\mathbf{A} = \mathbf{F}^{\mathbf{c}} d\mathbf{A}$$

where $\mathbf{F}^{\mathbf{c}}$ is the cofactor tensor of the deformation gradient.

# Examples

1. A body moves according to the function,

$$\mathbf{x} = \left(X_1 + 2X_2 \sin t + \frac{X_3}{2}\right)\mathbf{E}_1 - \left(\frac{X_1}{3} - X_2 + X_3 \sin t\right)\mathbf{E}_2 + \left(X_1^2 \sin 2t + \frac{3}{2}X_3\right)\mathbf{E_3}$$

Find (a) The deformation gradient, (b) Left and Right Stretch Tensors, (c.) Volume Ratio (d) Area change and (e.) Rotation tensor at the material point (1,1,0) at time t=0.3 secs. What is the deformed location of this material point?