SSG 516
Mechanics of Continua

A Mathematical Study of Material Response

Omotayo Fakinlede
oafak.com

Texts

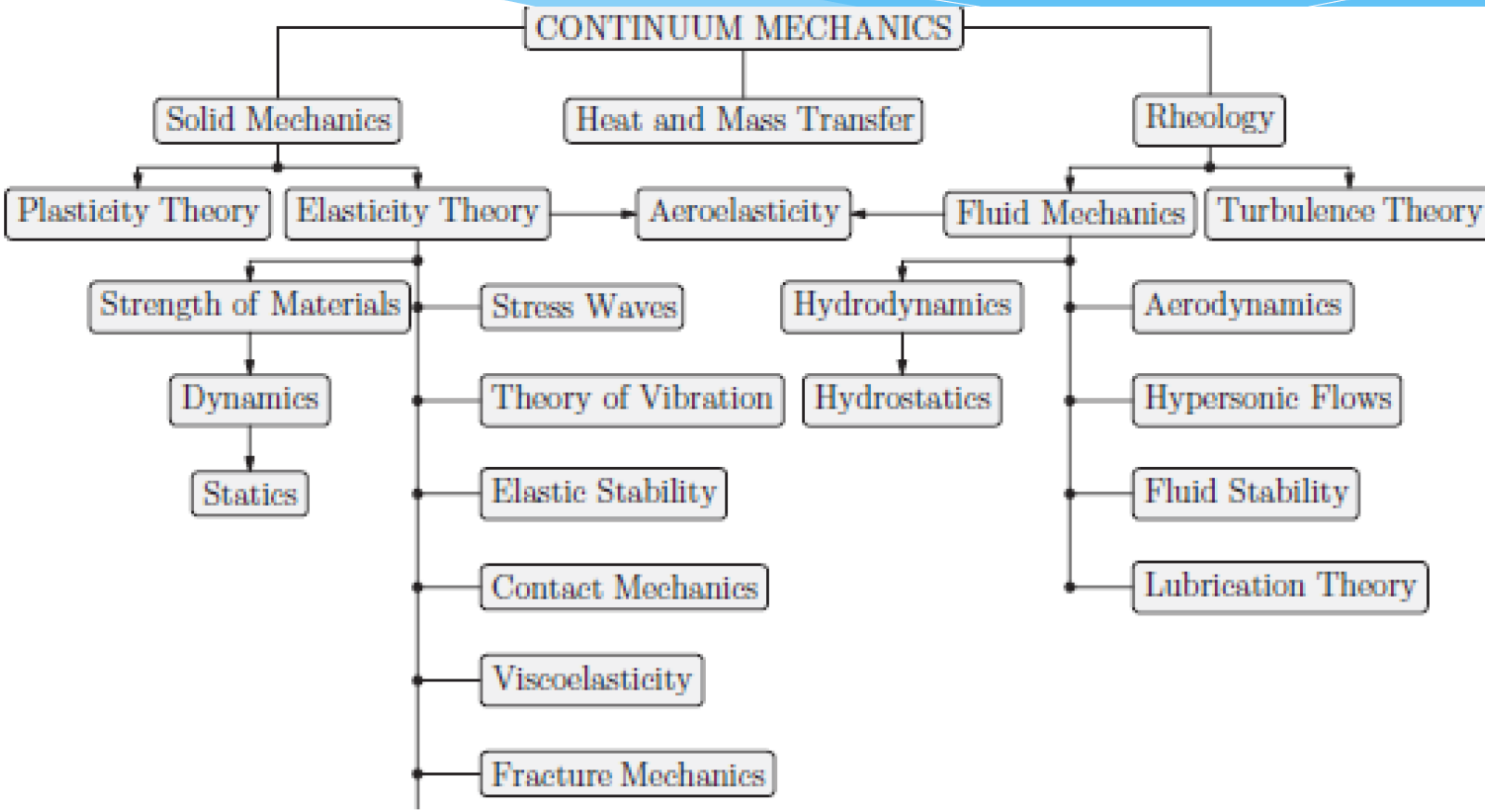
* Reddy, JN, **Principles of Continuum Mechanics,** Cambridge University Press, www.cambridge.org 2012
* Gurtin, ME, Fried, E & Anand, L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, www.cambridge.org 2010
* Tadmore, E, Miller, R & Elliott, R, **Continuum Mechanics and Thermodynamics From Fundamental Concepts to Governing Equations**, Cambridge University Press, www.cambridge.org , 2012
* Heinbockel, JH, **Introduction to Tensor Calculus and Continuum Mechanics**, Trafford, 2003

Sofware

* It is necessary to be proficient with the following software in order to be successful in this course.
  * Wolfram Research, Mathematica
    * I use Mathematica to give practical demonstrations in several situations where the theoretical background required may stretch beyond what you are normally expected to comprehend at this level.
    * If you deny yourself of this advantage, you are not likely to do well. It is therefore incumbent on each student to be good with it.
  * Autodesk, Fusion 360
    * To make this course practical, we use it for design. The facilities to carry this to a logical conclusion is given by Fusion 360.

Introduction

* Continuum Mechanics is a mathematical approach to the study of the behavior of the materials in our ambient environment.
* Several courses such as Mechanics of Solids, Strength of Materials, Mechanics of Fluids, Dynamics, Elasticity, Plasticity, Rheology, etc. have more in common that you may have supposed.
* The objective of Continuum Mechanics is to present the common grounds in a unified manner so as to avoid unnecessary repetition and gain a deeper insight into how to use this knowledge in design of the built environment and of products and services.


CONTINUUM MECHANICS
Solid Mechanics
Heat and Mass Transfer
Rheology
Plasticity Theory
Elasticity Theory
Aeroelasticity
Fluid Mechanics
Turbulence Theory
Strength of Materials
Stress Waves
Hydrodynamics
Aerodynamics
Dynamics
Theory of Vibration
Hydrostatics
Hypersonic Flows
Statics
Elastic Stability
Fluid Stability
Contact Mechanics
Lubrication Theory
Viscoelasticity
Fracture Mechanics

Elasticity

* By far, the most successful continuum theory is the Mathematical Theory of Elasticity. The linear theory has been attracted the best Mathematical minds in nearly two hundred years.
* In recent times, the availability of computational power and new interest has made the nonlinear theory increasingly successful.
* Most non-elastic theories are inherently nonlinear as we shall see later. It is instructive to hear from an expert, what the study of elasticity is all about.

Elasticity

"The Mathematical Theory of Elasticity is occupied with an attempt to reduce to calculation the **state of strain,** or relative displacement, within a **solid body** which is subject to the action of an equilibrating system of forces, or in a state of **slight internal relative motion,** and with endeavors to obtain **results which shall be practically important in applications** to architecture, engineering, and other useful arts in which the material of construction is solid"

AEH Love,
Sedleian Professor of Natural Philosophy,
Oxford University, 1926

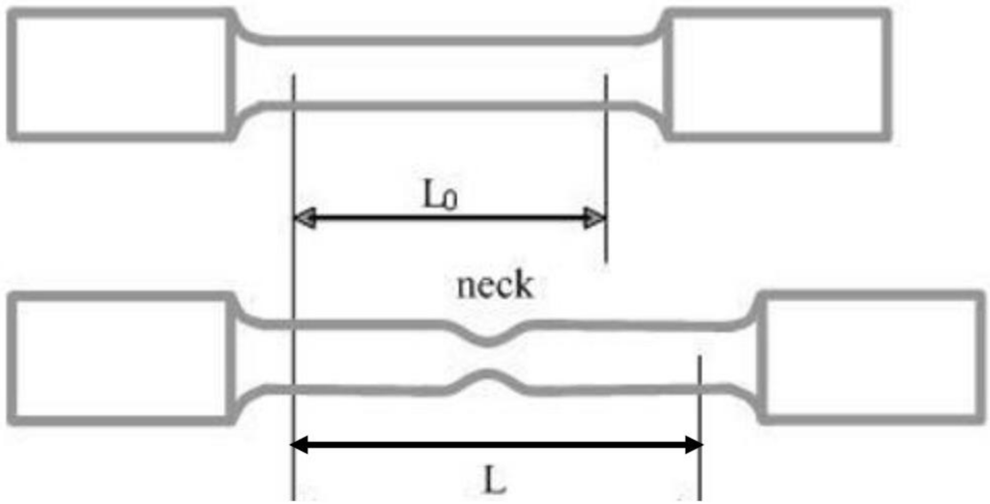
What is Elasticity

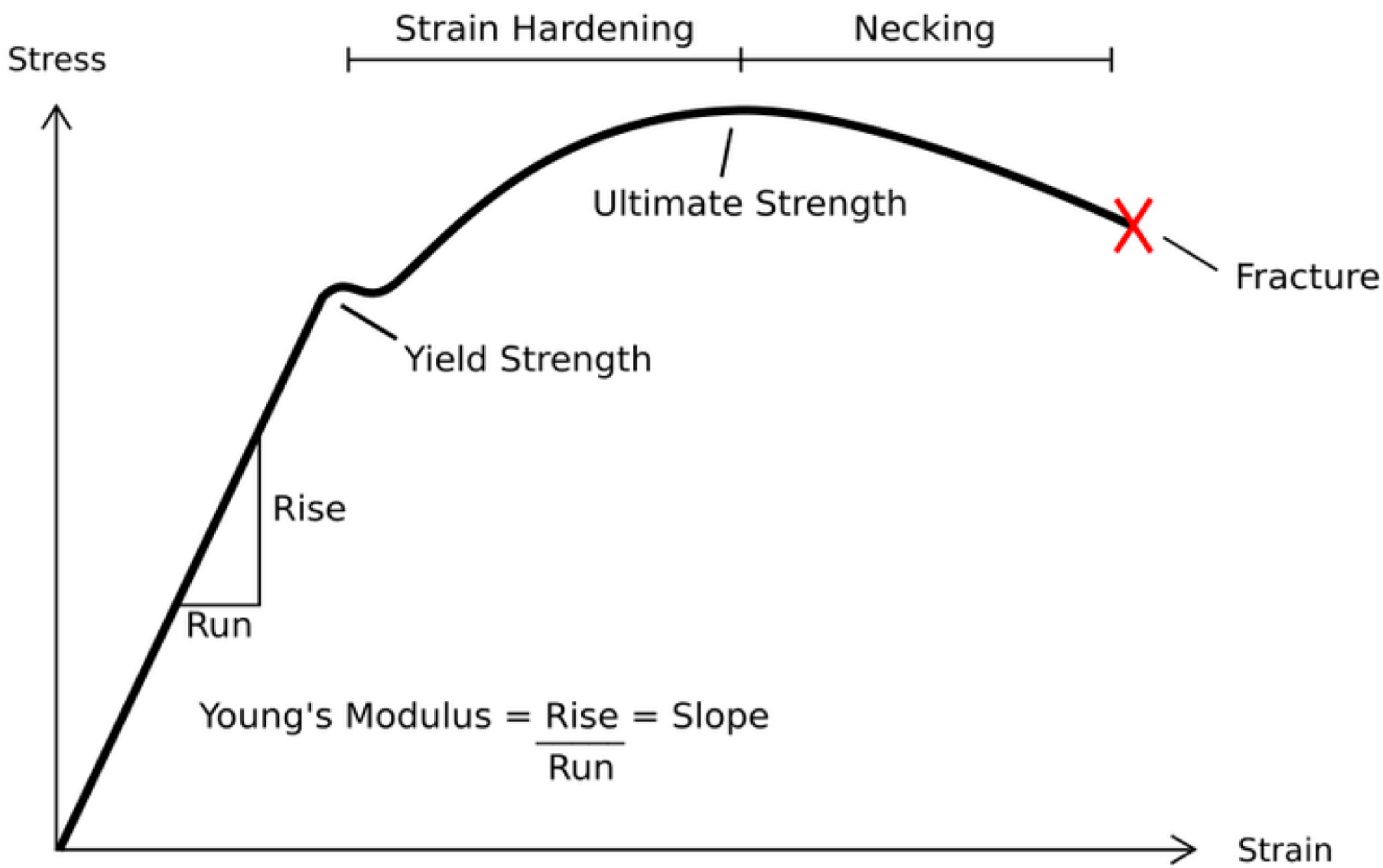
* Elasticity is the simplest of the several continuum models that Engineers and Physicists use.
* In our context, the ability to store energy and release the same with no dissipation is what we call elasticity. This shows up in the fact that the changes in shape that are caused by the application of mechanical or other kinds of loading disappear once the loads are removed.
* Give examples of loads on a body
* Strength of Materials results are approximations of some of the results of linear elasticity.

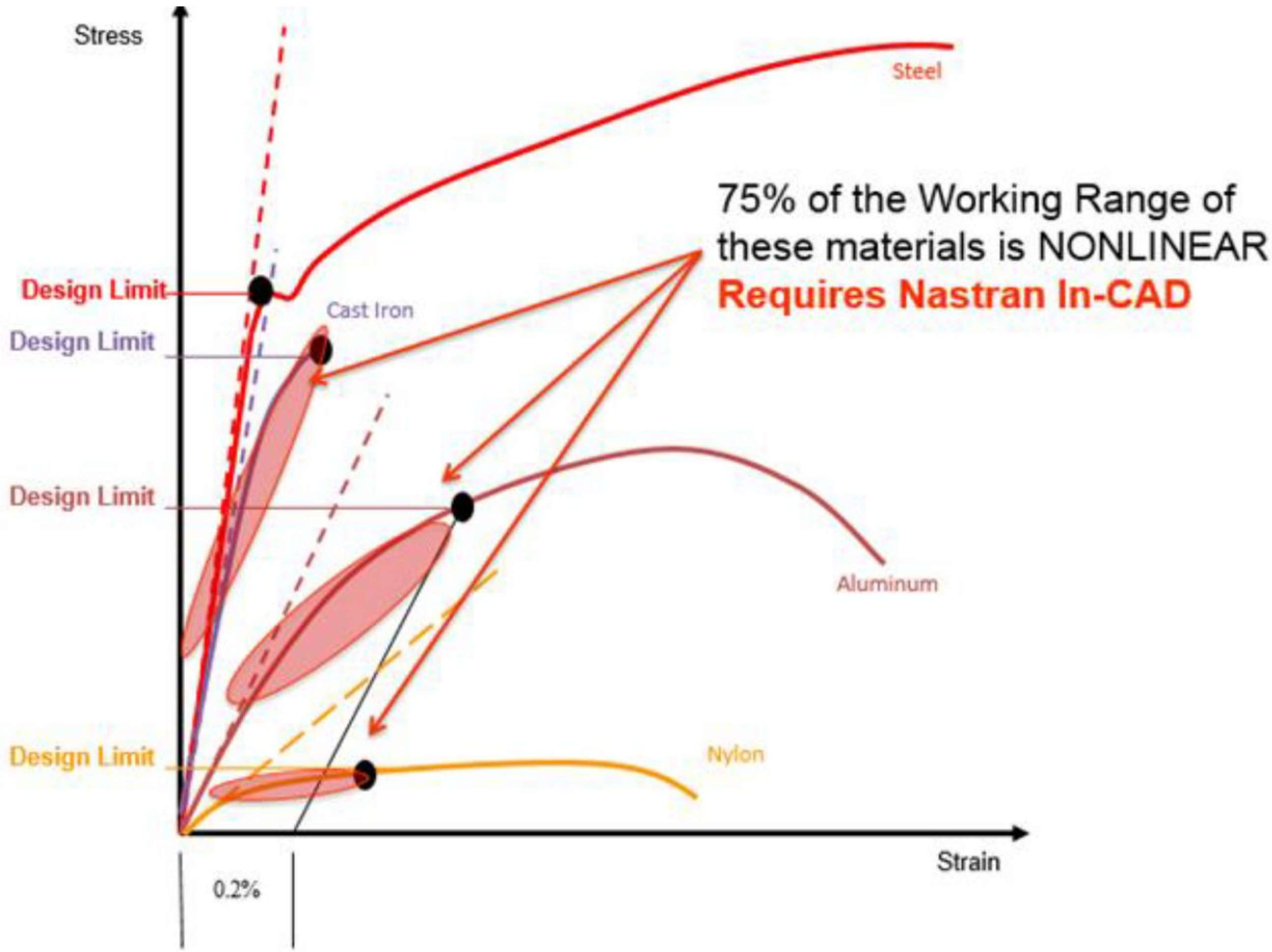
Linearity & Elasticity

* A material where the strain response to an applied load is proportional is said to be linear.
  * There are materials that are linear and NOT elastic
  * There are those that are elastic but NONlinearly so.
  * It is important to distinguish between linearity and elasticity.
* The linear theory of elasticity also assumes that the strains are small. This simplifies the analysis considerably.
  * Once the strains are large, the constitutive linearity of the material notwithstanding, we are already in the realm of nonlinearity!
  * Nonlinearity caused by large deformations are geometric.
  * Those caused by the material itself are constitutive.
* You will find in the following pages some examples to illustrate the issues. Notice that we are yet to define all our terms.

Tensile Test Specimen




Strain Hardening
Necking
Stress
Ultimate Strength
Fracture
Yield Strength
Rise
Run
Young's Modulus = Rise = Slope
Run
Strain



Stress
Steel
75% of the Working Range of these materials is NONLINEAR
Requires Nastran In-CAD
Design Limit
Design Limit
Cast Iron
Design Limit
Aluminum
Design Limit
Nylon
Strain
0.2%

Computation? How Else?

* We want to know beforehand, if a certain kind of load can be sustained by a structure; We also want to know how it will respond.
    * Does the structure fail under the load? How far can we load without the danger of failure?
    * If it does not fail, how large a displacement are we going to have?
    * If we have a clear constitutive relation, the state of strain will immediately tell us the state of stress.
* We could also arrive at this knowledge by experimentation or via a physical, microstructural theory rather than a mathematical approach.
    * These exist; they are costlier and have not been as successful as the Mathematical theory.
    * In practice, there is complementarity in these approaches.
    * The built environment as we know it today is a tribute to the success of the Mathematical theory.

Continuum Mechanics

In order to described our objective in the study of continuum mechanics, we need to paraphrase Love's statement in the following ways:

1. **Solid, Liquid or Gaseous Bodies.** We are looking here at all bodies that may interact with each other in our environment. Consequently, we are not only looking at the state of strain but may also be concerned about strain rates and deformation rates.
2. **Relative motion may be large** and we may no longer be able to make assumption that higher order terms are insignificant. Even for linear elasticity, this kinematical situation creates its own nonlinearity.

Continuum Mechanics

* One irreducible minimum to bear in mind is that, the purpose of reducing these quantities into mathematical computation in order "to obtain results which shall be practically important in applications" is a more eloquent way of saying that the goal is to design and create useful products.
* We want to get into the mindset that, all our efforts in this endeavor must be directed at that goal.
* It is your duty to never get lost in the mathematics or computation to the extent that that goal is lost.

Converging Courses

* Two sub-specialties of continuum mechanics give a good understanding why it is necessary to take a course in this subject.
  * In your strength of materials, you have considered simply loaded beams. Why?
  * In fluid mechanics, you have been looking at flows around simple geometries such as cylinders or simple plates. Why?
* Considerations are restricted to such simple situations in teaching and worked examples because of the computational complexity we want to avoid.

Converging Courses

* Today, availability of computers and powerful software can allow us to avoid such restrictions and engage problems that have practical consequences.
* These software are based on the more elaborate field equations that you are most easily taught within this framework.
* Without going into the details of all derivations, we can at least gain the language of the software and thereby derive more mileage in their use for design sooner than later.
* Continuum mechanics therefore meets us where the software, theory and practice converge. What you learn here, used with modern software systems, can be directly result in the design products to the point of manufacture.

Multiphysics

* You will learn to see the interaction of different physics and governing equations in what is now together called Multi-physics.
* Instead of seeing heat transfer and electricity flow, it is becoming more useful to see these phenomena in their natural habitat of multiphysical interactions as the heat flowing through your conductor may have been produced by the electrical resistance to the current going into it.
* Unification of these disparate phenomena is the way of the future; Continuum mechanics is a useful way to learn this approach early.

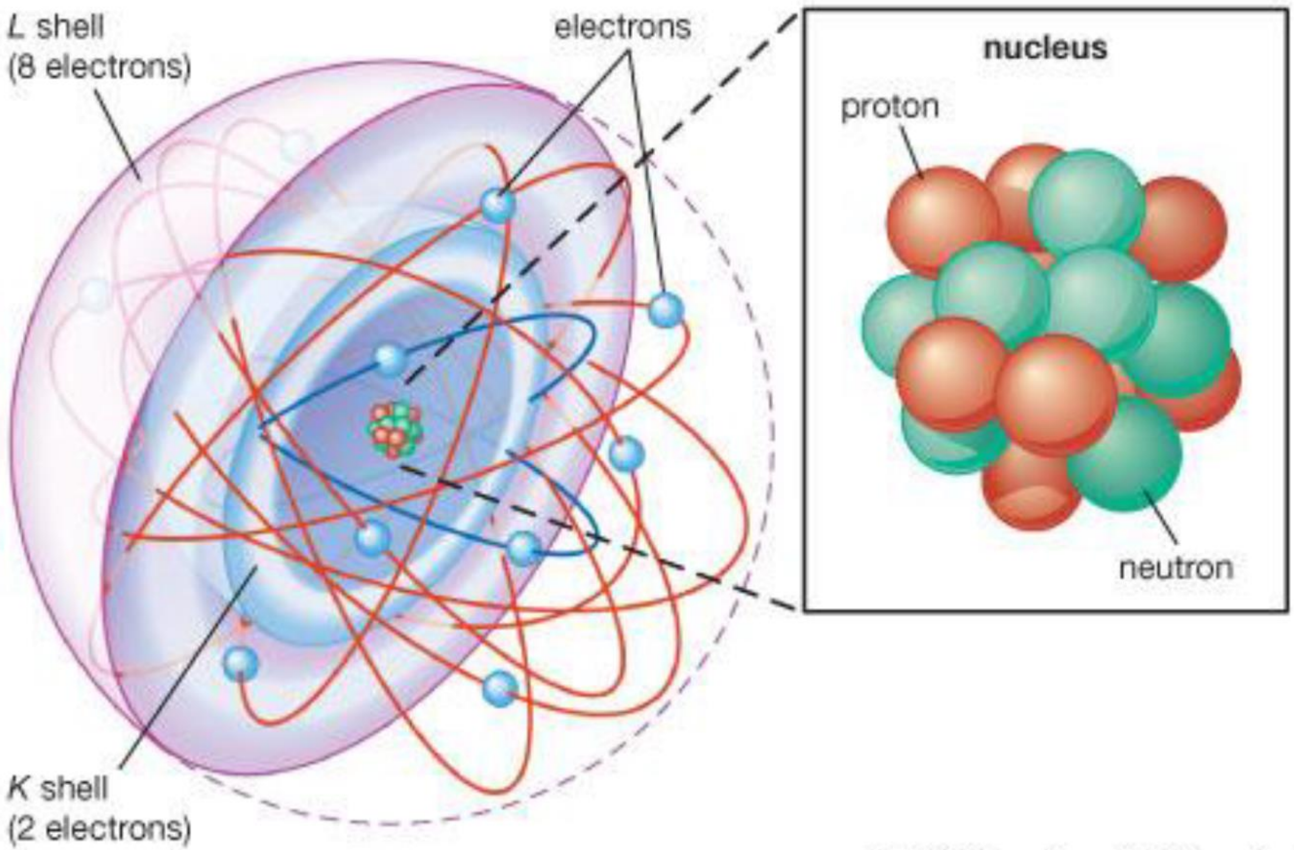
Continuum Theories

* The basis of Elasticity and all continuum theories is the assumption that MATTER IS A CONTINUUM.
  * That matter is infinitely divisible and exhibits the same properties even as we continue to divide them into small bits to infinitesimal sizes.
* Such assumptions allow us to use the laws of calculus with little modification to analyze the materials. We are enabled to define such things as limits, differentials, stress at a point, etc.

Discrete Theories

* The concept of molecules and atoms tells us that on the infinitesimal scale, matter is not continuous at all!
  * In fact, the molecule of steel is further divisible to molecules of iron and that of carbon.
  * Carbon and Iron have properties differing from steel! Furthermore, the molecule of iron contains iron atoms with subatomic particles: electrons, neutrons, protons, etc. These are in motion even for a specimen that appears at rest, etc.

Rutherford Atomic Model



Irreconcilable Differences?

- Theories of material behavior based on discrete models have been attempted. Results for the materials we are primarily concerned with have not been as successful as the results of continuum mechanics.
- Results of continuum theories, so long as we are in the macroscopic range have been so successful that the built environment have been based on them.
- The analyses tools we shall use for this course are based on them.

Background Materials

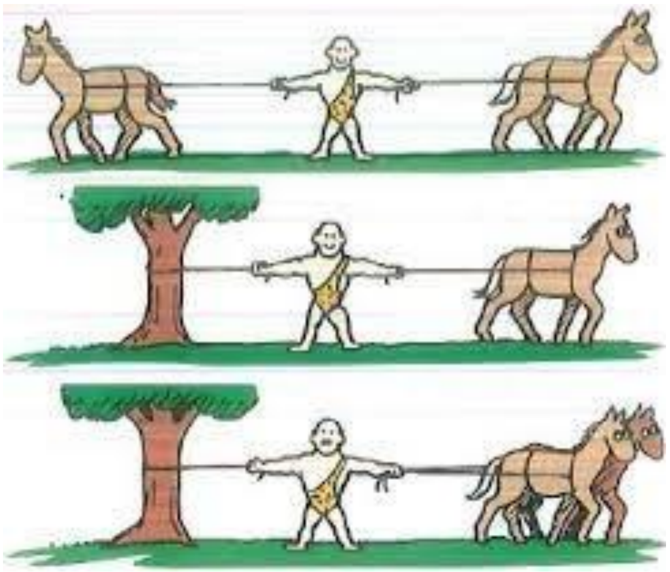
* In dealing with the response to "equilibrating systems of forces", we must define our problem in its important elements. An attempt to do this leads to the intrusion of the Mathematics at three levels:
    1. Differential Geometry (called Tensor Theory in modern texts). The geometry of the resulting shape changes and the definition of the elements that arise in exact terms; stress, strain, displacement, stretch, deformation gradient, etc.
    2. Differential Equations. Expresses in mathematical form, the conditions in the responding bodies, and
    3. Numerical Analyses: No closed form solutions in important cases: We have to appeal to numerical methods

Deformation, Strain & Motion

* The above terms have to become clearly defined notions for us to make any progress in Continuum Mechanics.
* We shall give practical examples that important for analysis and design to illustrate what we mean and to begin to clarify terms:
  Tension/Compression, Shear, Bending, Torsion

Tension

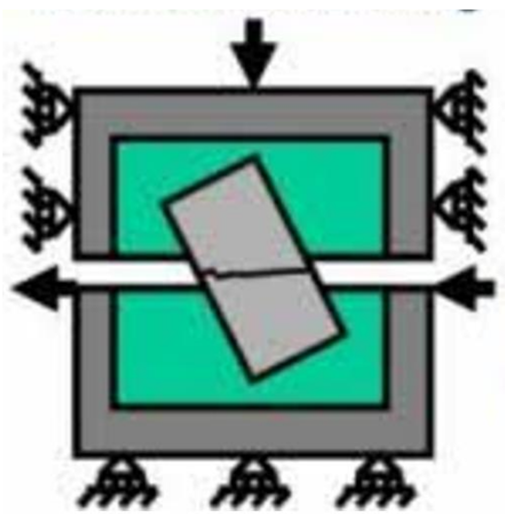
- Tension refers to a pulling apart causing axial transmission of loads by the means of a string, cable, chain. Compression is the opposite. In more complicated cases, you neither have pure tension nor compression especially in bars or multidimensional elements.
- Want to know what tension is? Put yourself in the position of this unfortunate fellow! He is obviously in tension.
    - Assuming the horses exert equal forces, where will the greatest tension be?



Shear

* Our first contact with a shearing situation may be when we first tried to use a hoe in removing weeds from the land. The top part is “sheared” from the lower part.

* The figure below creates a shearing situation on the inner block as the top part is being sheared from the lower.



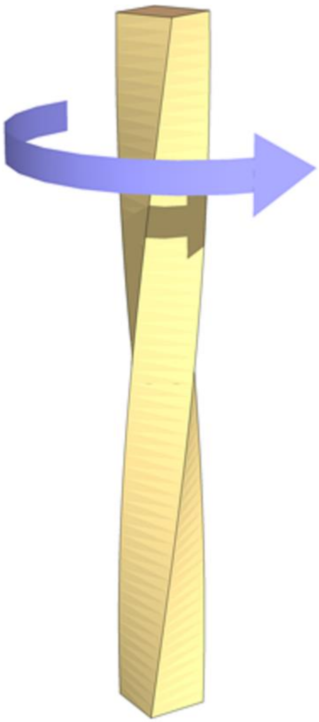
Shaving or other frictional forces can also cause shearing. The removal of outer coats from sheep is a shearing action.

Bending

* Bending is important on horizontal members such as beams. The primary action of bending is to tend to change the curvature of the member.
* In this figure, the bending is achieved by three forces that produce the bending moments along the pipe.
* Two moments applied at the edges could also produce changes in curvature and hence lead to bending. Try and bend your ruler.

Torsion

- Torsion occurs when there is a relative twist along an axis. In the figure shown, torsion occurs when a torque is applied to a bar that had been rigidly supported at its base.
- Shafts that rotate (through gears or other couplings) other members are also subject to torsion from the inertia of the members they are connected to.
- Torsion is seen in the twisting it generates; it also causes warping in other dimensions. Non-circular bars create warping functions when subjected to torsional loads.



Kinematics

The Study of Deformation & Motion

Definition

"… the various possible types of motion in themselves, leaving out … the causes to which the initiation of motion may be ascribed … constitutes the **science of Kinematics**."—ET Whittaker

"**Kinematics** does not deal with predicting the deformation resulting from a given loading, but rather with the machinery for describing all possible deformations a body can undergo" — EB Tadmore et al.

Context

There are three major aspects that interest us of the behavior of a continuously distributed body. The first subject of this chapter, kinematics, is an organized geometrical description of its displacement and motion. We shall also look at a mathematical description of internal forces. In the next chapter we shall look at basic balance laws and the second law of thermodynamics which describes the inbalance of entropy. The emphasis here is the fact that these principles are independent of the material considered. While we may use the terminology of solid mechanics, these laws are valid for any continuously distributed material.

Balance Laws and the Theory of Stress

All materials respond to external influences by obeying these same laws. The differences observed in their responses are results of their constitution. Such constitutive models distinguish between solids and fluids, elastic and inelastic or time independent and materials with time dependent behaviors. We shall endeavor to engage general principles in their most general forms.

Balance Laws and the Theory of Stress

Many books that engineering students encounter at this point treat the three levels of relations (kinematic, balance laws and constitutive models) differently for different materials. The reality is that only the constitutive models differ. The kinematics, transmission of forces and balance laws are material independent.

Placement of Bodies

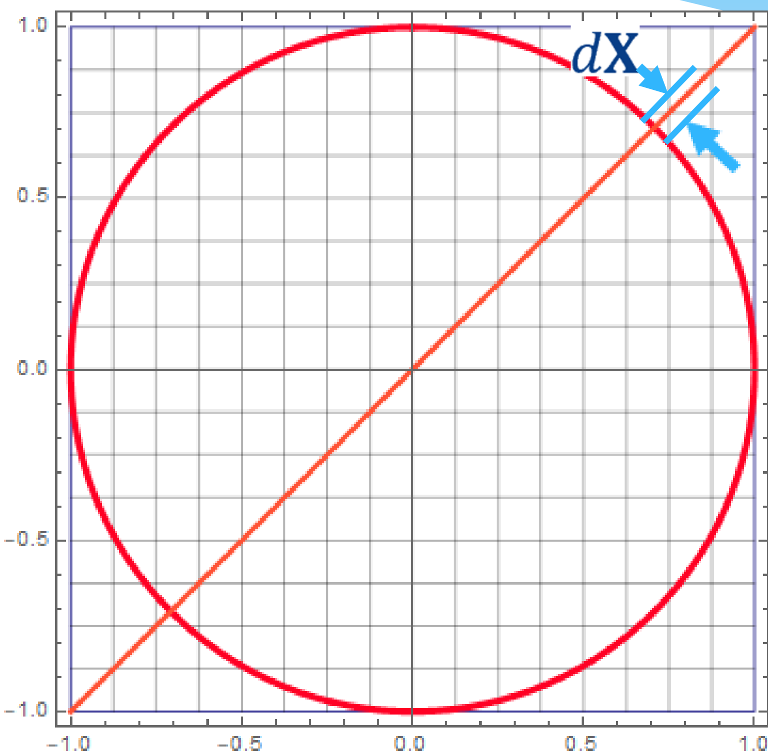
The abstract material body will be considered as a three-dimensional manifold with boundary, consisting of points, which we call material (in contrast to spatial points). The body becomes observable by us when it moves through the space. Mathematically, such a motion is a time-dependent embedding into the Euclidean space. We assume that at each instant, there is a mapping of each point in the body to $\mathscr{R}^3$ and that all coordinate changes are differentiable.

Deformation of a rectangular object

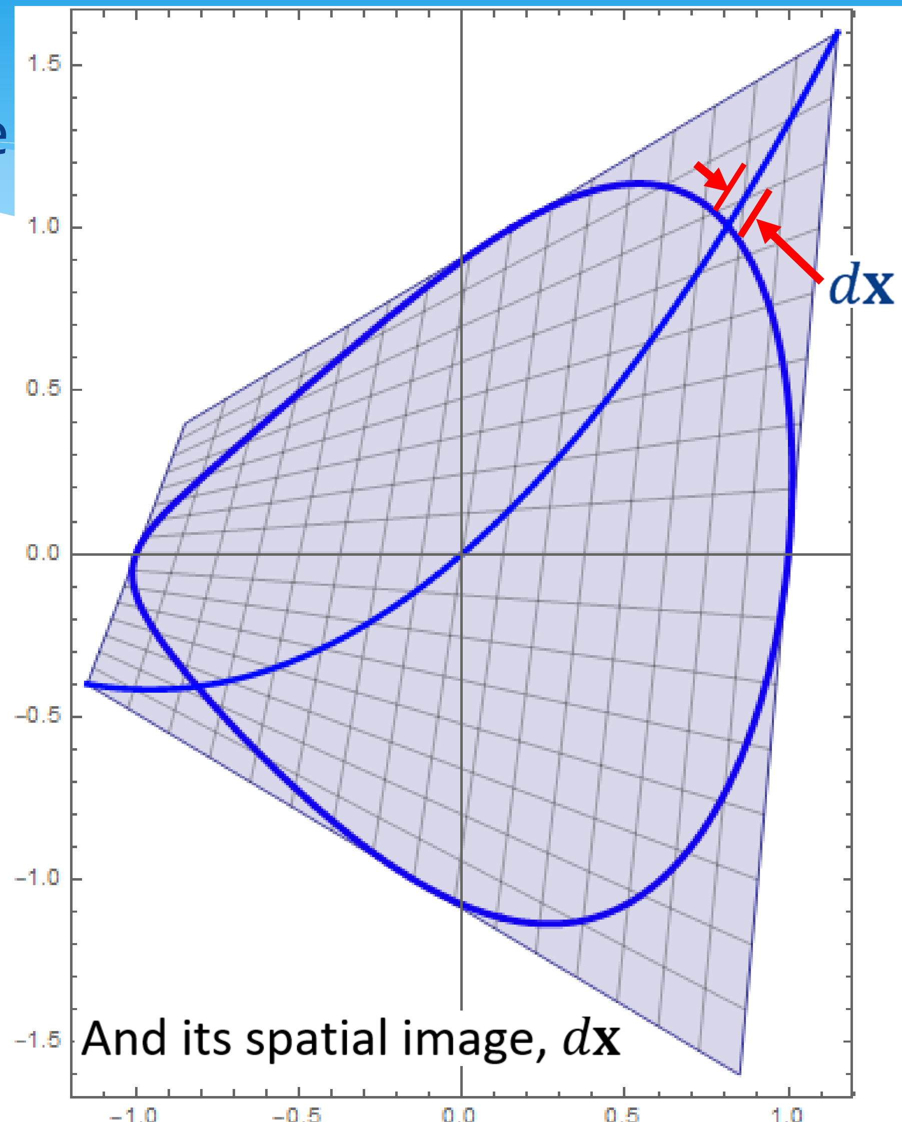
- Consider the rectangular object shown in the picture below.
  - For illustration, let us draw a circle and a diagonal line on this object and observe what happens when the whole rectangle is subjected to a deformation as shown.
  - We will take a small undeformed element, $d\mathbf{X}$ and observe what happens to that element as a result of the deformation.
  - It is now the spatial element, $d\mathbf{x}$. We select this element to be so small that it is approximately a small vector in each case:

Referential, Spatial

Imagine the shape change

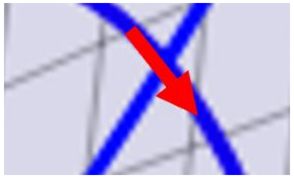


Look at a small element on the undeformed circle, $d\mathbf{X}$



Referential and Spatial Vectors

- Look at the referential vector $d\mathbf{X}$,
- Which, as a result of the transformation now turns to the spatial vector $d\mathbf{x}$



- Our objective is to define, mathematically, the transformation that causes this change. You will see that our concepts of displacement, strain, stretch, rotation, are geometric and will come from that transformation.

Deformation Gradient

* First note that this vector rotates and stretches. Now we seek the relationship,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}$$

where, $d\mathbf{x}$ is the infinitesimal spatial vector, and $d\mathbf{X}$ is the infinitesimal referential vector.

  * The tensor $\mathbf{F}$ that transforms the referential vector to the spatial is called the deformation gradient.
  * It contains all the essential geometric information on the shape changes that are of interest.
* Here, we have used the word “referential” to mean the original state of the body.
  * Strictly speaking, that is just a matter of convenience. Let us leave it like that to keep things simple.
  * Recall we could take a photo of this and keep with us while we allow the body to go through its displacement and motion.

Gradient

* If the actual mathematical transformation from the referential to the spatial is,

$$\mathbf{x} = \chi(\mathbf{X})$$

In which case, $\chi(\mathbf{X})$ is the mathematical function that we can use to compute the spatial position $\mathbf{x}$, given any referential point $\mathbf{X}$, then, we can easily show that the deformation gradient $\mathbf{F}$ we seek (that can account for the transformation of vector elements) is the referential gradient of the function, $\chi(\mathbf{X})$

$$\mathbf{F} = \text{Grad}\ \chi(\mathbf{X})$$

Deformation

$$\mathbf{x} = \chi(\mathbf{X})$$

* On the RHS, we have the typical point in the deformed, spatial configuration (placement). The LHS gives us the function, with each point in the referential frame as an argument, that we use in calculating $\mathbf{x}$.
* We can also reverse this process; if there is a one-to-one mapping from the referential to the spatial, the inverse of this transformation can be found:

$$\mathbf{X} = \chi^{-1}(\mathbf{x})$$

A reference map from which we may compute the referential location of any point $\mathbf{x}$ on the deformed space

Motion

* For the next demonstration, please copy and run the Mathematica code in this link: https://1drv.ms/u/s!AgbbD-KyVrKGhdIy4-7GYETFcm2eiw
* You will agree here that no longer are we dealing with a once for all change from the referential to the spatial. In fact, the spatial is now a time dependent function of the referential. We can account for this time variation by amending the deformation equation now and obtain the motion,

$$\mathbf{x} = \chi_t(\mathbf{X}) \equiv \chi(\mathbf{X}, t)$$

Motion Example

* From the example given in Mathematica code, it is easy to see that,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$
$$\mathbf{X} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$$

We use different base vectors because we are not compelled to refer the referential and spatial to the same base. Let us for simplicity use the same base for now, we can see that the functional relationship is,

$$x_1 = X_1 + X_2 t; x_2 = 4X_1X_2 t + X_2; x_3 = X_3$$

Which, in this case, is the vector equation or motion,

$$\mathbf{x} = \chi_t(\mathbf{X}) \equiv \chi(\mathbf{X}, t)$$

Deformation Gradient

* Now issue the mathematica command,

Grad[myMap[X_1,X_2,t],{X_1,X_2}]

You will easily see that, the deformation gradient in this case is,

$$\{\mathbf{F}(X_1, X_2, X_3, t)\} = \begin{bmatrix} 1 & t & 0 \\ 4tX_2 & 1 + 4tX_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is the matrix of the components of the deformation gradient tensor

Constraints on Deformation Gradient

* Let $J \equiv |\text{Grad}\, \chi(\mathbf{X})|$, the determinant of the deformation gradient $\mathbf{F}$. Let us examine the situation,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{o}$$

the zero vector. What can this mean? Mathematically, the Jacobian (determinant of $\mathbf{F}$) of the transformation is zero.

* We were able to find a non-trivial (not a zero tensor) transformation tensor that transforms a real vector into nothingness! We, by a deformation transformation destroyed matter!
* Our physical considerations precludes this possibility. We exclude from consideration such a possibility. And since we cannot have $J = 0$, we can therefore conclude that

$$J > 0$$

* The only allowable transformations have a positive determinant.