1. For an arbitrary unit vector $\mathbf{e}$, show that the skew tensor, $\mathbf{W} = (\mathbf{e}\,\times)$ is such that $\mathbf{W}^2 \equiv (\mathbf{e}\,\times)(\mathbf{e}\,\times) = (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}$

Let $\mathbf{e} = a_i\, \mathbf{g}_i$ where $\mathbf{g}_i, i = 1{,}2{,}3$ are the Cartesian basis vectors, $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ respectively.

$$
\begin{aligned}
(\mathbf{e}\,\times)(\mathbf{e}\,\times) &= \left(e_{ijk} a_j \mathbf{g}_i \otimes \mathbf{g}_k\right)\left(e_{\alpha\beta\gamma} a_\beta \mathbf{g}_\alpha \otimes \mathbf{g}_\gamma\right) \\
&= e_{ijk} e_{\alpha\beta\gamma} a_j a_\beta \mathbf{g}_i \otimes \mathbf{g}_\gamma \delta_{k\alpha} \\
&= e_{ijk} e_{k\beta\gamma} a_j a_\beta \mathbf{g}_i \otimes \mathbf{g}_\gamma \\
&= \left(\delta_{i\beta}\delta_{j\gamma} - \delta_{j\beta}\delta_{i\gamma}\right) a_j a_\beta \mathbf{g}_i \otimes \mathbf{g}_\gamma \\
&= a_j a_\beta \mathbf{g}_\beta \otimes \mathbf{g}_j - a_\beta a_\beta \mathbf{g}_i \otimes \mathbf{g}_i \\
&= a_j a_\beta \mathbf{g}_\beta \otimes \mathbf{g}_j - (\mathbf{e} \cdot \mathbf{e}) \mathbf{g}_i \otimes \mathbf{g}_i \\
&= (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}
\end{aligned}
$$

upon noting that the dot product of the unit vector with itself is unity.

2. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times) \equiv \mathbf{W}$ is the vector cross of $\mathbf{e}$. Show that for, $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)$ [Note that $\mathbf{e} \otimes \mathbf{e} = \mathbf{W}^2 + \mathbf{I}$]

Using the noted result,

$$\begin{aligned}\mathbf{Q}(\theta) &= \cos\theta\,\mathbf{I} + (1 - \cos\theta)\mathbf{e} \otimes \mathbf{e} + \sin\theta\,(\mathbf{e}\,\times) \\ &= \cos\theta\,\mathbf{I} + (1 - \cos\theta)(\mathbf{W}^2 + \mathbf{I}) + \mathbf{W}\sin\theta \\ &= \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)\end{aligned}$$

3. Use the fact that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)$ where $\mathbf{W} \equiv (\mathbf{e}\,\times)$ - the vector cross of the unit tensor, rotates every vector about the axis of $\mathbf{e}$ by the angle $\theta$ to find the tensor that rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_3\}$.

Clearly, the rotation axis here is the unit vector $\mathbf{e}_3$ and the angle of rotation is $\frac{\pi}{2}$. Consequently, since $\mathbf{e}_3 = \{0,0,1\}$,

$$\mathbf{W} \equiv (\mathbf{e}_3 \times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{W}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{Q}\left(\frac{\pi}{2}\right) &= \mathbf{I} + \mathbf{W}\sin\frac{\pi}{2} + \mathbf{W}^2\left(1 - \cos\frac{\pi}{2}\right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This same tensor can be found directly by recognizing that the tensor, $\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ so that the tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Given that $\mathbf{e}_1 = \{1,0,0\}, \mathbf{e}_2 = \{0,1,0\}, \mathbf{e}_3 = \{0,0,1\}, \mathbf{e}_4 = \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \mathbf{e}_5 = \left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \mathbf{e}_6 = \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}$, Find the tensor that transforms from $\{\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_3\}$ to $\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$.

Tensor, $\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. The tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_4 \otimes \mathbf{e}_2 + \mathbf{e}_5 \otimes \mathbf{e}_1 - \mathbf{e}_6 \otimes \mathbf{e}_3$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

5. Find the rotation tensor around an axis parallel to the unit vector, $\mathbf{e} = \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$ through an angle $\frac{\pi}{3}$.

The skew tensor $(\mathbf{e}\,\times) = \mathbf{W} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$.

$$(\mathbf{e}\,\times)^2 = \mathbf{W}^2 = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{Q}\left(\frac{\pi}{6}\right) = \mathbf{I} + \mathbf{W}\sin\frac{\pi}{6} + \mathbf{W}^2\left(1 - \cos\frac{\pi}{6}\right)$$

$$
= \begin{pmatrix} 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & -\frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \\ \frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) \\ -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) & -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & 1+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} 0.888354 & -0.430577 & 0.159465 \\ 0.385919 & 0.888354 & 0.248782 \\ -0.248782 & -0.159465 & 0.955341 \end{pmatrix}
$$

The inverse of this tensor is its transpose and its determinant is unity. Clearly, it is the rotation tensor we seek.

6. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\,\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the rotation tensor around the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axes through an angle $\theta$.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\,\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

Along the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis, $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2}$.

Consequently, $\mathbf{w} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$. $\mathbf{W}\left(\frac{\pi}{4},\frac{\pi}{2}\right) = (\mathbf{w}\,\times) =$

$$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \mathbf{W}^2\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

And the rotation tensor for this axis is,

$$\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$$

$$= \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{2}(1-\cos\theta) & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{2}(1+\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{pmatrix}.$$

7. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the general rotation through an angle $\theta$.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

$$\mathbf{Q}(\alpha,\beta,\theta) = \mathbf{I} + \mathbf{W}(\alpha,\beta)\sin\theta + \mathbf{W}^2(\alpha,\beta)(1-\cos\theta) =$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 1:

$$\{(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\beta))+1,$$
$$\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))-\cos(\beta)\sin(\theta),$$
$$\sin(\alpha)\sin(\beta)\sin(\theta)+\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 2:

$$\{\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))+\cos(\beta)\sin(\theta),$$
$$(1-\cos(\theta))\left(-\cos^2(\alpha)\sin^2(\beta)-\cos^2(\beta)\right)+1,$$
$$\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\cos(\alpha)\sin(\beta)\sin(\theta)\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 3

$$\{\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\sin(\alpha)\sin(\beta)\sin(\theta),$$
$$\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))+\cos(\alpha)\sin(\beta)\sin(\theta),$$
$$(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\alpha)\sin^2(\beta))+1\}$$

8. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Given that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$, and that, for any scalar $\alpha$, $\mathbf{Q}(\theta)(\alpha\mathbf{e}) = \alpha\mathbf{e}$, What is the physical meaning of $\mathbf{Q}(\theta)$?

> $\mathbf{Q}(\theta)$ is a rotation about the vector $\mathbf{e}$ counterclockwise through an angle $\theta$. It therefore does not alter the magnitude or direction of any vector in the direction of $\mathbf{e}$; for any other vector, it has no effect on the magnitude but affects direction.

9. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Show that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$. What is the physical meaning of $\mathbf{Q}(\theta)$?

> Let the scalar $x \equiv \mathbf{e} \cdot \mathbf{u}$ be the projection of $\mathbf{u}$ onto the unit

vector **e**. The square of the magnitude of **v** is $|\mathbf{v}|^2$

$$= \mathbf{v}\cdot\mathbf{v} = \big(\cos\theta(\mathbf{1u}) + (1-\cos\theta)(\mathbf{e}\otimes\mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)$$

$$\cdot\big(\cos\theta(\mathbf{1u}) + (1-\cos\theta)(\mathbf{e}\otimes\mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)\cdot$$

$$= \big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)^2$$

$$= (\mathbf{u}\cos\theta)\cdot\big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)$$

$$+x\mathbf{e}\cdot\big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)(1-\cos\theta)$$

$$+(\mathbf{e}\times\mathbf{u})\cdot\big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\big)\sin\theta$$

$$= \mathbf{u}^{\mathbf{2}}\cos^2\theta + 2(\cos\theta - \cos^2\theta)x^2$$

$$+\ 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2x^2$$

$$+\ 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{e}\times\mathbf{u})^2$$

$$= \mathbf{u}^{\mathbf{2}}\cos^2\theta + 2(\cos\theta - \cos^2\theta)x^2$$

$$+\ 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2x^2$$

$$+\ 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{u}^2 - x^2)$$

$$= \mathbf{u}^2(\cos^2\theta + \sin^2\theta)$$
$$+ x^2[2(\cos\theta - \cos^2\theta) + (1 - \cos\theta)^2 - \sin^2\theta]$$
$$= \mathbf{u}^2$$

As the term in square brackets vanish when expanded.

**10.** If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that for arbitrary $0 < \alpha, \beta \leq 2\pi$, that $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$.

It is convenient to write $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ in terms of their $i, j$ components; we assume that the unit vector $\mathbf{e} = (x_1, x_2, x_3)$:

$$[\mathbf{Q}(\alpha)]_{ij} = \cos\alpha\,\delta_{ij} + (1 - \cos\alpha)x_i x_j - \sin\alpha\,\epsilon_{ijk}x_k$$

Consequently, we can write for the product $\mathbf{Q}(\alpha)\mathbf{Q}(\beta)$,

$$[\mathbf{Q}(\alpha)\mathbf{Q}(\beta)]_{ij} = [\mathbf{Q}(\alpha)]_{ik}[\mathbf{Q}(\beta)]_{kj} =$$

$$
\begin{aligned}
&= [\cos\alpha\,\delta_{ik} + (1-\cos\alpha)x_i x_k - \sin\alpha\,\epsilon_{ikl}x_l][\cos\beta\,\delta_{kj} \\
&\qquad + (1-\cos\beta)x_k x_j - \sin\beta\,\epsilon_{kjn}x_n] \\
&= \cos\alpha\cos\beta\,\delta_{ik}\delta_{kj} + \cos\alpha\,(1-\cos\beta)\delta_{ik}x_k x_j \\
&\qquad - \cos\alpha\sin\beta\,\delta_{ik}\epsilon_{kjn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_k\delta_{kj} \\
&\qquad + (1-\cos\alpha)(1-\cos\beta)x_i x_k^2 x_j \\
&\qquad - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ikl}x_l\delta_{kj} \\
&\qquad - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l \\
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n \\
&\qquad + (1-\cos\alpha)\cos\beta\,x_i x_j + (1-\cos\alpha)(1-\cos\beta)x_i x_j \\
&\qquad - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l \\
&\qquad - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l
\end{aligned}
$$

$$
\begin{aligned}
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n \\
&\qquad + (1-\cos\alpha)\cos\beta\,x_i x_j + (1-\cos\alpha)(1-\cos\beta)x_i x_j \\
&\qquad - \boxed{(1-\cos\alpha)\sin\beta\,x_i x_k x_n \epsilon_{kjn}} - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l \\
&\qquad - \boxed{\sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j} \\
&\qquad + \sin\alpha\sin\beta\left(\delta_{lj}\delta_{in} - \delta_{ln}\delta_{ji}\right)x_n x_l \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\delta_{ij} \\
&\qquad + [1 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]x_i x_j \\
&\qquad - [(\cos\alpha\sin\beta - \sin\alpha\cos\beta)]\epsilon_{ijn}x_n \\
&= [\mathbf{Q}(\alpha+\beta)]_{ij}
\end{aligned}
$$

With the boxed terms vanishing on account of antisymmetric contraction with symmetry.

**11.** If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Show that . $\mathbf{Q}(\theta)$ is a periodic tensor function with period $2\pi$. [Hint: $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$

Since $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$ we can write that $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi)$. But a direct substitution shows that,
$\mathbf{Q}(0) = \mathbf{Q}(2\pi) = \mathbf{1}$. We therefore have that, $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi) = \mathbf{Q}(\alpha)$
which completes the proof. The above results show that $\mathbf{Q}(\alpha)$ is a rotation along the unit vector $\mathbf{e}$ through an angle $\alpha$.