VECTOR TRANSFORMATIONS

- In Graphics Packages, including Fusion 360 that we have been using, transformations are done in response to your commands. We consider in these slides, the computations that underly some of these transformations.
- Two of the most basic graphics transformations are translations and rotations of vectors and tensors.
- We have already seen that a vector can be rotated from any given set of coordinates to another using the tensor,

$$\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$$

2 COORDINATE ROTATIONS

Which can be used to rotate any vector or tensor from the coordinates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. To rotate any vector $\mathbf{v}$ from one coordinate to the other, we simply operate the rotation tensor on the vector $\mathbf{Qv}$. If we were to rotate a tensor $\boldsymbol{S}$ in the same way, the correct computation is, $\mathbf{QSQ}^{\mathrm{T}}$

Usually, we are not given the two sets of coordinates from where we are coming and the final set to which we want to move. What we need is to be able to move a vector or tensor object around a particular line (given by a unit vector direction) through a specific scalar angle in degrees or radians.

The computation behind this kind of transformation requires a small amount of background in vector analysis.

We begin here with a repetition of some of the basic issues.

3

Product Operations Between Vectors

- Consider two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_i\mathbf{e}_i$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b_j\mathbf{e}_j$. We already know that we can form three different products from these two vectors:

1. A scalar called: Inner, Dot or Scalar product $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$ where θ is the angle contained between the two vectors. This is also equal to the scalar sum, $a_1b_1 + a_2b_2 + a_3b_3 = a_ib_i$
2. A vector called: Vector or Cross product: $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 = e_{ijk}a_jb_k\mathbf{e}_i$

4 Product Operations Between Vectors

3. A tensor called: Tensor, Kronecker or Outer product:

$$\mathbf{a} \otimes \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \otimes (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$$

$$= a_1 b_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + a_1 b_2 \mathbf{e}_1 \otimes \mathbf{e}_2 + a_1 b_3 \mathbf{e}_1 \otimes \mathbf{e}_3 + a_2 b_1 \mathbf{e}_2 \otimes \mathbf{e}_1 + a_2 b_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + a_3 b_1 \mathbf{e}_3 \otimes \mathbf{e}_1 + a_3 b_2 \mathbf{e}_3 \otimes \mathbf{e}_2 + a_3 b_3 \mathbf{e}_3 \otimes \mathbf{e}_3$$

$$= a_i b_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

- Which we can express in matrix form of components as,

$$\begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}$$

5 PRODUCT OPERATIONS

- Which we could have obtained by treating $\mathbf{a}$ as a column vector and taking its product with the components of the row vector $\mathbf{b}$. This equivalency often makes us think that the tensor is a matrix.
- The component form of a tensor is very useful for computations. However, it is important for us to note that, despite the appearances, the tensor itself is NOT the matrix. In fact, the components may change when we look at the same tensor from another system of coordinates whereas, the tensor itself remains the same. It is not a trivial matter to continue to distinguish between the tensor and its components.

SUMMATION CONVENTION

- When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

7 SUMMATION CONVENTION

- Consider transformation equations such as,

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

- We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j$$

$$y_2 = \sum_{j=1}^{n} a_{2j}x_j$$

$$y_3 = \sum_{j=1}^{n} a_{3j}x_j$$

8 SUMMATION CONVENTION

- In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j$$
$$y_2 = a_{2j}x_j$$
$$y_3 = a_{3j}x_j$$

- Finally, we observe that $y_1, y_2,$ and y_3 can be represented as we have been doing by $y_i,\ i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \qquad i = 1,2,3$$

SUMMATION CONVENTION

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij}x_j$$

to represent compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

10 SUMMATION CONVENTION

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

for $i = 1, k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1, k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1, k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2, k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2, k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2, k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3, k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3, k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3, k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

11 SUMMATION CONVENTION

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}$$

In this same way, we could have also proved that,

$$a_{ij}a_{kj} = b_{ik}$$

- Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

12 VECTOR COMPONENTS IN ONB

Consider an orthogonal normalized basis (ONB)

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \qquad or\ \mathbf{e}_i, i = 1,2,3$$

The fact that $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$, and also that the norm on each, that is, $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$ make the computation of the components on any vector referred to this bases easy.

This is the main attraction of the ONB, also called Cartesian Coordinates.

VECTOR COMPONENTS IN ONB

Given a known vector $\mathbf{F}$, (known because its magnitude and direction are given) we can find its components in the ONB in these simple steps:

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

Where $F_1, F_2\ and\ F_3$, the scalar quantities to be determined are called the components of $\mathbf{F}$ in the ONB $\mathbf{e}_i, i = 1,2,3$

Take the scalar product of the above equation with $\mathbf{e}_1$, we have,

14 VECTOR COMPONENTS IN ONB

$$\mathbf{F} \cdot \mathbf{e}_1 = F_1 \mathbf{e}_1 \cdot \mathbf{e}_1 + F_2 \mathbf{e}_2 \cdot \mathbf{e}_1 + F_3 \mathbf{e}_3 \cdot \mathbf{e}_1$$
$$= F_1 \mathbf{e}_1 \cdot \mathbf{e}_1 + \boxed{F_2 \boldsymbol{e}_2 \cdot \boldsymbol{e}_1} + \boxed{F_3 \mathbf{e}_3 \cdot \mathbf{e}_1}$$

The boxed items vanish on account of orthogonality while the first term simply becomes F_1 on account of normality. We therefore find that $F_1 = \mathbf{F} \cdot \mathbf{e}_1$. We can repeat this process by taking the dot product with the other basis vectors and find also that, $F_2 = \mathbf{F} \cdot \mathbf{e}_2$ and that $F_3 = \mathbf{F} \cdot \mathbf{e}_3$.

In index notation, we could write the nine orthonormality equations as,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$

i	j	$\mathbf{e}_i \cdot \mathbf{e}_j$	δ_{ij}	Value
1	1	$\mathbf{e}_1 \cdot \mathbf{e}_1$	δ_{11}	1
1	2	$\mathbf{e}_1 \cdot \mathbf{e}_2$	δ_{12}	0
1	3	$\mathbf{e}_1 \cdot \mathbf{e}_3$	δ_{13}	0
2	1	$\mathbf{e}_2 \cdot \mathbf{e}_1$	δ_{21}	0
2	2	$\mathbf{e}_2 \cdot \mathbf{e}_2$	δ_{22}	1
2	3	$\mathbf{e}_2 \cdot \mathbf{e}_3$	δ_{23}	0
3	1	$\mathbf{e}_3 \cdot \mathbf{e}_1$	δ_{31}	0

16 VECTOR COMPONENTS IN ONB

Furthermore, the result of calculating the components, that

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 \\ &= (\mathbf{F}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{F}\cdot\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{F}\cdot\mathbf{e}_3)\mathbf{e}_3\end{aligned}$$

Can also be written as,

$$\mathbf{F} = F_i\mathbf{e}_i = \left(\mathbf{F}\cdot\mathbf{e}_j\right)\mathbf{e}_j$$

Where the repetition of the indices indicate summation.

In the above computation, everything was easy because the orthogonality and normality conditions made some terms vanish and the other terms were products with unity. It is this ease of computation that makes the ONB very attractive.

VECTOR COMPONENTS IN ONB

When we relax the conditions on our basis vectors, these nice properties disappear. Yet, there is an extension of this situation, which, if understood deeply, will make things relatively easy also.

It is easy to show that orthogonality and normality assumptions we have made thus far are only for convenience. We can define coordinate systems that do not make those assumptions. All we need are three vectors that are linearly independent.

CARTESIAN VECTOR COMPONENTS

It is convenient for us to represent the vectors of the Cartesian system of coordinates as $\mathbf{e}_i, i = 1,2,3$ which is just a shorthand for writing $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$instead of calling these unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

This change may look trivial at first, but when combined with Einstein's convention we will introduce, allows us the benefit of reducing the number of terms in expressions in a unique way necessary to express tensor terms.

19 VECTOR COMPONENTS

Clearly, addition and linearity of the vector space ⇒

$$\mathbf{v} + \mathbf{w} = (v_i + w_i)\mathbf{e}_i$$
$$= (v_1 + w_1)\mathbf{e}_1 + (v_2 + w_2)\mathbf{e}_2 + (v_3 + w_3)\mathbf{e}_3$$

Multiplication by scalar rule implies that if $\alpha \in \mathscr{R}, \forall \mathbf{v} \in \mathscr{V}$,

$$\alpha\, \mathbf{v} = (\alpha v_i)\mathbf{e}_i$$
$$= (\alpha v_1)\mathbf{e}_1 + (\alpha v_2)\mathbf{e}_2 + (\alpha v_3)\mathbf{e}_3$$

Because of the implicit summation implied by the repetition of indices.

KRONECKER DELTA

Kronecker Delta: δ_{ij} has the following properties:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$
$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$
$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

As is obvious, these are obtained by allowing the indices to attain all possible values in the range. The Kronecker delta is defined by the fact that when the indices explicit values are equal, it has the value of unity. Otherwise, it is zero. The above nine equations can be written more compactly as,

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

21 LEVI CIVITA SYMBOL

- The Levi-Civita Symbol: e_{ijk}
- $e_{111} = 0, e_{112} = 0, e_{113} = 0, e_{121} = 0, e_{122} = 0, e_{123} = 1, e_{131} = 0, e_{132} = -1, e_{133} = 0$
  $e_{211} = 0, e_{212} = 0, e_{213} = -1, e_{221} = 0, e_{222} = 0, e_{223} = 0, e_{231} = 1, e_{232} = 0, e_{233} = 0$
  $e_{311} = 0, e_{312} = 1, e_{313} = 0, e_{321} = -1, e_{322} = 0, e_{323} = 0, e_{331} = 0, e_{332} = 0, e_{333} = 0$

22 LEVI CIVITA SYMBOL

- While the above equations might look arbitrary at first, a closer look shows there is a simple logic to it all. In fact, note that whenever the value of an index is repeated, the symbol has a value of zero. Furthermore, we can see that once the indices are an even arrangement (permutation) of 1,2, and 3, the symbols have the value of 1, When we have an odd arrangement, the value is -1. Again, we desire to avoid writing twenty seven equations to express this simple fact. Hence we use the index notation to define the Levi-Civita symbol as follows:

- $$e_{ijk} = \begin{cases} 1 \text{ if } i, j \text{ and } k \text{ are an even permutation of } 1,2 \text{ and } 3 \\ -1 \text{ if } i, j \text{ and } k \text{ are an odd permutation of } 1,2 \text{ and } 3 \\ 0 \;\; \text{In all other cases} \end{cases}$$

EVEN AND ODD PERMUTATIONS

- First understand that by permutations, we simply mean arrangements we make by swapping pairs. Beginning with the sequence 1,2,3 the following table shows permutations we can make in a number of times:

Swap	Result	Number	Even/Odd
1,2	2,1,3	1	Odd
2,3	3,1,2	2	Even
1,2	3,2,1	3	Odd
3,2	2,3,1	4	Even
3,1	2,1,3	5	Odd
2,1	1,2,3	6	Even
2,3	1,3,2	7	Odd

24 CROSS AND TRIPLE PRODUCTS OF BASIS VECTORS

By relying on the definition of the cross product from last lecture, we can easily see that,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k.$$

Which is simply a shorthand form for writing the nine results,

Product	Value	Product	Value	Product	Value
$\mathbf{e}_1 \times \mathbf{e}_1$	0	$\mathbf{e}_2 \times \mathbf{e}_2$	0	$\mathbf{e}_3 \times \mathbf{e}_3$	0
$\mathbf{e}_1 \times \mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_2 \times \mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_3 \times \mathbf{e}_1$	$\mathbf{e}_2$
$\mathbf{e}_2 \times \mathbf{e}_1$	$-\mathbf{e}_3$	$\mathbf{e}_3 \times \mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{e}_1 \times \mathbf{e}_3$	$-\mathbf{e}_2$

If we take the dot product of both sides above with $\mathbf{e}_l$, we obtain,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l = e_{ijk}\mathbf{e}_k \cdot \mathbf{e}_l$$

$$= e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l$$

- When $l = 1$, *the above expression becomes,* e_{ij1}; when $l = 2$, *we get* e_{ij2}; *and* when $l = 3$, *it becomes* e_{ij3}. *Motivated by this behavior, we can write that*

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l = e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l = e_{ijl}$$

CROSS AND TRIPLE PRODUCTS OF BASIS VECTORS

- From the above,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l = e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l = e_{ijl}$$

- We can easily see that,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = \mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = e_{ijk}$$

- Since the interchange of the cross and the dot has no effect on a triple product.

THE VECTOR CROSS

- We are now in a position to introduce the important concept of the vector cross. Before we go into this matter, notice that once the vector cross is understood, the concept of a cross or vector product will no longer be necessary. Its results will be contained in the more useful vector cross.
- As before, consider the Cartesian based vector,
  - $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$
- Using this vector, let us form a tensor using only the components of the above tensor. The tensor we shall form will be fully determined once we know the above vector. We shall be interested in the relationship between the vector and the tensor we can form from its components:

27 VECTOR CROSS

- Take a look at the tensor,
  - $-a_3\mathbf{e}_1 \otimes \mathbf{e}_2 + a_2\mathbf{e}_1 \otimes \mathbf{e}_3 + a_3\mathbf{e}_2 \otimes \mathbf{e}_1 - a_1\mathbf{e}_2 \otimes \mathbf{e}_3 - a_2\mathbf{e}_3 \otimes \mathbf{e}_1 + a_1\mathbf{e}_3 \otimes \mathbf{e}_2$
- Or, to get greater clarity,
  - $\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$
- When expressed in terms of only its components as a matrix. Observe that you can always form the above tensor if you are given any vector. The converse is not true because a standard tensor can have nine independent components. It will not always be possible to form a unique vector from a tensor while it is ALWAYS possible to form a tensor from a vector.

- The vector we have formed from a vector is called the Vector Cross for the particular vector from which it is formed. We represent what we have just done, notionally by writing,

VECTOR CROSS

$$\mathbf{A} = \mathbf{a} \times$$

- That is, the tensor $\mathbf{A}$ is the Vector Cross of the vector $\mathbf{a}$. Compactly, using the Levi-Civital symbol and the Einstein summation convention, we can write,

$$\mathbf{A} = e_{ijk} a_j \mathbf{e}_i \otimes \mathbf{e}_k$$

- Consequence: Given any vector $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$, the cross product $\mathbf{a} \times \mathbf{b}$ can be obtained without taking any cross product at all! We simply operate the tensor $\mathbf{A} = e_{ijk} a_j \mathbf{e}_i \otimes \mathbf{e}_k$ on $\mathbf{b}$:

$$\mathbf{Ab} = e_{ijk} a_j (\mathbf{e}_i \otimes \mathbf{e}_k)\mathbf{b} = \mathbf{a} \times \mathbf{b}$$