RECOMMENDED TEXTS

- This course was prepared with several textbooks and papers. They will be listed below. However, the main course text is: Gurtin ME, Fried E & Anand L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, www.cambridge.org 2010
- The course will cover pp1-240 of the book. You can view the course as a way to assist your reading and understanding of this book.
- Reddy, JN, **Principles of Continuum Mechanics**, Cambridge University Press, www.cambridge.org 2012

2 HOMEWORK 1 DUE MARCH 26

1. Given that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors, find the values of scalars α and β in the equation, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}$

2. (a) Given that $\forall\ \mathbf{v} \in \mathscr{V}, \boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\boldsymbol{a} = \mathbf{b}$. (b) Given that $\forall\ \mathbf{v} \in \mathscr{V}$, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$, show that $\boldsymbol{a} = \mathbf{b}$

3. Given that $\mathbf{n}$ is a unit vector, use the fact that $\mathbf{n} \cdot \mathbf{u}$ is the projection of the vector $\mathbf{u}$ in the direction of $\mathbf{n}$ to represent $\mathbf{u}$ as $(\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ or $(\mathbf{n} \otimes \mathbf{n})\mathbf{u} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$.

4. Simplify the following by employing the substitution properties of the Kronecker delta and the definition of the Levi Civita Symbol.
$(a) e_{ijk}\delta_{kn}$, $(b) e_{ijk}\delta_{is}\delta_{jm}$ $(c) e_{ijk}\delta_{is}\delta_{jm}$ $(d)\ a_{ij}\delta_{in}$ $(e)\delta_{ij}\delta_{jn}$ $(f)\delta_{ij}\delta_{jn}\delta_{ni}$

3 HOMEWORK CONTINUED

5. Show that the Cylindrical Polar basis vectors,

$$\mathbf{e}_r(r,\phi,z) = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$$
$$\mathbf{e}_\phi(r,\phi,z) = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$
$$\mathbf{e}_z(r,\phi,z) = \mathbf{k}$$

constitute an orthonormal system.

6. Given that the position vector in spherical coordinates is given by $\mathbf{R} = \rho\mathbf{e}_\rho(\theta,\phi)$, where $\mathbf{e}_\rho = \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$ show that the set $\left\{\frac{\partial\mathbf{R}}{\partial\rho}, \frac{\partial\mathbf{R}}{\partial\theta}, \frac{\partial\mathbf{R}}{\partial\phi}\right\}$ forms a basis set of orthogonal vectors. Is the set orthonormal?

Note: A subset of vectors in a vector space forms a basis if all other vectors in the space can be written as a linear combination of vectors in the subset. T is possible to have several bases in a given space.

UNFINISHED BUSINESS: SUMMATION CONVENTION

- We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we deal with higher ranked objects, such notational conveniences become even more important. We shall often deal with coordinate transformations.

5 SUMMATION CONVENTION

- When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

6 SUMMATION CONVENTION

- Consider transformation equations such as,

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

- We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j$$

$$y_2 = \sum_{j=1}^{n} a_{2j}x_j$$

$$y_3 = \sum_{j=1}^{n} a_{3j}x_j$$

SUMMATION CONVENTION

- In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j$$
$$y_2 = a_{2j}x_j$$
$$y_3 = a_{3j}x_j$$

- Finally, we observe that $y_1, y_2,$ and y_3 can be represented as we have been doing by $y_i,\ i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \qquad i = 1,2,3$$

8 SUMMATION CONVENTION

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij} x_j$$

to represent compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

9 SUMMATION CONVENTION

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

for $i = 1, k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1, k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1, k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2, k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2, k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2, k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3, k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3, k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3, k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

10 SUMMATION CONVENTION

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}$$

In this same way, we could have also proved that,

$$a_{ij}a_{kj} = b_{ik}$$

- Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

VECTOR COMPONENTS IN ONB

Consider an orthogonal normalized basis (ONB)

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \qquad or\ \mathbf{e}_i, i = 1,2,3$$

The fact that $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$, and also that the norm on each, that is, $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$ make the computation of the components on any vector referred to this bases easy.

This is the main attraction of the ONB, also called Cartesian Coordinates.

12 VECTOR COMPONENTS IN ONB

Given a known vector $\mathbf{F}$, (known because its magnitude and direction are given) we can find its components in the ONB in these simple steps:

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

Where F_1, F_2 and F_3, the scalar quantities to be determined are called the components of $\mathbf{F}$ in the ONB $\mathbf{e}_i, i = 1,2,3$

Take the scalar product of the above equation with $\mathbf{e}_1$, we have,

13 VECTOR COMPONENTS IN ONB

$$\mathbf{F}\cdot\mathbf{e}_1 = F_1\mathbf{e}_1\cdot\mathbf{e}_1 + F_2\mathbf{e}_2\cdot\mathbf{e}_1 + F_3\mathbf{e}_3\cdot\mathbf{e}_1$$
$$= F_1\mathbf{e}_1\cdot\mathbf{e}_1 + \boxed{F_2\boldsymbol{e}_2\cdot\boldsymbol{e}_1} + \boxed{F_3\mathbf{e}_3\cdot\mathbf{e}_1}$$

The boxed items vanish on account of orthogonality while the first term simply becomes F_1 on account of normality. We therefore find that $F_1 = \mathbf{F}\cdot\mathbf{e}_1$. We can repeat this process by taking the dot product with the other basis vectors and find also that, $F_2 = \mathbf{F}\cdot\mathbf{e}_2$ and that $F_3 = \mathbf{F}\cdot\mathbf{e}_3$.

In index notation, we could write the nine orthonormality equations as,

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$$

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$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$

i	j	$\mathbf{e}_i \cdot \mathbf{e}_j$	δ_{ij}	Value
1	1	$\mathbf{e}_1 \cdot \mathbf{e}_1$	δ_{11}	1
1	2	$\mathbf{e}_1 \cdot \mathbf{e}_2$	δ_{12}	0
1	3	$\mathbf{e}_1 \cdot \mathbf{e}_3$	δ_{13}	0
2	1	$\mathbf{e}_2 \cdot \mathbf{e}_1$	δ_{21}	0
2	2	$\mathbf{e}_2 \cdot \mathbf{e}_2$	δ_{22}	1
2	3	$\mathbf{e}_2 \cdot \mathbf{e}_3$	δ_{23}	0
3	1	$\mathbf{e}_3 \cdot \mathbf{e}_1$	δ_{31}	0

15 VECTOR COMPONENTS IN ONB

Furthermore, the result of calculating the components, that

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 \\ &= (\mathbf{F}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{F}\cdot\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{F}\cdot\mathbf{e}_3)\mathbf{e}_3\end{aligned}$$

Can also be written as,

$$\mathbf{F} = F_i\mathbf{e}_i = \left(\mathbf{F}\cdot\mathbf{e}_j\right)\mathbf{e}_j$$

Where the repetition of the indices indicate summation.

In the above computation, everything was easy because the orthogonality and normality conditions made some terms vanish and the other terms were products with unity. It is this ease of computation that makes the ONB very attractive.

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VECTOR COMPONENTS IN ONB

When we relax the conditions on our basis vectors, these nice properties disappear. Yet, there is an extension of this situation, which, if understood deeply, will make things relatively easy also.

It is easy to show that orthogonality and normality assumptions we have made thus far are only for convenience. We can define coordinate systems that do not make those assumptions. All we need are three vectors that are linearly independent.

CARTESIAN VECTOR COMPONENTS

It is convenient for us to represent the vectors of the Cartesian system of coordinates as $\mathbf{e}_i, i = 1,2,3$ which is just a shorthand for writing $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$instead of calling these unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

This change may look trivial at first, but when combined with Einstein's convention we will introduce, allows us the benefit of reducing the number of terms in expressions in a unique way necessary to express tensor terms.

18 VECTOR COMPONENTS

Clearly, addition and linearity of the vector space ⇒

$$\mathbf{v} + \mathbf{w} = (v_i + w_i)\mathbf{e}_i$$
$$= (v_1 + w_1)\mathbf{e}_1 + (v_2 + w_2)\mathbf{e}_2 + (v_3 + w_3)\mathbf{e}_3$$

Multiplication by scalar rule implies that if $\alpha \in \mathscr{R}, \forall \mathbf{v} \in \mathscr{V}$,

$$\alpha\, \mathbf{v} = (\alpha v_i)\mathbf{e}_i$$
$$= (\alpha v_1)\mathbf{e}_1 + (\alpha v_2)\mathbf{e}_2 + (\alpha v_3)\mathbf{e}_3$$

Because of the implicit summation implied by the repetition of indices.

19 KRONECKER DELTA

Kronecker Delta: δ_{ij} has the following properties:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$
$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$
$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

As is obvious, these are obtained by allowing the indices to attain all possible values in the range. The Kronecker delta is defined by the fact that when the indices explicit values are equal, it has the value of unity. Otherwise, it is zero. The above nine equations can be written more compactly as,

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

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LEVI CIVITA SYMBOL

- The Levi-Civita Symbol: e_{ijk}
- $e_{111} = 0, e_{112} = 0, e_{113} = 0, e_{121} = 0, e_{122} = 0, e_{123} = 1, e_{131} = 0, e_{132} = -1, e_{133} = 0$
  $e_{211} = 0, e_{212} = 0, e_{213} = -1, e_{221} = 0, e_{222} = 0, e_{223} = 0, e_{231} = 1, e_{232} = 0, e_{233} = 0$
  $e_{311} = 0, e_{312} = 1, e_{313} = 0, e_{321} = -1, e_{322} = 0, e_{323} = 0, e_{331} = 0, e_{332} = 0, e_{333} = 0$

21 LEVI CIVITA SYMBOL

- While the above equations might look arbitrary at first, a closer look shows there is a simple logic to it all. In fact, note that whenever the value of an index is repeated, the symbol has a value of zero. Furthermore, we can see that once the indices are an even arrangement (permutation) of 1,2, and 3, the symbols have the value of 1, When we have an odd arrangement, the value is -1. Again, we desire to avoid writing twenty seven equations to express this simple fact. Hence we use the index notation to define the Levi-Civita symbol as follows:

- $e_{ijk} = \begin{cases} 1 \text{ if } i, j \text{ and } k \text{ are an even permutation of 1,2 and 3} \\ -1 \text{ if } i, j \text{ and } k \text{ are an odd permutation of 1,2 and 3} \\ 0 \quad \text{In all other cases} \end{cases}$

EVEN AND ODD PERMUTATIONS

- First understand that by permutations, we simply mean arrangements we make by swapping pairs. Beginning with the sequence 1,2,3 the following table shows permutations we can make in a number of times:

Swap	Result	Number	Even/Odd
1,2	2,1,3	1	Odd
2,3	3,1,2	2	Even
1,2	3,2,1	3	Odd
3,2	2,3,1	4	Even
3,1	2,1,3	5	Odd
2,1	1,2,3	6	Even
2,3	1,3,2	7	Odd

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CROSS AND TRIPLE PRODUCTS OF BASIS VECTORS

By relying on the definition of the cross product from last lecture, we can easily see that,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k.$$

Which is simply a shorthand form for writing the nine results,

Product	Value	Product	Value	Product	Value
$\mathbf{e}_1 \times \mathbf{e}_1$	0	$\mathbf{e}_2 \times \mathbf{e}_2$	0	$\mathbf{e}_3 \times \mathbf{e}_3$	0
$\mathbf{e}_1 \times \mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_2 \times \mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_3 \times \mathbf{e}_1$	$\mathbf{e}_2$
$\mathbf{e}_2 \times \mathbf{e}_1$	$-\mathbf{e}_3$	$\mathbf{e}_3 \times \mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{e}_1 \times \mathbf{e}_3$	$-\mathbf{e}_2$

If we take the dot product of both sides above with $\mathbf{e}_l$, we obtain,

$$\begin{aligned}\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l &= e_{ijk}\mathbf{e}_k \cdot \mathbf{e}_l \\ &= e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l\end{aligned}$$

- When $l = 1$, *the above expression becomes,* e_{ij1}; when $l = 2$, *we get* e_{ij2}; *and* when $l = 3$, *it becomes* e_{ij3}. *Motivated by this behavior, we can write that*

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l = e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l = e_{ijl}$$

CROSS AND TRIPLE PRODUCTS OF BASIS VECTORS

- From the above,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_l = e_{ij1}\mathbf{e}_1 \cdot \mathbf{e}_l + e_{ij2}\mathbf{e}_2 \cdot \mathbf{e}_l + e_{ij3}\mathbf{e}_3 \cdot \mathbf{e}_l = e_{ijl}$$

- We can easily see that,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = \mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = e_{ijk}$$

- Since the interchange of the cross and the dot has no effect on a triple product.

25 **Show that the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$** is $a_i b_j e_{ijk} \mathbf{e}_k$ **where a_i, b_j are the components of $\boldsymbol{a}$ and $\boldsymbol{b}$** in the Cartesian system.

Express vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as contravariant components: $\boldsymbol{a} = a_i \mathbf{e}_i$, and $\boldsymbol{b} = b_j \mathbf{e}_j$.

Using the above result, we can write that,

$$\boldsymbol{a} \times \boldsymbol{b} = (a_i \mathbf{e}_i) \times \left(b_j \mathbf{e}_j\right)$$
$$= a_i b_j (\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j e_{ijk} \mathbf{e}_k.$$

The last expression coming from the fact that $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k$.

- Show that
- $e_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{j1} & \delta_{k1} \\ \delta_{i2} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$

Given that,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Show that $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$

Expanding the equation, we have:

$$e_{rsk}e_{ijk} = \delta_{ki}\begin{vmatrix} \delta_{rj} & \delta_{rk} \\ \delta_{sj} & \delta_{sk} \end{vmatrix} - \delta_{kj}\begin{vmatrix} \delta_{ri} & \delta_{rk} \\ \delta_{si} & \delta_{sk} \end{vmatrix} + 3\begin{vmatrix} \delta_{ri} & \delta_{rj} \\ \delta_{si} & \delta_{sj} \end{vmatrix}$$

$$= \delta_{ki}(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}) - \delta_{kj}(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk})$$

$$+ 3(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj})$$

$$= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj})$$

$$= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$$

Show that $e_{rjk}e_{ijk} = 2\delta_{ri}$

Contracting one more index, we have:

$$e_{rjk}e_{ijk} = \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} = 3\delta_{ri} - \delta_{ri} = 2\delta_{ri}$$

These results are useful in several situations.

29 CONTENTS

TENSOR ALGEBRA

TENSORS AS LINEAR MAPPINGS

31 SECOND ORDER TENSOR

A second Order Tensor $\boldsymbol{T}$ is a linear mapping from a vector space to itself. Given $\boldsymbol{u} \in \mathscr{V}$ the mapping,

$$\boldsymbol{T}: \mathscr{V} \to \mathscr{V}$$

states that $\exists\, \boldsymbol{w} \in \mathscr{V}$ such that,

$$\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{w}.$$

Every other definition of a second order tensor can be derived from this simple definition. The tensor character of an object can be established by observing its action on a vector.

EXAMPLE

- $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = ?$

33 LINEARITY

- The mapping is linear. This means that if we have two runs of the process, we first input $\boldsymbol{u}$ and later input $\mathbf{v}$. The outcomes $\boldsymbol{T}(\boldsymbol{u})$ and $\boldsymbol{T}(\mathbf{v})$, added would have been the same as if we had added the inputs $\boldsymbol{u}$ and $\mathbf{v}$ first and supplied the sum of the vectors as input. More compactly, this means,

$$\boldsymbol{T}(\boldsymbol{u}+\mathbf{v})=\boldsymbol{T}(\boldsymbol{u})+\boldsymbol{T}(\mathbf{v})$$

34 EXAMPLE

- $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right] =$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

35 LINEARITY

Linearity further means that, for any scalar α and tensor $\boldsymbol{T}$

$$\boldsymbol{T}(\alpha\boldsymbol{u}) = \alpha\boldsymbol{T}(\boldsymbol{u})$$

The two properties can be added so that, given $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$, then

$$\boldsymbol{T}(\alpha\boldsymbol{u} + \beta\mathbf{v}) = \alpha\boldsymbol{T}(\boldsymbol{u}) + \beta\boldsymbol{T}(\mathbf{v})$$

The sum of two tensors is the tensor that will give an output which will be the sum of the outputs of the two tensors when each is given that input.

36 EXAMPLE

- $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \left[\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right] = \alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
- $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \left[\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right] =$

  $\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \beta \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

37 VECTOR SPACE

In general, $\alpha, \beta \in \mathscr{R}$; $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$ and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\alpha \boldsymbol{S}\boldsymbol{u} + \beta \boldsymbol{T}\boldsymbol{u} = (\alpha \boldsymbol{S} + \beta \boldsymbol{T})\boldsymbol{u}$$

With the definition above, the set of tensors constitute a vector space with its rules of addition and multiplication by a scalar. It will become obvious later that it also constitutes a Euclidean vector space with its own rule of the inner product.

SPECIAL TENSORS

Notation.

It is customary to write the tensor mapping without the parentheses. Hence, we can write,

$$\boldsymbol{T}\boldsymbol{u} \equiv \boldsymbol{T}(\boldsymbol{u})$$

For the mapping by the tensor $\boldsymbol{T}$ on the vector variable and dispense with the parentheses unless when needed.

39 ZERO TENSOR OR ANNIHILATOR

The annihilator $\boldsymbol{O}$ is defined as the tensor that maps all vectors to the zero vector, $\boldsymbol{o}$:

$$\boldsymbol{O}u = \boldsymbol{o}, \qquad \forall \boldsymbol{u} \in \mathscr{V}$$

40 THE IDENTITY

The identity tensor $\boldsymbol{I}$ is the tensor that leaves every vector unaltered. $\forall \boldsymbol{u} \in \mathscr{V}$,

$$\boldsymbol{I}\boldsymbol{u} = \boldsymbol{u}$$

Furthermore, $\forall \alpha \in \mathscr{R}$, the tensor, $\alpha\mathbf{1}$ is called a spherical tensor.

The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\boldsymbol{T} \in \mathscr{T}$ and $\boldsymbol{u} \in \mathscr{V}$, the mapping $\boldsymbol{w} \equiv \boldsymbol{T}\boldsymbol{u}$ produces a vector.

EXAMPLE

- $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

THE INVERSE

Consider a linear mapping that, operating on $\boldsymbol{w}$, produces our original argument, $\boldsymbol{u}$, if we can find it:

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{u}$$

As a linear mapping, operating on a vector, clearly, $\boldsymbol{Y}$ is a tensor. It is called the inverse of $\boldsymbol{T}$ because,

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{Y}\boldsymbol{T}\boldsymbol{u} = \boldsymbol{u}$$

So that the composition $\boldsymbol{Y}\boldsymbol{T} = \mathbf{1}$, the identity mapping. For this reason, we write,

$$\boldsymbol{Y} = \boldsymbol{T}^{-1}$$

43 INVERSE

It is easy to show that if $\boldsymbol{YT} = \mathbf{1}$, then $\boldsymbol{TY} = \boldsymbol{YT} = \mathbf{1}$.

- ***HW: Show this.***

The set of invertible sets is closed under composition. It is also closed under inversion. It forms a group with the identity tensor as the group's neutral element

44 TRANSPOSITION OF TENSORS

Given $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot \left(\boldsymbol{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of A.

A tensor indistinguishable from its transpose is said to be symmetric.

INVARIANTS

There are certain mappings from the space of tensors to the real space. Such mappings are called Invariants of the Tensor. Three of these, called Principal invariants play key roles in the application of tensors to design and analysis. We shall define them shortly.

The definition given here is free of any association with a coordinate system. It is a good practice to derive any other definitions from these fundamental ones:

46 THE TRACE

If we write

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

- where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors.

For any second order tensor $\boldsymbol{T}$, and linearly independent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the linear mapping I_1: $\mathscr{T} \to \mathscr{R}$

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{T}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \boldsymbol{T}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Is independent of the choice of the basis vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is called the First Principal Invariant of $\boldsymbol{T}$ or Trace of $\boldsymbol{T} \equiv \mathrm{tr}(\boldsymbol{T}) \equiv I_1(\boldsymbol{T})$

47 THE TRACE

Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary independent vectors let us choose the Cartesian Basis vectors, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

For any second order tensor $\boldsymbol{T}$,

$$I_1(\boldsymbol{T}) \equiv \text{tr}(\boldsymbol{T}) = [\boldsymbol{T}\mathbf{i}, \mathbf{j}, \mathbf{k}] + [\mathbf{i}, \mathbf{Tj}, \mathbf{k}] + [\mathbf{i}, \mathbf{j}, \mathbf{Tk}]$$

Since $[\mathbf{i}, \mathbf{j}, \mathbf{k}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1.$

48 THE TRACE

The trace is a linear mapping. It is easily shown that $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\mathrm{tr}(\alpha\boldsymbol{S} + \beta\boldsymbol{T}) = \alpha\mathrm{tr}(\boldsymbol{S}) + \beta\mathrm{tr}(\boldsymbol{T})$$

HW. Show this by appealing to the linearity of the vector space.

While the trace of a tensor is linear, the other two principal invariants are nonlinear. We now proceed to define them

SQUARE OF THE TRACE

The second principal invariant $I_2(\boldsymbol{S})$ is related to the trace. In fact, you may come across books that define it so. However, the most common definition is that

$$I_2(\boldsymbol{S}) = \frac{1}{2}\left[I_1^2(\boldsymbol{S}) - I_1(\boldsymbol{S}^2)\right]$$

Independently of the trace, we can also define the second principal invariant as,

50 SECOND PRINCIPAL INVARIANT

The Second Principal Invariant, $I_2(\boldsymbol{T})$, using the same notation as above is

$$\frac{[(\boldsymbol{Ta}),(\boldsymbol{Tb}),\boldsymbol{c}]+[\boldsymbol{a},(\boldsymbol{Tb}),(\boldsymbol{Tc})]+[(\boldsymbol{Ta}),\boldsymbol{b},(\boldsymbol{Tc})]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

This quantity remains unchanged for any arbitrary selection of linearly independent vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

THE DETERMINANT

The third mapping from tensors to the real space underlying the tensor is the determinant of the tensor. While you may be familiar with that operation and can easily extract a determinant from a matrix, it is important to understand the definition for a tensor that is independent of the component expression. The latter remains relevant even when we have not expressed the tensor in terms of its components in a particular coordinate system.

52 THE DETERMINANT

As before, For any second order tensor $\boldsymbol{T}$, and any linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

- The determinant of the tensor $\boldsymbol{T}$,

$$\det(\boldsymbol{T}) = \frac{[(\boldsymbol{Ta}), (\boldsymbol{Tb}), (\boldsymbol{Tc})]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

(In the special case when the chosen vectors are orthonormal, the denominator is unity)

53 OTHER PRINCIPAL INVARIANTS

- It is good to note that there are other principal invariants that can be defined. The ones we defined here are the ones you are most likely to find in other texts.
- An invariant is a scalar derived from a tensor that remains unchanged in any coordinate system. Mathematically, it is a mapping from the tensor space to the real space. Or simply **a scalar valued function of the tensor**.

54 DEVIATORIC TENSORS

- When the trace of a tensor is zero, the tensor is said to be traceless. A traceless tensor is also called a deviatoric tensor.
- Given any tensor $\mathbf{S}$, A deviatoric tensor may be created from $\mathbf{S}$ by the following process:

$$\mathbf{S}_0 \equiv \text{dev}\,\mathbf{S} \equiv \mathbf{S} - \frac{1}{3}(\text{tr}\,\mathbf{S})\mathbf{1} = \mathbf{S} - s\mathbf{1}$$

So that $s = \frac{1}{3}(\text{tr}\,\mathbf{S})$; $s\mathbf{1}$ is called the spherical part, and $\mathbf{S}_0$ as defined here is called the deviatoric part of $\mathbf{S}$.

Every tensor thus admits the decomposition,

$$\mathbf{S} = \mathbf{S}_0 + s\mathbf{1}$$

55 SYMMETRIC, ANTISYMMETRIC PARTS

Every second order tensor can be split into its symmetric and antisymmetric parts:

$$\frac{1}{2}\left(\mathbf{S}+\mathrm{S}^{\mathrm{T}}\right)+\frac{1}{2}\left(\mathbf{S}-\mathbf{S}^{\mathrm{T}}\right)\equiv \operatorname{sym}\mathrm{S}+\operatorname{skw}\mathrm{S}$$

This decomposition is unique. The component representation of these two parts will be given shortly.

56 INNER PRODUCT OF TENSORS

The trace provides a simple way to define the inner product of two second-order tensors. Given $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

The trace,

$$\operatorname{tr}(\boldsymbol{S}^T\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Is a scalar, independent of the coordinate system chosen to represent the tensors. This is defined as the inner or scalar product of the tensors $\boldsymbol{S}$ and $\boldsymbol{T}$. That is,

$$\boldsymbol{S}:\boldsymbol{T} \equiv \operatorname{tr}(\boldsymbol{S}^T\boldsymbol{T}) = \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

57 THE TENSOR PRODUCT

A product mapping from two vector spaces to $\mathscr{T}$ is defined as the tensor product. It has the following properties:

$$"\otimes": \mathscr{V} \times \mathscr{V} \rightarrow \mathscr{T}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

It is an ordered pair of vectors. It acts on any other vector by creating a new vector in the direction of its first vector as shown above. This product of two vectors is called a tensor product or a simple dyad.

58 EXAMPLE

- $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1 \quad y_2 \quad y_3) = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}$
- $(\mathbf{x} \otimes \mathbf{y})\mathbf{u} = \left[\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1 \quad y_2 \quad y_3) \right] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

$$= \begin{pmatrix} x_1 y_1 u_1 + x_1 y_2 u_2 + x_1 y_3 u_3 \\ x_2 y_1 u_1 + x_2 y_2 u_2 + x_3 y_1 u_1 \\ x_3 y_1 u_1 + x_3 y_2 u_2 + x_3 y_3 u_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1 u_1 + y_2 u_2 + y_3 u_3) = \mathbf{x}(\mathbf{y} \cdot \mathbf{u})$$

59 DYAD PROPERTIES

The tensor product is linear in its two factors.

Based on the obvious fact that for any tensor $\boldsymbol{T}$ and

$$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$$

$$\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{T}\boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = [(\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}]\boldsymbol{w}$$

It is clear that $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}$

Furthermore, the contraction,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes (\boldsymbol{T}^T \boldsymbol{v})$$

A fact that can be established by operating each side on the same vector.

60 TRANSPOSE OF A DYAD

For $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot \left(\boldsymbol{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of $\boldsymbol{A}$. Now let $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ a dyad.

$$\begin{aligned}
\mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w}) = & \\
&= \mathbf{v} \cdot [(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w}] = \mathbf{v} \cdot [\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{w})] \\
&= (\mathbf{v} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{b})(\mathbf{v} \cdot \boldsymbol{a}) \\
&= \boldsymbol{w} \cdot (\boldsymbol{b} \otimes \boldsymbol{a})\mathbf{v}
\end{aligned}$$

So that $(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}$

Showing that the transpose of a dyad is simply a reversal of its factors.

61 COMPOSITION WITH TENSORS

Operate on the vector $\boldsymbol{z}$ and let $\boldsymbol{Tz} = \boldsymbol{w}$. On the LHS,

$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{Tz} = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$. On the RHS, we have:

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T\boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}\left((\boldsymbol{T}^T\boldsymbol{v}) \cdot \boldsymbol{z}\right) = \boldsymbol{u}\left(\boldsymbol{z} \cdot (\boldsymbol{T}^T\boldsymbol{v})\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\boldsymbol{u} \otimes \left(\boldsymbol{z} \cdot (\boldsymbol{T}^T\boldsymbol{v})\right) = \boldsymbol{u} \otimes \left(\boldsymbol{v} \cdot (\boldsymbol{Tz})\right)$$

follows from the definition of transposition. Hence,

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T\boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

62 DYAD ON DYAD COMPOSITION

For $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathscr{V}$, We can show that the dyad composition,

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \otimes \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$$

Again, the proof is to show that both sides produce the same result when they act on the same vector. Let $\mathbf{y} \in \mathscr{V}$, then the LHS on $\mathbf{y}$ yields:

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x})\mathbf{y} = (\mathbf{u} \otimes \mathbf{v})[\mathbf{w}(\mathbf{x} \cdot \mathbf{y})] = \mathbf{u}(\mathbf{v} \cdot \mathbf{w})(\mathbf{x} \cdot \mathbf{y})$$

Which is obviously the result from the RHS also.

This therefore makes it straightforward to contract dyads by breaking and joining as seen above.

63 TRACE OF A DYAD

Show that the trace of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v}$.

Given any three independent vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, (No loss of generality in letting the three independent vectors be the curvilinear basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$). Using the above definition of trace, we can write that,

64 TRACE OF A DYAD

$$\mathrm{tr}(\mathbf{u}\otimes\mathbf{v}) = \frac{[\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_1\},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_2\},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_2,\{(\mathbf{u}\otimes\mathbf{v})\mathbf{e}_3\}]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,]}$$

$$= \frac{1}{e_{123}}[\{v_1\mathbf{u}\},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,\{v_2\mathbf{u}\},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_2,\{\boldsymbol{v}_3\mathbf{u}\}]$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{23i}\mathbf{e}_i) + (e_{31i}\mathbf{e}_i)\cdot(v_2\,\mathbf{u}) + (e_{12i}\mathbf{e}_i)\cdot(v_3\mathbf{u})\}$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{231}\mathbf{e}_1) + (e_{312}e_2)\cdot(v_2\mathbf{u}) + (e_{123}e_3)\cdot(v_3\mathbf{u})\} = \mathbf{u}\cdot\mathbf{v}$$

65 OTHER INVARIANTS OF A DYAD

- It is easy to show that for a tensor product

$$\boldsymbol{D} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V}$$

$$\boldsymbol{I}_2(\boldsymbol{D}) = \boldsymbol{I}_3(\boldsymbol{D}) = 0$$

HW. Show that this is so.

We proved earlier that $\boldsymbol{I}_1(\boldsymbol{D}) = \boldsymbol{u} \cdot \boldsymbol{v}$

Furthermore, if $\boldsymbol{T} \in \mathscr{T}$, then,

$$\mathrm{tr}(\boldsymbol{Tu} \otimes \boldsymbol{v}) = \mathrm{tr}(\boldsymbol{w} \otimes \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{Tu} \cdot \boldsymbol{v}$$

66 COMPONENT REPRESENTATION

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- The coefficient T_{ij} can be found by,

$$\begin{aligned} T_{ij} &= \mathbf{T}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= \mathrm{tr}\left(\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right) \\ &= \mathrm{tr}\left((\mathbf{T}\mathbf{e}_j) \otimes \mathbf{e}_i\right) \\ &= \left(\mathbf{T}\mathbf{e}_j\right) \cdot \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j \end{aligned}$$

- For the identity tensor, it is easy to show that,

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Showing that the Kronecker deltas are actually the coefficients of the identity tensor.

67 SYMMETRY

For tensor $\boldsymbol{T}$ in component form,

$$\boldsymbol{T} = \boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

The transpose,

$$\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T} = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$$
$$= T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

If the tensor is symmetrical,

$$\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}$$

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{T}^{\mathrm{T}} = T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that symmetry implies that,

$$T_{ij} = T_{ji}$$

68 ANTISYMMETRY

- A tensor is antisymmetric if its transpose is its negative. In product bases that are either covariant or contravariant, antisymmetry, like symmetry can be expressed in terms of the components:

If $\boldsymbol{T}$ is antisymmetric, then,

$$\boldsymbol{T} = -\boldsymbol{T}^{\mathrm{T}}$$

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -\boldsymbol{T}^{\mathrm{T}} = -T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that symmetry implies that,

$$T_{ij} = -T_{ji}$$

Antisymmetric tensors are also said to be skew-symmetric.

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SYMMETRIC & SKEW PARTS OF TENSORS

For any tensor $\mathbf{T}$, define the symmetric and skew parts

$\text{sym}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{\text{T}}\right)$, and $\text{skw}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^{\text{T}}\right)$. It is easy to show the following:

$\mathbf{T} = \text{sym}\,\mathbf{T} + \text{skw}\,\mathbf{T}$

$\text{skw}(\text{sym}\,\mathbf{T}) = \text{sym}(\text{skw}\,\mathbf{T}\,) = 0$. We can also write that,

$$\text{sym}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

and

$$\text{skw}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

70 COMPOSITION

Composition of tensors in component form follows the rule of the composition of dyads.

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\boldsymbol{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\begin{aligned}\boldsymbol{TS} &= \left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right) \\ &= T_{ij}S_{\alpha\beta}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right) \\ &= T_{ij}S_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\delta_{j\alpha} \\ &= T_{ij}S_{j\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta \\ &= T_{i\alpha}S_{\alpha j}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

ADDITION

- Addition of two tensors of the same order is the addition of their components provided they are refereed to the same product basis.

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\boldsymbol{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\boldsymbol{T} + \boldsymbol{S} = (T_{ij}+S_{ij})\mathbf{e}_i \otimes \mathbf{e}_j,$$

COMPONENT REPRESENTATION OF INVARIANTS

- Invoking the definition of the three principal invariants, we now find expressions for these in terms of the components of tensors in various product bases.
- First note that for $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, the triple product, $[\{\mathbf{Te}_1\}, \mathbf{e}_2, \mathbf{e}_3] = [\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3]$

$$= [\{T_{ij}\mathbf{e}_i\delta_{1j}\}, \mathbf{e}_2, \mathbf{e}_3] = T_{i1}[\mathbf{e}_i, \mathbf{e}_2, \mathbf{e}_3] = T_{i1}e_{i23}$$

73 THE TRACE

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\mathrm{tr}(\boldsymbol{T}) = \frac{[\mathbf{Te}_1, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

$$= \frac{\left[\left\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\right\}, \mathbf{e}_2, \mathbf{e}_3\right] + \left[\mathbf{e}_1, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_2, \mathbf{e}_3\right] + \left[\mathbf{e}_1, \mathbf{e}_2, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right]}{e_{123}}$$

$$= T_{i1}e_{i23} + T_{i2}e_{1i3} + T_{i3}e_{12i}$$

$$= T_{11} + T_{22} + T_{33}$$

$$= T_{ii}$$

74 SECOND INVARIANT

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{[\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

Which, in a similar way to the above, can be shown to be,

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji})$$

Which is half the square of the trace minus trace of the square of the tensor $\mathbf{T}$

75 DETERMINANT

The third invariant,

$$\frac{[(\boldsymbol{T}\mathbf{e}_{\mathbf{1}}),(\boldsymbol{T}\mathbf{e}_2),(\boldsymbol{T}\mathbf{e}_3)]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]} = e_{ijk}T_{i1}T_{j2}T_{k3}$$

$$= \det(\boldsymbol{T})$$

76 THE VECTOR CROSS

Given a vector $\boldsymbol{u} = u_i \mathbf{e}_i$, the tensor

$$(\boldsymbol{u}\times) \equiv \epsilon_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

is called a vector cross. The following relation is easily established between a the vector cross and its associated vector:

$$\forall \mathbf{v} \in \mathscr{V}, (\boldsymbol{u}\times)\mathbf{v} = \boldsymbol{u} \times \mathbf{v}$$

The vector cross is *traceless* and *antisymmetric*. *(HW. Show this)*

Traceless tensors are also called deviatoric or deviator tensors.

EXAMPLES

Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $(\mathbf{u}\,\times):(\mathbf{v}\,\times) = 2\mathbf{u}\cdot\mathbf{v}$. Hence show that $\|\mathbf{u}\,\times\| = \sqrt{2}\|\mathbf{u}\|$

ORTHOGONAL TENSORS

Given a Euclidean Vector Space $\mathscr{E}$, a tensor $\boldsymbol{Q}$ is said to be orthogonal if, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{E}$,

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$$

Specifically, we can allow $\boldsymbol{a} = \boldsymbol{b}$, so that

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{a}$$

Or

$$\|\boldsymbol{Q}\boldsymbol{a}\| = \|\boldsymbol{a}\|$$

In which case the mapping leaves the magnitude unaltered.

79 ORTHOGONAL TENSORS

Let $\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{a}$

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

By definition of the transpose, we have that,

$$\boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{q} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}$$

Clearly, $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{1}$

A condition necessary and sufficient for a tensor $\boldsymbol{Q}$ *to be orthogonal is that* $\boldsymbol{Q}$ *be invertible and its inverse equal to its transpose.*

ORTHOGONAL

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\boldsymbol{Q}^T\boldsymbol{Q}\right) = \left(\det \boldsymbol{Q}^T\right)(\det \boldsymbol{Q}) = (\det \boldsymbol{Q})^2 = 1$$

Which clearly shows that

$$(\det \boldsymbol{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "*proper orthogonal*".

ROTATION & REFLECTION

A rotation is a proper orthogonal tensor while a reflection is not.

82 ROTATION

- **Let $\boldsymbol{Q}$ be a rotation. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$**

This question is the same as showing that the cofactor of $\boldsymbol{Q}$ is $\boldsymbol{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\boldsymbol{T}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

where

$$\mathbf{T} = \mathrm{cof}(\boldsymbol{Q}) = \det(\boldsymbol{Q})\, \boldsymbol{Q}^{-\mathrm{T}}$$

Now that $\boldsymbol{Q}$ is a rotation, $\det(\boldsymbol{Q}) = 1$, and

$$\boldsymbol{Q}^{-\mathrm{T}} = (\boldsymbol{Q}^{-1})^{T} = (\boldsymbol{Q}^{\mathrm{T}})^{T} = \boldsymbol{Q}$$

This implies that $\boldsymbol{T} = \boldsymbol{Q}$ and consequently,

$$\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$