1. Show that the decomposition of a tensor into the symmetric and anti-symmetric parts is unique.

$$\mathbf{S} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{S} - \mathbf{S}^{\mathrm{T}}) = \operatorname{sym}\mathbf{S} + \operatorname{skw}\mathbf{S}$$

Suppose there is another decomposition into symmetric and antisymmetric parts similar to the above so that $\exists\mathbf{B}$ such that $\mathbf{S} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}})$. Now take the inner product of the two expressions for the tensor $\mathbf{S}$ and a symmetric tensor $\mathbf{D}$

$$\begin{aligned}
\mathbf{S}:\mathbf{D} &= (\mathbf{sym}\,\mathbf{S} + \mathbf{skw}\,\mathbf{S}):\mathbf{D} \\
&= (\mathbf{sym}\,\mathbf{S}):\mathbf{D} \\
&= \left(\frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}})\right):\mathbf{D} \\
&= \left(\frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}})\right):\mathbf{D}
\end{aligned}$$

since the inner products of symmetric and anti-symmetric tensors vanish. The above equation leads to,

$$(\operatorname{sym}\mathbf{S}):\mathbf{D} - \left(\frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}})\right):\mathbf{D} = \left(\operatorname{sym}\mathbf{S} - \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}})\right):\mathbf{D} = \mathbf{0}$$

Since $\mathbf{D}$ is arbitrary, it is clear that,

$$\operatorname{sym}\mathbf{S} - \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) = \mathbf{0}$$

or, $\operatorname{sym}\mathbf{S} \equiv \frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}})$ showing uniqueness of the symmetrical part. The uniqueness of the anti-symmetrical part is arrived at by taking the inner product with an antisymmetric tensor at the beginning.

2. The trilinear mapping, $[\,.\,,\,.\,,\,.\,]: \mathscr{V} \times \mathscr{V} \times \mathscr{V} \to \mathscr{R}$ from the product set $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to real space is defined by: $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\,\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

In component form,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon^{ijk} a_i b_j c_k$$

Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield,

$$\begin{aligned}\epsilon^{jki} b_j c_k a_i &= [\mathbf{b}, \mathbf{c}, \mathbf{a}] = \epsilon^{ijk} b_i c_j a_k \\ &= \epsilon^{kij} c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = \epsilon^{ijk} c_i a_j b_k\end{aligned}$$

The other results follow from antisymmetric arrangements and the nature of ϵ^{ijk}.

3. Given that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are linearly dependent.

Suppose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that,

$$\begin{aligned}[\mathbf{a}, \mathbf{b}, \mathbf{c}] &= \epsilon^{ijk} a_i b_j c_k = \epsilon^{ijk}(\alpha b_i + \beta c_i) b_j c_k \\ &= \alpha\epsilon^{ijk} b_i b_j c_k + \beta\epsilon^{ijk} c_i b_j c_k \\ &= 0\end{aligned}$$

Note that $b_i b_j c_k$ is symmetric in i and j, $c_i b_j c_k$ is symmetric in i and k and ϵ^{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.

4. Show that the product of a symmetric and an antisymmetric object vanishes.

Consider the product sum, $\epsilon^{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and ϵ^{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here.

$$\epsilon^{ijk} b_i b_j c_k = -\epsilon^{jik} b_i b_j c_k = -\epsilon^{jik} b_j b_i c_k = -\epsilon^{ijk} b_i b_j c_k = 0$$

The first equality on account of the antisymmetry of ϵ^{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself.

This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.

5. Show that the product $\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathbf{B}$ Can be written in indicial notation as, $a_{ij}a_{jk} = b_{ik}$.

To show this, apply summation convention and see that,

for $i = 1,\ k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1,\ k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1,\ k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2,\ k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2,\ k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2,\ k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3,\ k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3,\ k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3,\ k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

It is necessary to go through this manual process to gain the experience of seeing this transparently in future. Similary, $\mathbf{AA^T} = \mathbf{B}$ can be written in indicial notation as, $a_{ij}a_{kj} = b_{ik}$ which again becomes clear after a manual expansion after invoking the

summation convention.

6. Given that vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$, $\mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore,

$$\det \mathbf{T} = \frac{[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \neq 0$$

Consequently, $[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}] \neq 0$. Which shows that $\mathbf{Tu}$, $\mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

7. Given that vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$, $\mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, if $\mathbf{Tu}$, $\mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly dependent, then $\exists \alpha, \beta$ and γ all real such that $\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{o}$. But $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent. This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \neq \mathbf{o}$.

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = \boldsymbol{o}.$$

This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!

8. Given that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

If $\mathbf{T}$ is not singular, if $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly dependent, then $\exists\alpha$, and β both real such that $\alpha\mathbf{Tu} + \beta\mathbf{Tv} = \mathbf{o}$. But $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent. This means that $\alpha\mathbf{u} + \beta\mathbf{v} \neq \mathbf{o}$.

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} = \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \boldsymbol{o}.$$

This means that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!

.

9. Given that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore the cofactor, $\mathbf{T}^{\mathrm{c}} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T} \neq 0$ also exists and is non-zero. The linear independence of $\mathbf{u}$ and $\mathbf{v}$ means that the parallelogram formed by them has a non-trivial area $\mathbf{u} \times \mathbf{v} \neq 0$. Now, the parallelogram formed by $\mathbf{Tu}$ and $\mathbf{Tv}$ is also non zero because,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v}) \neq 0$$

Hence $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

10. Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$

$$\begin{aligned}(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} - [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{w}]\mathbf{v} \\ &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} \\ &= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{w} \\ &= (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

Consequently,

$$\begin{aligned}[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] &= \mathbf{e} \cdot [(\mathbf{e} \times \mathbf{u}) \times (\mathbf{e} \times \mathbf{v})] \\ &= \mathbf{e} \cdot [(\mathbf{e} \otimes \mathbf{e})(\mathbf{u} \times \mathbf{v})] \\ &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{e} \otimes \mathbf{e})\mathbf{e} \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e} = [\mathbf{e}, \mathbf{u}, \mathbf{v}]\end{aligned}$$

making use of the symmetry of $(\mathbf{e} \otimes \mathbf{e})$.

11. Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, using the result, $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$, show that $[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2$

From the given result,

$$\begin{aligned}[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] &= -(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{u}) \\ &= -(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \otimes \mathbf{w})(\mathbf{v} \times \mathbf{u})\end{aligned}$$

$$= (\mathbf{u} \times \mathbf{v})\big((\mathbf{w} \cdot \mathbf{u} \times \mathbf{v})\mathbf{w}\big)$$
$$= [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2$$

12. For any tensor $\boldsymbol{A}$, define $\big(\mathrm{Sym}(\boldsymbol{A})\big)_{ij} = \frac{1}{2}\big(\boldsymbol{A}_{ij} + \boldsymbol{A}_{ji}\big)$. Show that $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A}$

Clearly, $\mathrm{Sym}(\boldsymbol{S}) = \frac{1}{2}(\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}})$

It also follows that,

$$\boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A} = \frac{1}{2}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}})\boldsymbol{A}$$
$$= \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{A})$$

But $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{A})$.

Hence $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A}$

13. Given that the trace of a dyad $\mathbf{a} \otimes \mathbf{b}$, $\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$. By expressing the tensors $\mathbf{T}$ and $\mathbf{S}$ in component form, show that $\mathrm{tr}(\mathbf{S}\mathbf{T}) = \mathrm{tr}(\mathbf{T}\mathbf{S}) = \mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$

In component form, $\mathbf{S} = S_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{T} = T_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$.

$$\mathbf{S}\mathbf{T} = S_{ij}T_{\alpha\beta}\big(\mathbf{g}^i \otimes \mathbf{g}^j\big)\big(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\big)$$

$$= S_{ij}T_{\alpha\beta}(\mathbf{g}^i \otimes \mathbf{g}^j)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$$
$$= S_{ij}T_{\alpha\beta}\mathbf{g}^i \otimes \mathbf{g}^\beta g^{j\alpha}$$
$$\mathrm{tr}(\mathbf{ST}) = S_{ij}T_{\alpha\beta}\mathbf{g}^i \cdot \mathbf{g}^\beta g^{j\alpha} = S_{ij}T_{\alpha\beta}g^{i\beta}g^{j\alpha}$$
$$= S_{ij}T^{ji}$$
$$\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = S_{ij}T_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\beta \otimes \mathbf{g}^\alpha) = S_{ij}T_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\alpha g^{i\beta}$$

so that

$$\mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = S_{ij}T_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\alpha g^{i\beta} = S_{ij}T_{\alpha\beta}g^{j\alpha}g^{i\beta}$$
$$= S_{ij}T^{ji}$$

Similar computations lead to the conclusion that

$$\mathrm{tr}(\mathbf{ST}) = \mathrm{tr}(\mathbf{TS}) = \mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$$

14. Given an arbitrary tensor $\mathbf{T}$ a skew tensor $\mathbf{W}$ and a symmetric tensor $\mathbf{S}$. Show that

$$\mathbf{S}:\mathbf{T} = \mathbf{S}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\mathrm{sym}\,\mathbf{T}$$
$$\mathbf{W}:\mathbf{T} = -\mathbf{W}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\mathrm{skw}\,\mathbf{T}$$
$$\mathbf{S}:\mathbf{W} = 0$$

Note that $\mathbf{T} = \mathrm{sym}\,\mathbf{T} + \mathrm{skw}\,\mathbf{T}$, and $\mathbf{T}^{\mathrm{T}} = \mathrm{sym}\,\mathbf{T} - \mathrm{skw}\,\mathbf{T}$. Also note that the inner product between a skew and a symmetric tensor vanishes. Consequently,

$$\mathbf{S}:\mathbf{T} = \mathbf{S}:(\mathrm{sym}\,\mathbf{T} + \mathrm{skw}\,\mathbf{T})$$
$$= \mathbf{S}:\mathrm{sym}\,\mathbf{T} + \mathbf{S}:\mathrm{skw}\,\mathbf{T}$$

$$
\begin{aligned}
&= \mathbf{S}\colon \operatorname{sym}\mathbf{T} \\
&= \mathbf{S}\colon \mathbf{T}^{\mathrm{T}} \\
\mathbf{W}\colon \mathbf{T} &= \mathbf{W}\colon (\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T}) \\
&= \mathbf{W}\colon \operatorname{sym}\mathbf{T} + \mathbf{W}\colon \operatorname{skw}\mathbf{T} \\
&= \mathbf{W}\colon \operatorname{skw}\mathbf{T} \\
&= -\mathbf{S}\colon \mathbf{T}^{\mathrm{T}}
\end{aligned}
$$

To show that $\mathbf{S}\colon \mathbf{W} = 0$. Observe that, in component form, $\mathbf{S} = S_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{W} = W_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$.

$$
\begin{aligned}
\mathbf{S}^{\mathrm{T}}\mathbf{W} &= S_{ij}W_{\alpha\beta}\left(\mathbf{g}^j \otimes \mathbf{g}^i\right)\left(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\right) \\
&= S_{ij}W_{\alpha\beta}\left(\mathbf{g}^j \otimes \mathbf{g}^i\right)\left(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\right) \\
&= S_{ij}W_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha} \\
\mathbf{S}\colon \mathbf{W} &= \operatorname{tr}\mathbf{S}^{\mathrm{T}}\mathbf{W} \\
&= S_{ij}W_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij}W_{\alpha\beta}g^{j\beta}g^{i\alpha} \\
&= S_{ij}W^{ij} = -S_{ij}W^{ji} \\
&= -S_{ji}W^{ji} = -S_{ij}W^{ij}
\end{aligned}
$$

Which vanishes because it is equal to the negative of itself.

15. Show that if for every skew tensor $\mathbf{W}$, the inner product $\mathbf{S}:\mathbf{W} = 0$, it must follow that $\mathbf{S}$ is symmetric.

$$\begin{aligned}
\mathbf{S}^{\mathrm{T}}\mathbf{W} &= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}W_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha} \\
\mathbf{S}:\mathbf{W} &= \operatorname{tr}\mathbf{S}^{\mathrm{T}}\mathbf{W} \\
&= S_{ij}W_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij}W_{\alpha\beta}g^{j\beta}g^{i\alpha} \\
&= S_{ij}W^{ij} = -S_{ij}W^{ji} = 0 = S_{ij}W^{ji}
\end{aligned}$$

Since all inner products $\mathbf{S}:\mathbf{W} = 0$. But,

$$S_{ij}W^{ij} = S_{ij}W^{ji} = S_{ji}W^{ij}$$

So that $(S_{ij} - S_{ji})W^{ij} = 0 \Rightarrow S_{ij} = S_{ji}$ Hence, $\mathbf{S}$ is symmetric.

16. Show that if for every symmetric tensor $\mathbf{S}$, the inner product $\mathbf{S}:\mathbf{W} = 0$, it must follow that $\mathbf{W}$ is anti-symmetric.

$$\begin{aligned}
\mathbf{S}^{\mathrm{T}}\mathbf{W} &= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}W_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha} \\
\mathbf{S}:\mathbf{W} &= \operatorname{tr}\mathbf{S}^{\mathrm{T}}\mathbf{W}
\end{aligned}$$

$$= S_{ij}W_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij}W_{\alpha\beta}g^{j\beta}g^{i\alpha}$$
$$= S_{ij}W^{ij} = S_{ji}W^{ij} = 0 = -S_{ji}W^{ji}$$

Since all inner products $\mathbf{S}:\mathbf{W} = 0$. But,

$$S_{ij}W^{ij} = S_{ji}W^{ji} = -S_{ji}W^{ji}$$

So that $S_{ij}\left(W^{ij} + W^{ji}\right) = 0 \;\Rightarrow W^{ij} = -W^{ji}$ Hence, $\mathbf{W}$ is anti-symmetric

17. If we can find α, β and γ unit tensor, $\mathbf{I} = \alpha\mathbf{a} \otimes \mathbf{b} + \beta\mathbf{b} \otimes \mathbf{c} + \gamma\mathbf{c} \otimes \mathbf{a}$, show that unless $\mathbf{b} \cdot \mathbf{a} = \alpha^{-1}$ then $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ cannot be linearly independent.

In the expression,

$$\mathbf{I} = \alpha\mathbf{a} \otimes \mathbf{b} + \beta\mathbf{b} \otimes \mathbf{c} + \gamma\mathbf{c} \otimes \mathbf{a}$$

Take a product on the right with vector $\mathbf{a}$,

$$\mathbf{Ia} = \alpha\mathbf{a}(\mathbf{b} \cdot \mathbf{a}) + \beta\mathbf{b}(\mathbf{c} \cdot \mathbf{a}) + \gamma\mathbf{c}(\mathbf{a} \cdot \mathbf{a})$$
$$\Rightarrow \mathbf{a}\left(1 - \alpha(\mathbf{b} \cdot \mathbf{a})\right) = \beta\mathbf{b}(\mathbf{c} \cdot \mathbf{a}) + \gamma\mathbf{c}(\mathbf{a} \cdot \mathbf{a})$$
$$\mathbf{a} = \frac{\beta\mathbf{b}(\mathbf{c} \cdot \mathbf{a})}{1 - \alpha(\mathbf{b} \cdot \mathbf{a})} + \frac{\gamma\mathbf{c}(\mathbf{a} \cdot \mathbf{a})}{\left(1 - \alpha(\mathbf{b} \cdot \mathbf{a})\right)}$$

So that this expression enables us to write $\mathbf{a}$ in terms of $\mathbf{b}$ and $\mathbf{c}$.

18. Divergence of a product: Given that φ is a scalar field and $\mathbf{v}$ a vector field, show that $\operatorname{div}(\varphi\mathbf{v}) = (\operatorname{grad}\varphi)\cdot\mathbf{v} + \varphi \operatorname{div}\mathbf{v}$

$$\begin{aligned}\operatorname{grad}(\varphi\mathbf{v}) &= \left(\varphi v^i\right)_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \varphi_{,j}\, v^i \mathbf{g}_i \otimes \mathbf{g}^j + \varphi v^i{}_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \mathbf{v} \otimes (\operatorname{grad}\varphi) + \varphi \operatorname{grad}\mathbf{v}\end{aligned}$$

Now, $\operatorname{div}(\varphi\mathbf{v}) = \operatorname{tr}\left(\operatorname{grad}(\varphi\mathbf{v})\right)$. Taking the trace of the above, we have:

$$\operatorname{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\operatorname{grad}\varphi) + \varphi \operatorname{div}\mathbf{v}$$

19. Show that $\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = (\operatorname{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u}$

$\mathbf{u}\cdot\mathbf{v} = u^i v_i$ is a scalar sum of components.

$$\begin{aligned}\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= \left(u^i v_i\right)_{,j}\,\mathbf{g}^j \\ &= u^i{}_{,j}\, v_i \mathbf{g}^j + u^i v_{i,j}\,\mathbf{g}^j\end{aligned}$$

Now $\operatorname{grad}\mathbf{u} = u^i{}_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j$ swapping the bases, we have that,

$$(\operatorname{grad}\mathbf{u})^{\mathrm{T}} = u^i{}_{,j}\left(\mathbf{g}^j \otimes \mathbf{g}_i\right).$$

Writing $\mathbf{v} = v_k\mathbf{g}^k$, we have that, $(\operatorname{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} = u^i{}_{,j}\, v_k\left(\mathbf{g}^j \otimes \mathbf{g}_i\right)\mathbf{g}^k = u^i{}_{,j}\, v_k \mathbf{g}^j \delta_i^k = u^i{}_{,j}\, v_i \mathbf{g}^j$

It is easy to similarly show that $u^i v_{i,j}\,\mathbf{g}^j = (\operatorname{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u}$. Clearly,

$$\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = \left(u^i v_i\right)_{,j}\,\mathbf{g}^j = u^i{}_{,j}\, v_i \mathbf{g}^j + u^i v_{i,j}\,\mathbf{g}^j$$

$$= (\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\text{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$$

As required.

20. Show that $\text{grad}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times)\text{grad}\,\mathbf{v} \; - \; (\mathbf{v} \times)\text{grad}\,\mathbf{u}$

$$\mathbf{u} \times \mathbf{v} = \epsilon^{ijk} u_j v_k \mathbf{g}_i$$

Recall that the gradient of this vector is the tensor,

$$\begin{aligned}
\text{grad}(\mathbf{u} \times \mathbf{v}) &= \left(\epsilon^{ijk} u_j v_k\right)_{,l} \mathbf{g}_i \otimes \mathbf{g}^l \\
&= \epsilon^{ijk} u_{j,l}\, v_k \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} u_j v_{k,l}\, \mathbf{g}_i \otimes \mathbf{g}^l \\
&= -\epsilon^{ikj} u_{j,l}\, v_k \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} u_j v_{k,l}\, \mathbf{g}_i \otimes \mathbf{g}^l \\
&= -\,(\mathbf{v} \times)\text{grad}\,\mathbf{u} + (\mathbf{u} \times)\text{grad}\,\mathbf{v}
\end{aligned}$$

21. Show that $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$

We already have the expression for $\text{grad}(\mathbf{u} \times \mathbf{v})$ above; remember that

$$\begin{aligned}
\text{div}\,(\mathbf{u} \times \mathbf{v}) &= \text{tr}[\text{grad}(\mathbf{u} \times \mathbf{v})] \\
&= -\epsilon^{ikj} u_{j,l}\, v_k \mathbf{g}_i \cdot \mathbf{g}^l + \epsilon^{ijk} u_j v_{k,l}\, \mathbf{g}_i \cdot \mathbf{g}^l \\
&= -\epsilon^{ikj} u_{j,l}\, v_k \delta_i^l + \epsilon^{ijk} u_j v_{k,l}\, \delta_i^l \\
&= -\epsilon^{ikj} u_{j,i}\, v_k + \epsilon^{ijk} u_j v_{k,i} = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}
\end{aligned}$$

22. Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that $\text{curl}\,(\phi\mathbf{v}) = \phi\,\text{curl}\,\mathbf{v} + (\text{grad}\,\phi) \times \mathbf{v}$.

$$\begin{aligned}
\text{curl}\,(\phi\mathbf{v}) &= \epsilon^{ijk}(\phi v_k)_{,j}\,\mathbf{g}_i \\
&= \epsilon^{ijk}\big(\phi_{,j}\,v_k + \phi v_{k,j}\big)\mathbf{g}_i \\
&= \epsilon^{ijk}\phi_{,j}\,v_k\mathbf{g}_i + \epsilon^{ijk}\phi v_{k,j}\,\mathbf{g}_i \\
&= (\text{grad}\,\phi) \times \mathbf{v} + \phi\,\text{curl}\,\mathbf{v}
\end{aligned}$$

23. Show that $\text{div}\,(\mathbf{u} \otimes \mathbf{v}) = (\text{div}\,\mathbf{v})\mathbf{u} + (\text{grad}\,\mathbf{u})\mathbf{v}$

$\mathbf{u} \otimes \mathbf{v}$ is the tensor, $u^i v^j \mathbf{g}_i \otimes \mathbf{g}_j$. The gradient of this is the third order tensor,

$$\text{grad}\,(\mathbf{u} \otimes \mathbf{v}) = \big(u^i v^j\big)_{,k}\,\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k$$

And by divergence, we mean the contraction of the last basis vector:

$$\begin{aligned}
\text{div}\,(\mathbf{u} \otimes \mathbf{v}) &= \big(u^i v^j\big)_k\big(\mathbf{g}_i \otimes \mathbf{g}_j\big)\mathbf{g}^k \\
&= \big(u^i v^j\big)_k\mathbf{g}_i\delta^k_j = \big(u^i v^j\big)_j\mathbf{g}_i \\
&= u^i{}_{,j}\,v^j\mathbf{g}_i + u^i v^j{}_{,j}\,\mathbf{g}_i \\
&= (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{div}\,\mathbf{v})\mathbf{u}
\end{aligned}$$

24. For a scalar field ϕ and a tensor field $\mathbf{T}$ show that grad $(\phi\mathbf{T}) = \phi\text{grad}\,\mathbf{T} + \mathbf{T} \otimes \text{grad}\phi$. Also show that div $(\phi\mathbf{T}) = \phi\ \text{div}\,\mathbf{T} + \mathbf{T}\text{grad}\phi$

$$\begin{aligned}\text{grad}(\phi\mathbf{T}) &= \left(\phi T^{ij}\right)_{,k}\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= \left(\phi_{,k}\,T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= \mathbf{T} \otimes \text{grad}\phi + \ \phi\text{grad}\,\mathbf{T}\end{aligned}$$

Furthermore, we can contract the last two bases and obtain,

$$\begin{aligned}\text{div}(\phi\mathbf{T}) &= \left(\phi_{,k}\,T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^k \\ &= \left(\phi_{,k}\,T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i\delta^k_j \\ &= T^{ik}\,\phi_{,k}\,\mathbf{g}_i + \ \phi T^{ik}{}_{,k}\,\mathbf{g}_i \\ &= \mathbf{T}\text{grad}\phi + \ \phi\ \text{div}\,\mathbf{T}\end{aligned}$$

25. For two arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$, show that $\text{grad}(\mathbf{ST}) = (\text{grad}\,\mathbf{S}^{\mathrm{T}})^{\mathrm{T}}\mathbf{T} + \mathbf{S}\,\text{grad}\,\mathbf{T}$

$$\begin{aligned}\text{grad}(\mathbf{ST}) &= \left(S_{ij}T^{jk}\right)_{,\alpha}\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\ &= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\ &= \left(T^{kj}S_{ji,\alpha} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\ &= (\text{grad}\,\mathbf{S}^{\mathrm{T}})^{\mathrm{T}}\mathbf{T} + \mathbf{S}\,\text{grad}\,\mathbf{T}\end{aligned}$$

26. For two arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$, show that $\text{div}(\mathbf{ST}) = (\text{grad}\,\mathbf{S}):\mathbf{T} + \mathbf{T}\,\text{div}\,\mathbf{S}$

$$\text{grad}(\mathbf{ST}) = \left(S_{ij}T^{jk}\right)_{,\alpha}\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha$$

$$
\begin{aligned}
&= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\
\operatorname{div}(\mathbf{ST}) &= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i(\mathbf{g}_k \cdot \mathbf{g}^\alpha) \\
&= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i\delta_k^\alpha \\
&= \left(S_{ij,k}T^{jk} + S_{ij}T^{jk}{}_{,k}\right)\mathbf{g}^i \\
&= (\operatorname{grad}\mathbf{S}) : \mathbf{T} + \mathbf{S}\operatorname{div}\mathbf{T}
\end{aligned}
$$

27. For two arbitrary vectors, $\mathbf{u}$ and $\mathbf{v}$, show that $\operatorname{grad}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times)\operatorname{grad}\mathbf{v} - (\mathbf{v} \times)\operatorname{grad}\mathbf{u}$

$$
\begin{aligned}
\operatorname{grad}(\mathbf{u} \times \mathbf{v}) &= \left(\epsilon^{ijk}u_j v_k\right)_{,l}\,\mathbf{g}_i \otimes \mathbf{g}^l \\
&= \left(\epsilon^{ijk}u_{j,l}\,v_k + \epsilon^{ijk}u_j v_{k,l}\right)\mathbf{g}_i \otimes \mathbf{g}^l \\
&= \left(u_{j,l}\,\epsilon^{ijk}v_k + v_{k,l}\,\epsilon^{ijk}u_j\right)\mathbf{g}_i \otimes \mathbf{g}^l \\
&= -(\mathbf{v} \times)\operatorname{grad}\mathbf{u} + (\mathbf{u} \times)\operatorname{grad}\mathbf{v}
\end{aligned}
$$

28. For a vector field $\mathbf{u}$, show that $\operatorname{grad}(\mathbf{u} \times)$ is a third ranked tensor. Hence or otherwise show that $\operatorname{div}(\mathbf{u} \times) = -\operatorname{curl}\mathbf{u}$.

The second-order tensor $(\mathbf{u} \times)$ is defined as $\epsilon^{ijk}u_j\mathbf{g}_i \otimes \mathbf{g}_k$. Taking the covariant derivative with an independent base, we have

$$
\operatorname{grad}(\mathbf{u} \times) = \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}^l
$$

This gives a third order tensor as we have seen. Contracting on the last two bases,

$$\begin{aligned}\operatorname{div}(\mathbf{u}\times) &= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \otimes \mathbf{g}_k \cdot \mathbf{g}^l \\ &= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \delta^l_k \\ &= \epsilon^{ijk} u_{j,k}\, \mathbf{g}_i \\ &= -\operatorname{curl} \mathbf{u}\end{aligned}$$

29. Show that div $(\phi\mathbf{I}) = \operatorname{grad}\phi$

Note that $\phi\mathbf{I} = (\phi g_{\alpha\beta})\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$. Also note that

$$\operatorname{grad}\phi\mathbf{I} = (\phi g_{\alpha\beta})_{,i}\, \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \otimes \mathbf{g}^i$$

The divergence of this third order tensor is the contraction of the last two bases:

$$\begin{aligned}\operatorname{div}(\phi\mathbf{I}) = \operatorname{tr}(\operatorname{grad}\phi\mathbf{I}) &= (\phi g_{\alpha\beta})_{,i}\,(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)\mathbf{g}^i = (\phi g_{\alpha\beta})_{,i}\, \mathbf{g}^\alpha g^{\beta i} \\ &= \phi_{,i}\, g_{\alpha\beta} g^{\beta i} \mathbf{g}^\alpha \\ &= \phi_{,i}\, \delta^i_\alpha \mathbf{g}^\alpha = \phi_{,i}\, \mathbf{g}^i = \operatorname{grad}\phi\end{aligned}$$

30. Show that curl $(\phi\mathbf{I}) = (\operatorname{grad}\phi)\times$

Note that $\phi\mathbf{I} = (\phi g_{\alpha\beta})\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$, and that $\operatorname{curl}\mathbf{T} = \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$ so that,

$$\begin{aligned}\operatorname{curl}(\phi\mathbf{I}) &= \epsilon^{ijk}(\phi g_{\alpha k})_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha \\ &= \epsilon^{ijk}(\phi_{,j}\, g_{\alpha k})\mathbf{g}_i \otimes \mathbf{g}^\alpha = \epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k \\ &= (\operatorname{grad}\phi)\times\end{aligned}$$

31. Show that the dyad $\mathbf{u} \otimes \mathbf{v}$ is NOT, in general symmetric: $\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \times \mathbf{v}) \times$

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \epsilon^{ijk} u_j v_k \, \mathbf{g}_i \\
\big((\mathbf{u} \times \mathbf{v}) \times\big) &= \epsilon_{\alpha i \beta} \epsilon^{ijk} u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= -\left(\delta^j_\alpha \delta^k_\beta - \delta^k_\alpha \delta^j_\beta\right) u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= \left(-u_\alpha v_\beta + u_\beta v_\alpha\right) \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}
\end{aligned}$$

32. Show that curl $(\mathbf{v} \times) = (\text{div}\, \mathbf{v})\mathbf{I} - \text{grad}\, \mathbf{v}$

$$(\mathbf{v} \times) = \epsilon^{\alpha\beta k} v_\beta \, \mathbf{g}_\alpha \otimes \mathbf{g}_k$$

$$\text{curl}\, \mathbf{T} = \epsilon^{ijk} T_{\alpha k, j} \, \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

so that

$$\begin{aligned}
\text{curl}\, (\mathbf{v} \times) &= \epsilon^{ijk} \epsilon^{\alpha\beta k} v_{\beta,j} \, \mathbf{g}_i \otimes \mathbf{g}_\alpha \\
&= \left(g^{i\alpha} g^{j\beta} - g^{i\beta} g^{j\alpha}\right) v_{\beta,j} \, \mathbf{g}_i \otimes \mathbf{g}_\alpha \\
&= v^j{}_{,j} \, \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha - v^i{}_{,j} \, \mathbf{g}_i \otimes \mathbf{g}^j \\
&= (\text{div}\, \mathbf{v})\mathbf{I} - \text{grad}\, \mathbf{v}
\end{aligned}$$

33. Show that div $(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\, \mathbf{u} - \mathbf{u} \cdot \text{curl}\, \mathbf{v}$

$$\text{div}\,(\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk} u_j v_k\right)_{,i}$$

Noting that the tensor ϵ^{ijk} behaves as a constant under a covariant differentiation, we can write,

$$\begin{aligned}\text{div}\,(\mathbf{u} \times \mathbf{v}) &= \left(\epsilon^{ijk} u_j v_k\right)_{,i} \\ &= \epsilon^{ijk} u_{j,i}\, v_k + \epsilon^{ijk} u_j v_{k,i} \\ &= \mathbf{v} \cdot \text{curl}\, \mathbf{u} - \mathbf{u} \cdot \text{curl}\, \mathbf{v}\end{aligned}$$

34. Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that curl $(\phi\mathbf{v}) = \phi\, \text{curl}\, \mathbf{v} + (\text{grad}\, \phi) \times \mathbf{v}$.

$$\begin{aligned}\text{curl}\,(\phi\mathbf{v}) &= \epsilon^{ijk} (\phi v_k)_{,j}\, \mathbf{g}_i \\ &= \epsilon^{ijk} \left(\phi_{,j}\, v_k + \phi v_{k,j}\right) \mathbf{g}_i \\ &= \epsilon^{ijk} \phi_{,j}\, v_k \mathbf{g}_i + \epsilon^{ijk} \phi v_{k,j}\, \mathbf{g}_i \\ &= (\text{grad}\, \phi) \times \mathbf{v} + \phi\, \text{curl}\, \mathbf{v}\end{aligned}$$

35. Show that curl $(\text{grad}\, \phi) = \mathbf{o}$

For any tensor $\mathbf{v} = v_\alpha \mathbf{g}^\alpha$

$$\text{curl}\, \mathbf{v} = \epsilon^{ijk} v_{k,j}\ \mathbf{g}_i$$

Let $\mathbf{v} =$ grad ϕ. Clearly, in this case, $v_k = \phi_{,k}$ so that $v_{k,j} = \phi_{,kj}$. It therefore follows

that,

$$\text{curl}\,(\text{grad}\,\phi) = \epsilon^{ijk}\phi_{,kj}\,\mathbf{g}_i = \mathbf{o}.$$

The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the scalar field is immaterial. This will be true only if the scalar field is continuous – a proposition we have assumed in the above.

36. Show that curl (grad $\mathbf{v}$) $= \mathbf{0}$

For any tensor $\mathbf{T} = \mathrm{T}_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

$$\text{curl}\,\mathbf{T} = \epsilon^{ijk}T_{\alpha k,j}\;\mathbf{g}_i \otimes \mathbf{g}^\alpha$$

Let $\mathbf{T} =$ grad $\mathbf{v}$. Clearly, in this case, $\mathrm{T}_{\alpha\beta} = v_{\alpha,\beta}$ so that $T_{\alpha k,j} = v_{\alpha,kj}$. It therefore follows that,

$$\text{curl}\,(\text{grad}\,\mathbf{v}) = \epsilon^{ijk}v_{\alpha,kj}\,\mathbf{g}_i \otimes \mathbf{g}^\alpha = \mathbf{0}.$$

The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the vector field is immaterial. This will be true only if the vector field is continuous – a proposition we have assumed in the above.

37. Show that curl $(\text{grad}\,\mathbf{v})^{\mathrm{T}} = \text{grad}(\text{curl}\,\mathbf{v})$

From previous derivation, we can see that, curl $\mathbf{T} = \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$. Clearly,

$$\text{curl}\,\mathbf{T}^{\mathrm{T}} = \epsilon^{ijk} T_{k\alpha,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

so that curl $(\text{grad}\,\mathbf{v})^{\mathrm{T}} = \epsilon^{ijk} v_{k,\alpha j}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha$. But curl $\mathbf{v} = \epsilon^{ijk} v_{k,j}\ \mathbf{g}_i$. The gradient of this is,

$$\text{grad}(\text{curl}\,\mathbf{v}) = \left(\epsilon^{ijk} v_{k,j}\right)_{,\alpha} \mathbf{g}_i \otimes \mathbf{g}^\alpha = \epsilon^{ijk} v_{k,j\alpha}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha = \text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}}$$

38. Show that div $(\text{grad}\,\phi \times \text{grad}\,\vartheta) = 0$

$$\text{grad}\,\phi \times \text{grad}\,\theta = \epsilon^{ijk} \phi_{,j}\, \theta_{,k}\, \mathbf{g}_i$$

The gradient of this vector is the tensor,

$$\begin{aligned}\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta) &= \left(\epsilon^{ijk} \phi_{,j}\, \theta_{,k}\right)_{,l} \mathbf{g}_i \otimes \mathbf{g}^l \\ &= \epsilon^{ijk} \phi_{,jl}\, \theta_{,k}\, \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} \phi_{,j}\, \theta_{,kl}\, \mathbf{g}_i \otimes \mathbf{g}^l\end{aligned}$$

The trace of the above result is the divergence we are seeking:

$$\begin{aligned}\text{div}\,(\text{grad}\,\phi \times \text{grad}\,\theta) &= \text{tr}[\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta)] \\ &= \epsilon^{ijk} \phi_{,jl}\, \theta_{,k}\, \mathbf{g}_i \cdot \mathbf{g}^l + \epsilon^{ijk} \phi_{,j}\, \theta_{,kl}\, \mathbf{g}_i \cdot \mathbf{g}^l \\ &= \epsilon^{ijk} \phi_{,jl}\, \theta_{,k}\, \delta_i^l + \epsilon^{ijk} \phi_{,j}\, \theta_{,kl}\, \delta_i^l \\ &= \epsilon^{ijk} \phi_{,ji}\, \theta_{,k} + \epsilon^{ijk} \phi_{,j}\, \theta_{,ki} = 0\end{aligned}$$

Each term vanishing on account of the contraction of a symmetric tensor with an

antisymmetric.

39. Show that curl curl $\mathbf{v} = \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$

Let $\mathbf{w} = \text{curl}\,\mathbf{v} \equiv \epsilon^{ijk} v_{k,j}\,\mathbf{g}_i$. But $\text{curl}\,\mathbf{w} \equiv \epsilon^{\alpha\beta\gamma} w_{\gamma,\beta}\,\mathbf{g}_\alpha$. Upon inspection, we find that $w_\gamma = g_{\gamma i}\epsilon^{ijk} v_{k,j}$ so that

$$\text{curl}\,\mathbf{w} \equiv \epsilon^{\alpha\beta\gamma}\left(g_{\gamma i}\epsilon^{ijk} v_{k,j}\right)_{,\beta}\,\mathbf{g}_\alpha = g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} v_{k,j\beta}\,\mathbf{g}_\alpha$$

Now, it can be shown that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$ so that,

$$\begin{aligned}\text{curl}\,\mathbf{w} &= \left(g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}\right) v_{k,j\beta}\,\mathbf{g}_\alpha \\ &= v^\beta{}_{,j\beta}\,\mathbf{g}^j - g^{\beta j} v^\alpha{}_{,j\beta}\,\mathbf{g}_\alpha \\ &= \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}\end{aligned}$$

Also recall that the Laplacian (grad^2) of a scalar field ϕ is, $\text{grad}^2\phi = g^{ij}\phi_{,ij}$. In Cartesian coordinates, this becomes,

$$\text{grad}^2\phi = g^{ij}\phi_{,ij} = \delta_{ij}\,\phi_{,ij} = \phi_{,ii}$$

as the unit (metric) tensor now degenerates to the Kronecker delta in this special case. For a vector field, $\text{grad}^2\mathbf{v} = g^{\beta j} v^\alpha{}_{,j\beta}\,\mathbf{g}_\alpha$.

Also note that while grad is a vector operator, the Laplacian (grad^2) is a scalar operator.

40. Given that $\varphi(t) = |\mathbf{A}(t)|$, Show that $\dot{\varphi}(t) = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$

$$\varphi^2 \equiv \mathbf{A} : \mathbf{A}$$

Now,

$$\frac{d}{dt}(\varphi^2) = 2\varphi \frac{d\varphi}{dt} = \frac{d\mathbf{A}}{dt} : \mathbf{A} + \mathbf{A} : \frac{d\mathbf{A}}{dt} = 2\mathbf{A} : \frac{d\mathbf{A}}{dt}$$

as inner product is commutative. We can therefore write that

$$\frac{d\varphi}{dt} = \frac{\mathbf{A}}{\varphi} : \frac{d\mathbf{A}}{dt} = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$$

as required.

41. Given a tensor field $\mathbf{T}$, obtain the vector $\mathbf{w} \equiv \mathbf{T}^{\mathrm{T}}\mathbf{v}$ and show that its divergence is $\mathbf{T} : (\nabla \mathbf{v}) + \mathbf{v} \cdot \operatorname{div} \mathbf{T}$

The gradient of $\mathbf{w}$ is the tensor, $(T_{ji} v^j)_{,k}\, \mathbf{g}^i \otimes \mathbf{g}^k$. Therefore, divergence of $\boldsymbol{w}$ (the trace of the gradient) is the scalar sum, $T_{ji} v^j{}_{,k}\, g^{ik} + T_{ji,k}\, v^j g^{ik}$. Expanding, we obtain,

$$\begin{aligned}
\operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{v}) &= T_{ji} v^j{}_{,k}\, g^{ik} + T_{ji,k}\, v^j g^{ik} \\
&= T_j^k{}_{,k}\, v^j + T_j^k v^j{}_{,k} \\
&= (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^{\mathrm{T}} \operatorname{grad} \mathbf{v}) \\
&= (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \mathbf{T} : (\operatorname{grad} \mathbf{v})
\end{aligned}$$

Recall that scalar product of two vectors is commutative so that

$$\operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{v}) = \mathbf{T}:(\operatorname{grad}\mathbf{v}) + \mathbf{v}\cdot\operatorname{div}\mathbf{T}$$

42. For a second-order tensor $\mathbf{T}$ define curl $\mathbf{T} \equiv \epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$ show that for any constant vector $\boldsymbol{a}$, $(\operatorname{curl}\mathbf{T})\,\boldsymbol{a} = \operatorname{curl}(\mathbf{T}^{\mathrm{T}}\boldsymbol{a})$

Express vector $\boldsymbol{a}$ in the invariant form with covariant components as $\boldsymbol{a} = a^\beta\mathbf{g}_\beta$. It follows that

$$\begin{aligned}(\operatorname{curl}\mathbf{T})\,\boldsymbol{a} &= \epsilon^{ijk}T_{\alpha k,j}\ (\mathbf{g}_i \otimes \mathbf{g}^\alpha)\boldsymbol{a}\\ &= \epsilon^{ijk}T_{\alpha k,j}\ a^\beta(\mathbf{g}_i \otimes \mathbf{g}^\alpha)\mathbf{g}_\beta\\ &= \epsilon^{ijk}T_{\alpha k,j}\ a^\beta\mathbf{g}_i\delta^\alpha_\beta\\ &= \epsilon^{ijk}(T_{\alpha k})_{,j}\ \mathbf{g}_i a^\alpha\\ &= \epsilon^{ijk}(T_{\alpha k}a^\alpha)_{,j}\ \mathbf{g}_i\end{aligned}$$

The last equality resulting from the fact that vector $\boldsymbol{a}$ is a constant vector. Clearly,

$$(\operatorname{curl}\mathbf{T})\,\boldsymbol{a} = \operatorname{curl}(\mathbf{T}^{\mathrm{T}}\boldsymbol{a})$$

43. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that curl $(\mathbf{u}\otimes\mathbf{v}) = [(\operatorname{grad}\mathbf{u})\mathbf{v}\times]^T + (\operatorname{curl}\mathbf{v})\otimes\boldsymbol{u}$ where $\mathbf{v}\times$ is the skew tensor $\epsilon^{ikj}v_k\ \mathbf{g}_i \otimes \mathbf{g}_j$.

Recall that the curl of a tensor $\boldsymbol{T}$ is defined by curl $\boldsymbol{T} \equiv \epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$. Clearly therefore,

$$
\begin{aligned}
\operatorname{curl}(\mathbf{u}\otimes\mathbf{v}) &= \epsilon^{ijk}(u_\alpha v_k)_{,j}\ \mathbf{g}_i\otimes\mathbf{g}^\alpha \\
&= \epsilon^{ijk}\left(u_{\alpha,j}\, v_k + u_\alpha v_{k,j}\right)\mathbf{g}_i\otimes\mathbf{g}^\alpha \\
&= \epsilon^{ijk}u_{\alpha,j}\, v_k\, \mathbf{g}_i\otimes\mathbf{g}^\alpha + \epsilon^{ijk}u_\alpha v_{k,j}\ \mathbf{g}_i\otimes\mathbf{g}^\alpha \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\right)\otimes\left(u_{\alpha,j}\,\mathbf{g}^\alpha\right) + \left(\epsilon^{ijk}v_{k,j}\ \mathbf{g}_i\right)\otimes\left(u_\alpha\mathbf{g}^\alpha\right) \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\otimes\mathbf{g}_j\right)\left(u_{\alpha,\beta}\,\mathbf{g}^\beta\otimes\mathbf{g}^\alpha\right) + \left(\epsilon^{ijk}v_{k,j}\ \mathbf{g}_i\right)\otimes\left(u_\alpha\mathbf{g}^\alpha\right) \\
&= -(\mathbf{v}\times)(\operatorname{grad}\mathbf{u})^{\boldsymbol{T}} + \ (\operatorname{curl}\mathbf{v})\otimes\mathbf{u} \\
&= [(\operatorname{grad}\mathbf{u})\mathbf{v}\times]^{\boldsymbol{T}} + \ (\operatorname{curl}\mathbf{v})\otimes\mathbf{u}
\end{aligned}
$$

upon noting that the vector cross is a skew tensor.

44. Show that $\operatorname{curl}(\mathbf{u}\times\mathbf{v}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u})$

The vector $\mathbf{w}\equiv\mathbf{u}\times\mathbf{v} = w_{\boldsymbol{k}}\mathbf{g}^{\boldsymbol{k}} = \epsilon_{k\alpha\beta}u^\alpha v^\beta\mathbf{g}^{\boldsymbol{k}}$ and $\operatorname{curl}\mathbf{w} = \boldsymbol{\epsilon}^{\boldsymbol{ijk}}w_{k,j}\,\mathbf{g}_i$. Therefore,

$$
\begin{aligned}
\operatorname{curl}(\mathbf{u}\times\mathbf{v}) &= \epsilon^{ijk}w_{k,j}\,\mathbf{g}_i \\
&= \epsilon^{ijk}\epsilon_{k\alpha\beta}\left(u^\alpha v^\beta\right)_{,j}\,\mathbf{g}_i \\
&= \left(\delta^i_\alpha\delta^j_\beta - \delta^i_\beta\delta^j_\alpha\right)\left(u^\alpha v^\beta\right)_{,j}\,\mathbf{g}_i \\
&= \left(\delta^i_\alpha\delta^j_\beta - \delta^i_\beta\delta^j_\alpha\right)\left(u^\alpha{}_{,j}\,v^\beta + u^\alpha v^\beta{}_{,j}\right)\mathbf{g}_i \\
&= \left[u^i{}_{,j}\,v^j + u^i v^j{}_{,j} - \left(u^j{}_{,j}\,v^i + u^j v^i{}_{,j}\right)\right]\mathbf{g}_i \\
&= \left[\left(u^i v^j\right)_{,j} - \left(u^j v^i\right)_{,j}\right]\mathbf{g}_i \\
&= \operatorname{div}(\mathbf{u}\otimes\mathbf{v}-\mathbf{v}\otimes\mathbf{u})
\end{aligned}
$$

since $\operatorname{div}(\mathbf{u} \otimes \mathbf{v}) = \left(u^i v^j\right)_{,\alpha} \mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^\alpha = \left(u^i v^j\right)_{,j} \mathbf{g}_i$.

45. Given a scalar point function ϕ and a second-order tensor field $\mathbf{T}$, show that $\operatorname{curl}(\phi\mathbf{T}) = \phi \operatorname{curl} \mathbf{T} + \big((\operatorname{grad}\phi)\times\big)\mathbf{T}^{\mathrm{T}}$ where $[(\operatorname{grad}\phi)\times]$ is the skew tensor $\epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k$

$$
\begin{aligned}
\operatorname{curl}(\phi\mathbf{T}) &\equiv \epsilon^{ijk}(\phi T_{\alpha k})_{,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\left(\phi_{,j}\, T_{\alpha k} + \phi T_{\alpha k,j}\right) \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\phi_{,j}\, T_{\alpha k}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha + \phi\epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \left(\epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k\right)\left(T_{\alpha\beta}\mathbf{g}^\beta \otimes \mathbf{g}^\alpha\right) + \phi\epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \phi \operatorname{curl} \mathbf{T} + \big((\operatorname{grad}\phi)\times\big)\mathbf{T}^{\mathrm{T}}
\end{aligned}
$$

46. For a second-order tensor field $\boldsymbol{T}$, show that $\operatorname{div}(\operatorname{curl}\mathbf{T}) = \operatorname{curl}(\operatorname{div}\mathbf{T}^{\mathrm{T}})$

Define the second order tensor S as

$$\operatorname{curl}\mathbf{T} \equiv \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha = S^i_{.\alpha}\mathbf{g}_i \otimes \mathbf{g}^\alpha$$

The gradient of $\boldsymbol{S}$ is $S^i_{.\alpha,\beta}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta = \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

Clearly,

$$
\begin{aligned}
\operatorname{div}(\operatorname{curl}\mathbf{T}) &= \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i\, g^{\alpha\beta} \\
&= \epsilon^{ijk} T^\beta{}_{k,j\beta}\ \mathbf{g}_i = \operatorname{curl}(\operatorname{div}\mathbf{T}^{\mathrm{T}})
\end{aligned}
$$

47. The position vector in Cartesian coordinates $\mathbf{r} = x_i \mathbf{e}_i$. Show that (a) div $\mathbf{r} = \mathbf{3}$, **(b)** div $(\mathbf{r} \otimes \mathbf{r}) = \mathbf{4r}$, **(c)** div $\mathbf{r} = 3$, and (d) grad $\mathbf{r} = \mathbf{1}$ and (e) curl $(\mathbf{r} \otimes \mathbf{r}) = -\mathbf{r} \times$

$$
\begin{aligned}
\operatorname{grad} \boldsymbol{r} &= x_{i,\boldsymbol{j}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \\
&= \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} = \mathbf{1} \\
\operatorname{div} \boldsymbol{r} &= x_{i,\boldsymbol{j}}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_{\boldsymbol{j}} \\
&= \delta_{ij}\delta_{ij} = \delta_{jj} = 3. \boldsymbol{r} \otimes \boldsymbol{r} = x_i \boldsymbol{e}_i \otimes x_j \boldsymbol{e}_j = x_i x_j \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \operatorname{grad}(\boldsymbol{r} \otimes \boldsymbol{r}) \\
&= (x_i x_j)_{,k}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \otimes \boldsymbol{e}_{\boldsymbol{k}} = (x_{i,k}\, x_j + x_i x_{j,k})\boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \cdot \boldsymbol{e}_{\boldsymbol{k}} \\
&= (\delta_{ik} \boldsymbol{x}_{\boldsymbol{j}} + \boldsymbol{x}_{\boldsymbol{i}} \delta_{jk})\delta_{jk} \boldsymbol{e}_i = (\delta_{ik} \boldsymbol{x}_{\boldsymbol{k}} + \boldsymbol{x}_{\boldsymbol{i}} \delta_{jj})\boldsymbol{e}_i \\
&= 4 x_i \boldsymbol{e}_i = 4\boldsymbol{r} \\
\operatorname{curl}(\boldsymbol{r} \otimes \boldsymbol{r}) &= \epsilon_{\alpha\beta\gamma} (x_i x_\gamma)_{,\beta}\, \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha\beta\gamma} (x_{i,\beta}\, x_\gamma + x_i x_{\gamma,\beta})\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha\beta\gamma} (\delta_{i\beta} x_\gamma + x_i \delta_{\gamma\beta})\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha i\gamma} x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} + \epsilon_{\alpha\beta\beta} x_i \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} = -\epsilon_{\alpha\gamma i} x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} = -\boldsymbol{r} \times
\end{aligned}
$$

48. Define the magnitude of tensor $\mathbf{A}$ as, $|\mathbf{A}| = \sqrt{\operatorname{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}})}$ Show that $\frac{\partial |\mathbf{A}|}{\partial \mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$

By definition, given a scalar α, the derivative of a scalar function of a tensor $f(\mathbf{A})$ is

$$
\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} f(\mathbf{A} + \alpha \mathbf{B})
$$

for any arbitrary tensor $\mathbf{B}$.

In the case of $f(\mathbf{A}) = |\mathbf{A}|$,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}}:\mathbf{B} = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}|\mathbf{A}+\alpha\mathbf{B}|$$

$$|\mathbf{A}+\alpha\mathbf{B}| = \sqrt{\mathrm{tr}(\mathbf{A}+\alpha\mathbf{B})(\mathbf{A}+\alpha\mathbf{B})^{\mathrm{T}}} = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{B}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{A}\mathbf{B}^{\mathrm{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}$$

Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write,

$$\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}|\mathbf{A}+\alpha\mathbf{B}| = \lim_{\alpha\to 0}\frac{\mathrm{tr}\,(\mathbf{B}\mathbf{A}^{\mathrm{T}})+\mathrm{tr}\,(\mathbf{A}\mathbf{B}^{\mathrm{T}})+2\alpha\mathrm{tr}\,(\mathbf{B}\mathbf{B}^{\mathrm{T}})}{2\sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{B}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{A}\mathbf{B}^{\mathrm{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}} = \frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}} = \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

So that,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}}:\mathbf{B} = \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

or,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

as required since $\mathbf{B}$ is arbitrary.

49. Show that $\frac{\partial \boldsymbol{I}_3(\boldsymbol{S})}{\partial\boldsymbol{S}} = \frac{\partial\det(\boldsymbol{S})}{\partial\boldsymbol{S}} = \mathbf{S}^{\mathrm{c}}$ the cofactor of $\mathbf{S}$.

Clearly $\mathbf{S}^{\mathrm{c}} = \det(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}} = \boldsymbol{I}_3(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}}$. Details of this for the contravariant

components of a tensor is presented below. Let

$$\det(\mathbf{S}) \equiv |\mathbf{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha\delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha\delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha\delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $[S_{\alpha\beta}]$ or $\boldsymbol{S}$

50. For a scalar variable α, if the tensor $\mathbf{T} = \mathbf{T}(\alpha)$ and $\dot{\mathbf{T}} \equiv \frac{d\mathbf{T}}{d\alpha}$, Show that $\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\mathrm{tr}(\dot{\mathbf{T}}\mathbf{T}^{-1})$

Let $\boldsymbol{A} \equiv \dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$ so that, $\dot{\boldsymbol{T}} = \boldsymbol{A}\boldsymbol{T}$. In component form, we have $\dot{T}_j^i = A_m^i T_j^m$. Therefore,

$$\frac{d}{d\alpha}\det(\boldsymbol{T}) = \frac{d}{d\alpha}\left(e^{ijk}T_i^1T_j^2T_k^3\right) = e^{ijk}\left(\dot{T}_i^1T_j^2T_k^3 + T_i^1\dot{T}_j^2T_k^3 + T_i^1T_j^2\dot{T}_k^3\right)$$

$$= e^{ijk}\left(A_l^1T_i^lT_j^2T_k^3 + T_i^1A_m^2T_j^mT_k^3 + T_i^1T_j^2A_n^3T_k^n\right)$$

$$= e^{ijk}\left[\left(A_1^1T_i^1 + \boxed{A_2^1T_i^2} + \boxed{A_3^1T_i^3}\right)T_j^2T_k^3 + T_i^1\left(\boxed{A_1^2T_j^1} + A_2^2T_j^2 + \boxed{A_3^2T_j^3}\right)T_k^3 + T_i^1T_j^2\left(\boxed{A_1^3T_k^1} + \boxed{A_2^3T_k^2} + A_3^3T_k^3\right)\right]$$

All the boxed terms in the above equation vanish on account of the contraction of a symmetric tensor with an antisymmetric one.

(For example, the first boxed term yields, $e^{ijk}A_2^1T_i^2T_j^2T_k^3$

Which is symmetric as well as antisymmetric in i and j. It therefore vanishes. The same is true for all other such terms.)

$$\frac{\partial}{\partial\alpha}\det(\mathbf{T}) = e^{ijk}\left[\left(A_1^1T_i^1\right)T_j^2T_k^3 + T_i^1\left(A_2^2T_j^2\right)T_k^3 + T_i^1T_j^2\left(A_3^3T_k^3\right)\right]$$
$$= A_m^m e^{ijk}T_i^1T_j^2T_k^3 = \mathrm{tr}\left(\dot{\mathbf{T}}\mathbf{T}^{-1}\right)\det(\mathbf{T})$$

as required.

51. Prove Liouville's Theorem that for a scalar variable α, if the tensor $\mathbf{T} = \mathbf{T}(\alpha)$ and $\dot{\mathbf{T}} \equiv \frac{d\mathbf{T}}{d\alpha}$, $\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\mathrm{tr}(\dot{\mathbf{T}}\mathbf{T}^{-1})$ by direct methods.

We choose three constant, linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ so that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\det(\mathbf{T}) = [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]$$

Differentiating both sides, noting that the RHS is a product,

$$
\begin{aligned}
&[\mathbf{a}, \mathbf{b}, \mathbf{c}] \frac{d}{d\alpha} \det(\mathbf{T}) \\
&= \frac{d}{d\alpha} [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] \\
&= \left[\frac{d\mathbf{T}}{d\alpha}\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}\right] + \left[\mathbf{Ta}, \frac{d\mathbf{T}}{d\alpha}\mathbf{b}, \mathbf{Tc}\right] + \left[\mathbf{Ta}, \mathbf{Tb}, \frac{d\mathbf{T}}{d\alpha}\mathbf{c}\right] \\
&= \left[\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}\right] + \left[\mathbf{Ta}, \frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Tb}, \mathbf{Tc}\right] + \left[\mathbf{Ta}, \mathbf{Tb}, \frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Tc}\right] \\
&= \operatorname{tr}\left(\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\right)[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]
\end{aligned}
$$

Clearly, $\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T}) \operatorname{tr}\left(\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\right)$

52. If $\mathbf{T}$ is invertible, show that $\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T})) = \mathbf{T}^{-\mathrm{T}}$

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T})) &= \frac{\partial(\log\det(\mathbf{T}))}{\partial\det(\mathbf{T})} \frac{\partial\det(\mathbf{T})}{\partial \mathbf{T}} \\
&= \frac{1}{\det(\mathbf{T})}\mathbf{T}^{\mathrm{c}} = \frac{1}{\det(\mathbf{T})}\det(\mathbf{T})\,\mathbf{T}^{-\mathrm{T}} \\
&= \mathbf{T}^{-\mathrm{T}}
\end{aligned}
$$

53. If $\mathbf{T}$ is invertible, show that $\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathrm{T}}$

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T}^{-1})) &= \frac{\partial(\log\det(\mathbf{T}^{-1}))}{\partial\det(\mathbf{T}^{-1})}\frac{\partial\det(\mathbf{T}^{-1})}{\partial \mathbf{T}^{-1}}\frac{\partial \mathbf{T}^{-1}}{\partial \mathbf{T}} \\
&= \frac{1}{\det(\mathbf{T}^{-1})}\mathbf{T}^{-\mathbf{c}}(-\mathbf{T}^{-2}) \\
&= \frac{1}{\det(\mathbf{T}^{-1})}\det(\mathbf{T}^{-1})\,\mathbf{T}^{\mathrm{T}}(-\mathbf{T}^{-2}) \\
&= -\mathbf{T}^{-\mathrm{T}}
\end{aligned}$$

54. Given that $\mathbf{A}$ is a constant tensor, Show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \mathbf{A}^{\mathrm{T}}$

In invariant components terms, let $\mathbf{A} = \mathrm{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$ and let $\mathbf{S} = \mathrm{S}_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}$.

$$\begin{aligned}
\mathbf{AS} &= \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}(\mathbf{g}_i \otimes \mathbf{g}_j)(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}) \\
&= \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}(\mathbf{g}_i \otimes \mathbf{g}^{\beta})\delta_j^{\alpha} \\
&= \mathrm{A}^{ij}\mathrm{S}_{j\beta}(\mathbf{g}_i \otimes \mathbf{g}^{\beta})
\end{aligned}$$

$$\begin{aligned}
\mathrm{tr}(\mathbf{AS}) &= \mathrm{A}^{ij}\mathrm{S}_{j\beta}(\mathbf{g}_i \cdot \mathbf{g}^{\beta}) \\
&= \mathrm{A}^{ij}\mathrm{S}_{j\beta}\delta_i^{\beta} = \mathrm{A}^{ij}\mathrm{S}_{ji}
\end{aligned}$$

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \frac{\partial}{\partial S_{\alpha\beta}}\mathrm{tr}(\mathbf{AS})\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{\partial A^{ij}S_{ji}}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= A^{ij}\delta_j^\alpha \delta_i^\beta \mathbf{g}_\alpha \otimes \mathbf{g}_\beta = A^{ij}\mathbf{g}_j \otimes \mathbf{g}_i = \mathbf{A}^{\mathrm{T}} = \frac{\partial}{\partial \mathbf{S}}(\mathbf{A}^{\mathrm{T}}\colon \mathbf{S})$$

as required.

55. Given that **A** and **B** are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = \mathbf{A}^{\mathrm{T}}\mathbf{B}$

First observe that $\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS})$. If we write, $\mathbf{C} \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}$, it is obvious from the above that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{CS}) = \mathbf{C}^{\mathrm{T}}$. Therefore,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$$

56. Given that **A** and **B** are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathbf{B}^{\mathrm{T}}\mathbf{A}$

Observe that $\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS}^{\mathrm{T}}) = \mathrm{tr}[\mathbf{S}(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}] = \mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}]$
[The transposition does not alter trace; neither does a cyclic permutation. Ensure you understand why each equality here is true.] Consequently,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{A}\mathbf{S}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \frac{\partial}{\partial \mathbf{S}}\ \mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}] = [(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}]^{\mathbf{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$$

57. Let $\mathbf{S}$ be a symmetric and positive definite tensor and let $I_1(\mathbf{S}), I_2(\mathbf{S})andI_3(\mathbf{S})$ be the three principal invariants of $\mathbf{S}$ show that $(a)\ \frac{\partial \boldsymbol{I}_1(\mathbf{S})}{\partial \mathbf{S}} = \mathbf{I}$ the identity tensor, (b) $\frac{\partial \boldsymbol{I}_2(\mathbf{S})}{\partial \mathbf{S}} = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$ and (c) $\frac{\partial I_3(\mathbf{S})}{\partial \mathbf{S}} = I_3(\mathbf{S})\ \mathbf{S}^{-1}$

$\frac{\partial \boldsymbol{I}_1(\boldsymbol{S})}{\partial \boldsymbol{S}}$ can be written in the invariant component form as,

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

Recall that $I_1(\mathbf{S}) = \mathrm{tr}(\mathbf{S}) = S_\alpha^\alpha$ hence

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_i^j}\mathbf{g}_i \otimes \mathbf{g}^j = \frac{\partial S_\alpha^\alpha}{\partial S_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \delta_\alpha^i \delta_j^\alpha \mathbf{g}_i \otimes \mathbf{g}^j = \delta_j^i \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \mathbf{I}$$

which is the identity tensor as expected.

$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}}$ in a similar way can be written in the invariant component form as,

$$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2}\frac{\partial I_1(\mathbf{S})}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j$$

where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)]$. Consequently,

$$\begin{aligned}
\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} &= \frac{1}{2}\frac{\partial}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j \\
&= \frac{1}{2}\left[\delta_\alpha^i \delta_j^\alpha S_\beta^\beta + \delta_\beta^i \delta_j^\beta S_\alpha^\alpha - \delta_\beta^i \delta_j^\alpha S_\alpha^\beta - \delta_\alpha^i \delta_j^\beta S_\beta^\alpha\right]\mathbf{g}_i \otimes \mathbf{g}^j \\
&= \frac{1}{2}\left[\delta_j^i S_\beta^\beta + \delta_j^i S_\alpha^\alpha - S_i^j - S_i^j\right]\mathbf{g}_i \otimes \mathbf{g}^j = \left(\delta_j^i S_\alpha^\alpha - S_i^j\right)\mathbf{g}_i \otimes \mathbf{g}^j \\
&= I_1(\mathbf{S})\mathbf{1} - \mathbf{S}
\end{aligned}$$

$$\det(\boldsymbol{S}) \equiv |\boldsymbol{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\begin{aligned}
\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta &= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \\
&= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha \delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha \delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha \delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \\
&= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta
\end{aligned}$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $[S_{\alpha\beta}]$ or $\boldsymbol{S}$

58. For a tensor field $\boldsymbol{\Xi}$, The volume integral in the region $\Omega \subset \mathcal{E}$, $\int_\Omega (\mathrm{grad}\,\boldsymbol{\Xi})\,dv = \int_{\partial\Omega} \boldsymbol{\Xi} \otimes \boldsymbol{n}\,ds$ where $\boldsymbol{n}$ is the outward drawn normal to $\partial\Omega$ – the boundary of Ω. Show that for a vector field $\boldsymbol{f}$

$$\int_\Omega (\mathrm{div}\,\boldsymbol{f})\,dv = \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}\,ds$$

Replace $\boldsymbol{\Xi}$ by the vector field $\boldsymbol{f}$ we have,

$$\int_\Omega (\mathrm{grad}\,\boldsymbol{f})\,dv = \int_{\partial\Omega} \boldsymbol{f}\otimes\boldsymbol{n}\,ds$$

Taking the trace of both sides and noting that both trace and the integral are linear operations, therefore we have,

$$\int_\Omega (\mathrm{div}\,\boldsymbol{f})\,dv = \int_\Omega \mathrm{tr}(\mathrm{grad}\,\boldsymbol{f})\,dv$$

$$= \int_{\partial\Omega} \mathrm{tr}(\boldsymbol{f}\otimes\boldsymbol{n})\,ds$$

$$= \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\, ds$$

59. Show that for a scalar function Hence the divergence theorem becomes, $\int_{\Omega} (\text{grad}\ \phi)\, dv = \int_{\partial\Omega} \phi \boldsymbol{n}\, ds$

Recall that for a vector field, that for a vector field $\boldsymbol{f}$

$$\int_{\Omega} (\text{div}\, \boldsymbol{f})\, dv = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\, ds$$

if we write, $\mathbf{f} = \phi \boldsymbol{a}$ where $\boldsymbol{a}$ is an arbitrary constant vector, we have,

$$\int_{\Omega} (\text{div}[\phi \boldsymbol{a}])\, dv = \int_{\partial\Omega} \phi \boldsymbol{a} \cdot \mathbf{n}\, ds = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

For the LHS, note that, $\text{div}[\phi \boldsymbol{a}] = \text{tr}(\text{grad}[\phi \boldsymbol{a}])$

$$\text{grad}[\phi \boldsymbol{a}] = (\phi a^i)_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^j = a^i \phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^j$$

The trace of which is,

$$a^i \phi_{,j}\, \mathbf{g}_i \cdot \mathbf{g}^j = a^i \phi_{,j}\, \delta_i^j = a^i \phi_{,i} = \boldsymbol{a} \cdot \text{grad}\ \phi$$

For the arbitrary constant vector $\boldsymbol{a}$, we therefore have that,

$$\int_{\Omega} (\text{div}[\phi \boldsymbol{a}])\, dv = \boldsymbol{a} \cdot \int_{\Omega} \text{grad}\ \phi\ \ dv = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

$$\int_{\Omega} \text{grad}\, \phi \;\, dv = \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

60. For the tensor $\mathbf{T}$, given that $\boldsymbol{YT} = \boldsymbol{I}$, show that $\boldsymbol{TY} = \boldsymbol{YT} = \boldsymbol{I}$.

Consider $\boldsymbol{TYTu}$ where $\boldsymbol{u}$ is a vector. Since $\boldsymbol{YT} = \boldsymbol{I}$, it follows that $\boldsymbol{TYTu} = \boldsymbol{TIu} = \boldsymbol{Tu} \equiv \boldsymbol{v}$ where $\boldsymbol{v}$ is also a vector. Clearly,

$$\boldsymbol{TYTu} = \boldsymbol{TYv} = \boldsymbol{v}$$

which immediately shows that $\boldsymbol{TY} = \boldsymbol{I}$ as required to be shown.

61. Given the basis vectors $\mathbf{g}_i, i = 1,2,3$ show that for any tensor $\mathbf{T}$, the product

$$\mathbf{T}\mathbf{g}_i = T_{\alpha i}\mathbf{g}^{\alpha}$$

In component form, let $\mathbf{T} = T_{\alpha\beta}(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\alpha})$ so that

$$\begin{aligned}\mathbf{T}\mathbf{g}_i &= T_{\alpha\beta}\left(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}\right)\mathbf{g}_i \\ &= T_{\alpha\beta}\left(\mathbf{g}^{\beta} \cdot \mathbf{g}_i\right)\mathbf{g}^{\alpha} = T_{\alpha\beta}\delta_i^{\beta}\mathbf{g}^{\alpha} \\ &= T_{\alpha i}\mathbf{g}^{\alpha}\end{aligned}$$

62. Show that the trace of a tensor $\mathbf{T}$, in component form, can be written as $T_i^{\cdot i} = T_{ij}g^{ij} = T^{ij}g_{ij}$.

Observe that any three linearly independent vectors can be treated as a basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$\begin{aligned}
\operatorname{tr}(\mathbf{T}) \equiv I_1(\mathbf{T}) &= \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{[(T_{i1}\mathbf{g}^i), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}\mathbf{g}^j), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}\mathbf{g}^k)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{[(T_{i1}g^{\alpha i}\mathbf{g}_\alpha), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}g^{\beta j}\mathbf{g}_\beta), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}g^{\gamma k}\mathbf{g}_\gamma)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{T_1^\alpha[\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_3] + T_2^\beta[\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_3] + T_3^\gamma[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_\gamma]}{\epsilon_{123}} \\
&= \frac{T_1^\alpha\epsilon_{\alpha 23} + T_2^\beta\epsilon_{1\beta 3} + T_3^\gamma\epsilon_{12\gamma}}{\epsilon_{123}} \\
&= \frac{T_1^1\epsilon_{123} + T_2^2\epsilon_{123} + T_3^3\epsilon_{123}}{\epsilon_{123}} = T_i^{\cdot i} = T_{ij}g^{ij} = T^{ij}g_{ij}
\end{aligned}$$

63. Show that $[\mathbf{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{T}\mathbf{g}_k] = T^\alpha_\alpha [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]$

Note that,

$$
\begin{aligned}
&[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3] \\
&= [(T_{i1}\mathbf{g}^i), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}\mathbf{g}^j), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}\mathbf{g}^k)] \\
&= [(T_{i1}g^{\alpha i}\mathbf{g}_\alpha), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}g^{\beta j}\mathbf{g}_\beta), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}g^{\gamma k}\mathbf{g}_\gamma)] \\
&= T^\alpha_1 [\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_3] + T^\beta_2 [\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_3] + T^\gamma_3 [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_\gamma] \\
&= T^\alpha_1 \epsilon_{\alpha 23} + T^\beta_2 \epsilon_{1\beta 3} + T^\gamma_3 \epsilon_{12\gamma} \\
&= T^1_1 \epsilon_{123} + T^2_2 \epsilon_{123} + T^3_3 \epsilon_{123} \\
&= T^\alpha_\alpha \epsilon_{123} = T^\alpha_\alpha [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]
\end{aligned}
$$

Swapping $\mathbf{g}_2$ and $\mathbf{g}_3$, it is clear that,

$$
\begin{aligned}
&[\mathbf{T}\mathbf{g}_1, \mathbf{g}_3, \mathbf{g}_2] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_3, \mathbf{g}_2] + [\mathbf{g}_1, \mathbf{g}_3, \mathbf{T}\mathbf{g}_2] \\
&\qquad = -[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] - [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3] \\
&\qquad = T^\alpha_\alpha \epsilon_{132} = T^\alpha_\alpha [\mathbf{g}_1, \mathbf{g}_3, \mathbf{g}_2]
\end{aligned}
$$

Continuing, we have that

$$
[\mathbf{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{T}\mathbf{g}_k] = T^\alpha_\alpha [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] = T^\alpha_\alpha \epsilon_{ijk}
$$

64. Show that $I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{Ta},\boldsymbol{b},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{Tb},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{b},\boldsymbol{Tc}]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$ is independent of the choice of the linearly independent vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Let us refer each vector to a covariant basis so that, $\boldsymbol{a} = a^i\mathbf{g}_i$, $\boldsymbol{b} = b^j\mathbf{g}_j$, and $\boldsymbol{c} = c^k\mathbf{g}_k$. Hence,

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}(a^i\mathbf{g}_i), b^j\mathbf{g}_j, c^k\mathbf{g}_k] + [a^i\mathbf{g}_i, \boldsymbol{T}(b^j\mathbf{g}_j), c^k\mathbf{g}_k] + [a^i\mathbf{g}_i, b^j\mathbf{g}_j, \boldsymbol{T}(c^k\mathbf{g}_k)]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

$$= \frac{a^ib^jc^k([\boldsymbol{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \boldsymbol{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \boldsymbol{T}\mathbf{g}_k])}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

But $[\mathbf{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{T}\mathbf{g}_k] = T^\alpha_\alpha[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]$. We have that

$$I_1(\boldsymbol{T}) = \frac{a^ib^jc^kT^\alpha_\alpha[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]}{\epsilon_{ijk}a^ib^jc^k}$$

$$= \frac{\epsilon_{ijk}a^ib^jc^k}{\epsilon_{ijk}a^ib^jc^k}T^\alpha_\alpha = T^\alpha_\alpha$$

Which is obviously independent of the choice of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

65. Write the second tensor invariant in terms of components

As previously observed, any three linearly independent vectors can be treated as the

basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$I_2(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

The first of the numerator terms can be simplified as,

$$\begin{aligned}[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] &= \left[\left(T_{i1}\mathbf{g}^i\right), \left(T_{j2}\mathbf{g}^j\right), \mathbf{g}_3\right] \\ &= \left[\left(T_{i1}g^{\alpha i}\mathbf{g}_\alpha\right), \left(T_{j2}g^{\beta j}\mathbf{g}_\beta\right), \mathbf{g}_3\right] = T_1^\alpha T_2^\beta [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_3]\end{aligned}$$

The other terms are similarly simplified. Clearly,

$$\begin{aligned}I_2(\mathbf{T}) &= \frac{T_1^\alpha T_2^\beta [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_3] + T_2^\beta T_3^\gamma [\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_\gamma] + T_1^\alpha T_3^\gamma [\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_\gamma]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\ &= \frac{T_1^\alpha T_2^\beta \epsilon_{\alpha\beta 3} + T_2^\beta T_3^\gamma \epsilon_{1\beta\gamma} + T_1^\alpha T_3^\gamma \epsilon_{\alpha 2\gamma}}{\epsilon_{123}} \\ &= \frac{[(T_1^1 T_2^2 - T_1^2 T_2^1) + (T_2^2 T_3^3 - T_2^3 T_3^2) + (T_3^3 T_1^1 - T_3^1 T_1^3)]\epsilon_{123}}{\epsilon_{123}} \\ &= \frac{1}{2}\left(T_\alpha^\alpha T_\beta^\beta - T_\beta^\alpha T_\alpha^\beta\right)\end{aligned}$$

66. Using direct notation, show that the second invariant of a tensor is the trace of its cofactor.

Given three basis vectors, $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$

$$
\begin{aligned}
I_2(\mathbf{T}) &= \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{\mathbf{T}^{\mathrm{c}}(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 + \mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3) + [\mathbf{T}^{\mathrm{c}}(\mathbf{g}_3 \times \mathbf{g}_1) \cdot \mathbf{g}_2]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{T}^{\mathrm{cT}}\mathbf{g}_3 + (\mathbf{g}_2 \times \mathbf{g}_3) \cdot \mathbf{T}^{\mathrm{cT}}\mathbf{g}_1 + (\mathbf{g}_3 \times \mathbf{g}_1) \cdot \mathbf{T}^{\mathrm{cT}}\mathbf{g}_2}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}^{\mathrm{cT}}\mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_3, \mathbf{g}_1, \mathbf{T}^{\mathrm{cT}}\mathbf{g}_2]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= I_1(\mathbf{T}^{\mathrm{cT}}) = I_1(\mathbf{T}^{\mathrm{c}})
\end{aligned}
$$

Which is the trace of its cofactor as required.

67. Express the third tensor invariant in terms of its components.

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken for granted. Consequently, for any tensor $\mathbf{T}$,

$$I_3(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} = \frac{[(T_{i1}\mathbf{g}^i), (T_{j2}\mathbf{g}^j), (T_{k3}\mathbf{g}^k)]}{\epsilon_{123}}$$

$$= \frac{[(T_{i1}g^{\alpha i}\mathbf{g}_\alpha), (T_{j2}g^{\beta j}\mathbf{g}_\beta), (T_{k3}g^{\gamma k}\mathbf{g}_\gamma)]}{\epsilon_{123}} = \frac{T_1^\alpha T_2^\beta T_3^\gamma [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_\gamma]}{\epsilon_{123}}$$

$$= \frac{T_1^\alpha T_2^\beta T_3^\gamma \epsilon_{\alpha\beta\gamma}}{\epsilon_{123}} = e_{\alpha\beta\gamma} T_1^\alpha T_2^\beta T_3^\gamma = \det \mathbf{T}$$

68. For the tensor $\mathbf{T}$, the cofactor is defined as $\mathbf{T}^{\mathrm{c}} \equiv \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$. Using direct notation and the property of the cofactor that $\mathbf{T}\mathbf{a} \times \mathbf{T}\mathbf{b} = \mathbf{T}^{\mathrm{c}}(\mathbf{a} \times \mathbf{b})$ for any two vectors $\mathbf{a}, \mathbf{b}$, show that the third invariant of a tensor, defined as $I_3(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$ for any set of independent vectors, $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$, is its determinant.

$$\begin{aligned}
[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] &= \mathbf{T}\mathbf{g}_1 \cdot (\mathbf{T}\mathbf{g}_2 \times \mathbf{T}\mathbf{g}_3) \\
&= \mathbf{T}\mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3) \\
&= (\mathbf{g}_2 \times \mathbf{g}_3) \cdot (\mathbf{T}^{\mathrm{c}})^{\mathrm{T}} \mathbf{T}\mathbf{g}_1 \\
&= (\mathbf{g}_2 \times \mathbf{g}_3) \cdot \mathbf{T}^{\mathrm{cT}} \mathbf{T}\mathbf{g}_1 \\
&= (\mathbf{g}_2 \times \mathbf{g}_3) \cdot (\det \mathbf{T}) \mathbf{g}_1
\end{aligned}$$

so that

$$I_3(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \det \mathbf{T} \frac{(\mathbf{g}_2 \times \mathbf{g}_3) \cdot \mathbf{g}_1}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$
$$= \det \mathbf{T}$$

69. In component form, the third tensor invariant of a tensor $\mathbf{T}, I_3(\mathbf{T}) = e_{\alpha\beta\gamma} T_1^{\alpha} T_2^{\beta} T_3^{\gamma} = \det \mathbf{T}$. Show that $e_{ijk} T_{\alpha}^{i} T_{\beta}^{j} T_{\gamma}^{k} = e_{\alpha\beta\gamma} \det \mathbf{T}$.

We do this by first establishing the fact that the LHS is completely antisymmetric in α, β and γ. We note that the indices i, j and k are dummy and therefore,

$$e_{ijk} T_{\alpha}^{i} T_{\beta}^{j} T_{\gamma}^{k} = e_{kji} T_{\alpha}^{k} T_{\beta}^{j} T_{\gamma}^{i} = e_{kji} T_{\gamma}^{i} T_{\alpha}^{k} T_{\beta}^{j} = -e_{ijk} T_{\gamma}^{i} T_{\beta}^{j} T_{\alpha}^{k}$$

Showing that a simple swap of α and γ changes the sign. This is similarly true for the other pairs in the lower symbols. Thus we establish anti-symmetry in α, β and γ. Noting that both sides take the same values when α, β and γ are equal to 1, 2 and 3 respectively. The arrangement of the indices makes this value positive or negative in the same antisymmetric way. This completes the proof. Similarly we can write,

$$e^{ijk} T_{i}^{\alpha} T_{j}^{\beta} T_{k}^{\gamma} = e^{ijk} T_{i}^{1} T_{j}^{2} T_{k}^{3} e^{\alpha\beta\gamma} = e^{\alpha\beta\gamma} \det \mathbf{T}$$

70. Given the basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and their dual, $\mathbf{g}^1,\ \mathbf{g}^2,\ \mathbf{g}^3$, Show that for any other basis pair, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ and their dual, $\boldsymbol{\gamma}^1,\ \boldsymbol{\gamma}^2,\ \boldsymbol{\gamma}^3$, the relationship, $(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = \delta_\alpha^\beta$ holds.

By simply reversing the step, it is immediately obvious that,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta$$

Observe that the expression in the parentheses is the unit tensor – having no effect on a vector; it follows that,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta = \mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta_\alpha^\beta$$

71. Show that the first invariant has the same value in every coordinate system.

We have shown elsewhere that for any tensor $\boldsymbol{T}$ the first invariant, $I_1(\boldsymbol{T}) =$

$$\mathrm{tr}(\boldsymbol{T}) \equiv \frac{[\{\mathbf{T}\mathbf{g}_1\},\ \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{\mathbf{T}\mathbf{g}_2\}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, \{\mathbf{T}\mathbf{g}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]} = T_{ij}g^{ij} = T_i^{\ i}$$

We proceed to show that this quantity has the same value in any other independent set of basis vectors. Let $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)$ be another arbitrary set of linearly independent vectors. They therefore form a basis of a coordinate system. Let $(\boldsymbol{\gamma}^1,\ \boldsymbol{\gamma}^2,\ \boldsymbol{\gamma}^3)$ be the dual to the set. It is easy to establish that the new set will be related to the old in;

$$\boldsymbol{\gamma}_j = (\boldsymbol{\gamma}_j \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha, \text{ and } \boldsymbol{\gamma}^i = (\boldsymbol{\gamma}^i \cdot \mathbf{g}_\beta)\mathbf{g}^\beta$$

Let us write the $(i,\ j)^{th}$ components in the $\boldsymbol{\gamma}-$ bases as $\tilde{T}_i^{\cdot j}$.

Clearly,

$$\begin{aligned}\tilde{T}_i^{\cdot j} &\equiv \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^j = (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha \cdot \left[\mathbf{T}\left(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta\right)\mathbf{g}^\beta\right] \\ &= (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)\left(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta\right)\left[\mathbf{g}_\alpha \cdot \mathbf{T}\mathbf{g}^\beta\right] \\ &= \left[\left(\boldsymbol{\gamma}^j \otimes \boldsymbol{\gamma}_i\right)\mathbf{g}^\alpha\right] \cdot \mathbf{g}_\beta \left[T_\alpha^{\cdot\beta}\right]\end{aligned}$$

Contracting i with j, we obtain the sum,

$$\begin{aligned}\tilde{T}_i^{\cdot i} &= \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^i = \left[\left(\boldsymbol{\gamma}^i \otimes \boldsymbol{\gamma}_i\right)\mathbf{g}^\alpha\right] \cdot \mathbf{g}_\beta \left[T_\alpha^{\cdot\beta}\right] \\ &= \mathbf{g}^\alpha \cdot \mathbf{g}_\beta T_\alpha^{\cdot\beta} = \delta_\beta^\alpha T_\alpha^{\cdot\beta} = T_\alpha^{\cdot\alpha}\end{aligned}$$

So that the trace has the same value in any arbitrarily chosen coordinate system including curvilinear ones whether orthogonal or not.

72. Express the second invariant of a tensor in terms of traces only. Show that the second invariant has the same value in all coordinate systems.

For arbitrary vectors bases, $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ the second principal invariant of a tensor $\mathbf{T}$

$$I_2(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}, \text{or}$$

$$I_2(\mathbf{T}) = \frac{1}{2}\left(T_\alpha^\alpha T_\beta^\beta - T_\beta^\alpha T_\alpha^\beta\right) = \frac{1}{2}\left(I_1^2(\mathbf{T}) - I_1(\mathbf{T}^2)\right)$$

That is, half of the square of the trace minus the trace of the square. These therefore are unchanged by coordinate transformation because traces are unchanged bt coordinate transformation.

73. Show that the third tensor invariant is unaffected by a coordinate transformation.

We begin by showing that we can express the third invariant in terms of traces only. Given a tensor $\mathbf{S}$, by Cayley Hamilton theorem,

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3 = 0$$

Note that $I_1 = I_1(\mathbf{S}), I_2 = I_2(\mathbf{S})$ and $I_3 = I_3(\mathbf{S})$ the first, second and third invariants of $\mathbf{S}$ are all scalar functions of the tensor $\mathbf{S}$. Taking the trace of the above equation,

$$\begin{aligned} I_1(\mathbf{S}^3) &= I_1(\mathbf{S})I_1(\mathbf{S}^2) - I_2(\mathbf{S})I_1(\mathbf{S}) + 3I_3(\mathbf{S}) \\ &= I_1(\boldsymbol{S})\left(I_1^2(\mathbf{S}) - 2I_2(\mathbf{S})\right) - I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S}) \\ &= I_1^3(\mathbf{S}) - 3I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S}) \end{aligned}$$

Or, $I_3(\mathbf{S}) = \frac{1}{3}\left(I_1(\mathbf{S}^3) - I_1^3(\mathbf{S}) + 3I_1(\mathbf{S})I_2(\mathbf{S})\right)$

which show that the third invariant is itself expressible in terms of traces only. It is therefore invariant in value as a result of coordinate transformation.

74. For any tensor $\mathbf{T}$, the arbitrary vector $\mathbf{u}$ and the scalar λ, show that the eigenvalue problem, $\mathbf{Tu} = \lambda\mathbf{u}$ leads to the characteristic equation, $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$, where $I_1 = I_1(\mathbf{T}), I_2 = I_2(\mathbf{T})$ and $I_3 = I_3(\mathbf{T})$ the first, second and third invariants of $\mathbf{T}$.

Writing the tensor and vector in component forms, we have

$$\mathbf{Tu} = T_i^{\cdot j}(\mathbf{g}^i \otimes \mathbf{g}_j)u_k\mathbf{g}^k = \lambda\mathbf{u} = \lambda u_i\mathbf{g}^i$$

So that,

$$\begin{aligned}\mathbf{Tu} - \lambda\mathbf{u} &= T_i^{\cdot j}\mathbf{g}^i(\mathbf{g}_j \cdot \mathbf{g}^k)u_k - \lambda u_i\mathbf{g}^i = T_i^{\cdot j}\mathbf{g}^i u_j - \lambda u_i\mathbf{g}^i \\ &= T_i^{\cdot j}\mathbf{g}^i u_j - \lambda u_j\delta_i^j\mathbf{g}^i = (T_i^{\cdot j} - \lambda\delta_i^j)u_j\mathbf{g}^i = \boldsymbol{o}\end{aligned}$$

Which is possible only if the coefficient determinant, $|T_i^{\cdot j} - \lambda\delta_i^j|$ vanishes.

Expanding, we find that,

$$\begin{aligned}&-T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 + T_1^1T_2^2T_3^3 + T_2^1T_1^2\lambda - T_1^1T_2^2\lambda \\ &\qquad + T_3^1T_1^3\lambda + T_3^2T_2^3\lambda - T_1^1T_3^3\lambda - T_2^2T_3^3\lambda + T_1^1\lambda^2 + T_2^2\lambda^2 + T_3^3\lambda^2 - \lambda^3 \\ &= -T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 + T_1^1T_2^2T_3^3 \\ &\qquad + (T_2^1T_1^2 - T_1^1T_2^2 + T_3^1T_1^3 + T_3^2T_2^3 - T_1^1T_3^3 - T_2^2T_3^3)\lambda + (T_1^1 + T_2^2 + T_3^3)\lambda^2 \\ &\qquad - \lambda^3 = 0\end{aligned}$$

Or, $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$, as required.

75. For arbitrary vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, and a tensor $\mathbf{T}$ use the relationship, $[\lambda\mathbf{a} - \mathbf{Ta}, \lambda\mathbf{b} - \mathbf{Tb}, \lambda\mathbf{c} - \mathbf{Tc}\,] = \det(\lambda\mathbf{I} - \mathbf{T})[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ and the characteristic equation $\lambda^3 - I_1(\mathbf{T})\lambda^2 + I_2(\mathbf{T})\lambda - I_3(\mathbf{T}) = 0$ to find expressions for the for the invariants of $\mathbf{T}$.

The characteristic equation, $\det(\lambda\mathbf{I} - \mathbf{T}) = 0$ immediately implies that $[\lambda\mathbf{a} - \mathbf{Ta}, \lambda\mathbf{b} - \mathbf{Tb}, \lambda\mathbf{c} - \mathbf{Tc}\,] = 0$.

Expanding the scalar triple product, we have

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\lambda^3 - ([\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}])\lambda^2 + ([\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}])\lambda - [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] = 0$$

From which we can see that,

$$I_1(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \text{and}$$

$$I_3(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Assuming we have carefully chosen $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$.

76. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, find the principal invariants of the dyad $\mathbf{u} \otimes \mathbf{v}$

The first principal invariant $I_1(\mathbf{u} \otimes \mathbf{v})$ is the trace, defined as,

$$\mathrm{tr}(\mathbf{u} \otimes \mathbf{v}) = \frac{[\{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_1\},\ \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_2\}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{1}{\epsilon_{123}}\{[v_1\mathbf{u}, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, v_2\mathbf{u}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, v_3\mathbf{u}]\}$$

$$= \frac{1}{\epsilon_{123}}\left\{(v_1\mathbf{u}) \cdot \left(\epsilon_{23i}\mathbf{g}^i\right) + \left(\epsilon_{31i}\mathbf{g}^i\right) \cdot (v_2\mathbf{u}) + \left(\epsilon_{12i}\mathbf{g}^i\right) \cdot (v_3\mathbf{u})\right\}$$

$$= \frac{1}{\epsilon_{123}}\{(v_1\mathbf{u}) \cdot (\epsilon_{231}\mathbf{g}^1) + (\epsilon_{312}\mathbf{g}^2) \cdot (v_2\mathbf{u}) + (\epsilon_{123}\mathbf{g}^3) \cdot (v_3\mathbf{u})\} = v_i u^i.$$

The second principal invariant $I_2(\mathbf{u} \otimes \mathbf{v}) =$

$$= \frac{[\{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_1\}, (\mathbf{u} \otimes \mathbf{v})\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_2\}, (\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3] + [(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_1,\ \mathbf{g}_2, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]}$$

$= \frac{1}{\epsilon_{123}}\{[v_1\mathbf{u}, v_2\mathbf{u}, \mathbf{g}_3] + [\mathbf{g}_1, v_2\mathbf{u}, v_3\mathbf{u}] + [v_1\mathbf{u},\ \mathbf{g}_2, v_3\mathbf{u}]\} = 0$. Vanishes – collinearity.

The third invariant $I_3(\mathbf{u} \otimes \mathbf{v}) = \frac{1}{\epsilon_{123}}[v_1\mathbf{u}, v_2\mathbf{u}, v_3\mathbf{u}]$ which also vanishes on account of collinearity

77. Define the cofactor of a tensor as, $\operatorname{cof}\mathbf{T} \equiv \mathbf{T}^{c} \equiv \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$. Show that, for any pair of independent vectors $\mathbf{u}$ and $\mathbf{v}$ the cofactor satisfies, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{c}(\mathbf{u} \times \mathbf{v})$

First note that if $\mathbf{T}$ is invertible, the independence of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$. Consequently, we can define the non-vanishing

$$\mathbf{n} \equiv \mathbf{Tu} \times \mathbf{Tv} \neq \mathbf{0}.$$

It follows that $\mathbf{n}$ must be on the perpendicular line to both $\mathbf{Tu}$ and $\mathbf{Tv}$. Therefore,

$$\mathbf{n} \cdot \mathbf{Tu} = \mathbf{n} \cdot \mathbf{Tv} = \mathbf{0}.$$

We can also take a transpose and write,

$$\mathbf{u} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{v} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{0}$$

Showing that the vector $\mathbf{T}^{\mathrm{T}}\mathbf{n}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It follows that $\exists\ \alpha \in \Re$ such that

$$\mathbf{T}^{\mathrm{T}}\mathbf{n} = \alpha(\mathbf{u} \times \mathbf{v})$$

Therefore, $\mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v})$.

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ so that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, then we can take a scalar product of the above equation and obtain,

$$\mathbf{w} \cdot \mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$$

The LHS is also $\mathbf{Tw} \cdot (\mathbf{Tu} \times \mathbf{Tv}) = \mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw}$. In the equation, $\mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw} = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$, it is clear that

$$\alpha = \det \mathbf{T}$$

We have that, $\mathbf{T}\mathbf{u} \times \mathbf{T}\mathbf{v} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v})$. And therefore, we have that,

$$\mathbf{T}\mathbf{u} \times \mathbf{T}\mathbf{v} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v}).$$

78. Find and expression for the components of the cofactor of tensor $\mathbf{T}$ in terms of the components of $\mathbf{T}$.

We now express the cofactor in its general components.

$$\begin{aligned}
\mathbf{T}^{\mathrm{c}} &= (\mathbf{T}^{\mathrm{c}})_i^{\alpha}\mathbf{g}_\alpha \otimes \mathbf{g}^i = (\mathbf{g}^\alpha \cdot \mathbf{T}^{\mathrm{c}}\mathbf{g}_i)\mathbf{g}_\alpha \otimes \mathbf{g}^i \\
&= \frac{1}{2}\epsilon_{ijk}\left[\mathbf{g}^\alpha \cdot \mathbf{T}^{\mathrm{c}}\left(\mathbf{g}^j \times \mathbf{g}^k\right)\right]\mathbf{g}_\alpha \otimes \mathbf{g}^i \\
&= \frac{1}{2}\epsilon_{ijk}\left[\mathbf{g}^\alpha \cdot \left(\mathbf{T}\mathbf{g}^j\right) \times \left(\mathbf{T}\mathbf{g}^k\right)\right]\mathbf{g}_\alpha \otimes \mathbf{g}^i.
\end{aligned}$$

The scalar in brackets,

$$\begin{aligned}
\mathbf{g}^\alpha \cdot \left(\mathbf{T}\mathbf{g}^j\right) \times \left(\mathbf{T}\mathbf{g}^k\right) &= \mathbf{g}^\alpha \cdot \epsilon^{lmn}\left(\mathbf{g}_m \cdot \mathbf{T}\mathbf{g}^j\right)\left(\mathbf{g}_n \cdot \mathbf{T}\mathbf{g}^k\right)\mathbf{g}_l \\
&= \delta_l^\alpha \epsilon^{lmn}\left(\mathbf{g}_m \cdot \mathbf{T}\mathbf{g}^j\right)\left(\mathbf{g}_n \cdot \mathbf{T}\mathbf{g}^k\right) \\
&= \delta_l^\alpha \epsilon^{lmn} T_m^j T_n^k = \epsilon^{\alpha mn} T_m^j T_n^k
\end{aligned}$$

Inserting this above, we therefore have, in invariant component form,

$$\begin{aligned}
\mathbf{T}^{\mathrm{c}} &= \frac{1}{2}\epsilon_{ijk}\left[\mathbf{g}^\alpha \cdot \left(\mathbf{T}\mathbf{g}^j\right) \times \left(\mathbf{T}\mathbf{g}^k\right)\right]\mathbf{g}_\alpha \otimes \mathbf{g}^i \\
&= \frac{1}{2}\epsilon_{ijk}\epsilon^{\alpha mn} T_m^j T_n^k \mathbf{g}_\alpha \otimes \mathbf{g}^i
\end{aligned}$$

$$= \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}_l \otimes \mathbf{g}^i$$

79. Given that $\operatorname{cof}\mathbf{T} = \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}_l \otimes \mathbf{g}^i$ if $\mathbf{T} = \mathbf{u}\otimes\mathbf{p} + \mathbf{v}\otimes\mathbf{q} + \mathbf{w}\otimes\mathbf{r}$, show that $\operatorname{cof}\mathbf{T} = \mathbf{u}\times\mathbf{v}\otimes\mathbf{p}\times\mathbf{q} + \mathbf{v}\times\mathbf{w}\otimes\mathbf{q}\times\mathbf{r} + \mathbf{w}\times\mathbf{u}\otimes\mathbf{r}\times\mathbf{p}$, $\operatorname{tr}\mathbf{T} = \mathbf{u}\cdot\mathbf{p} + \mathbf{v}\cdot\mathbf{q} + \mathbf{w}\cdot\mathbf{r}$ and $I_2(\mathbf{T}) = \mathbf{u}\times\mathbf{v}\cdot\mathbf{p}\times\mathbf{q} + \mathbf{v}\times\mathbf{w}\cdot\mathbf{q}\times\mathbf{r} + \mathbf{w}\times\mathbf{u}\cdot\mathbf{r}\times\mathbf{p}$

Clearly, $T_j^m = u_j p^m + v_j q^m + w_j r^m$. The cofactor

$$\begin{aligned}
\operatorname{cof}\mathbf{T} &= \frac{1}{2}\delta_{ijk}^{lmn}\left(u_j p^m + v_j q^m + w_j r^m\right)\left(u_k p^n + v_k q^n + w_k r^n\right)\mathbf{g}_l \otimes \mathbf{g}^i \\
&= \frac{1}{2}\delta_{ijk}^{lmn}\Big(\left(u_j p^m u_k p^n + v_j q^m v_k q^n + w_j r^m w_k r^n\right) \\
&\qquad + 2\left(u_j p^m v_k q^n + v_j q^m w_k r^n + u_k p^n w_j r^m\right)\Big)\mathbf{g}_l \otimes \mathbf{g}^i \\
&= \delta_{ijk}^{lmn}\left(u_j p^m v_k q^n + v_j q^m w_k r^n + u_k p^n w_j r^m\right)\mathbf{g}_l \otimes \mathbf{g}^i \\
&= \mathbf{u}\times\mathbf{v}\otimes\mathbf{p}\times\mathbf{q} + \mathbf{v}\times\mathbf{w}\otimes\mathbf{q}\times\mathbf{r} + \mathbf{w}\times\mathbf{u}\otimes\mathbf{r}\times\mathbf{p}
\end{aligned}$$

as the terms in the first inner parentheses vanish on account of symmetry and anti-symmetry in m, n, as well as j, k. The rest of the results follow by a change in the tensor product to the dot product once we take the traces of the tensor and its cofactor noting that

$$I_2(\mathbf{T}) = \operatorname{tr}(\operatorname{cof}\mathbf{T}) = \mathbf{u}\times\mathbf{v}\cdot\mathbf{p}\times\mathbf{q} + \mathbf{v}\times\mathbf{w}\cdot\mathbf{q}\times\mathbf{r} + \mathbf{w}\times\mathbf{u}\cdot\mathbf{r}\times\mathbf{p}$$

80. Find an expression for the inverse of the tensor $\mathbf{T}$ in terms of the components of $\mathbf{T}$.

The cofactor $\mathbf{T}^{\mathrm{c}} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$. Transposing the equation, we have,

$$\begin{aligned}\mathbf{T}^{-1} &= \frac{1}{\det \mathbf{T}} (\mathbf{T}^{\mathrm{c}})^{\mathrm{T}} \\ &= \frac{1}{2 \det \mathbf{T}} \delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}^i \otimes \mathbf{g}_l\end{aligned}$$

81. Let $\mathbf{T}$ be invertible and assume we can select the vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{v} \cdot \mathbf{T}^{-1}\mathbf{u} \neq 1$, Show that $(\mathbf{T} + \mathbf{u} \otimes \mathbf{v})^{-1} = \mathbf{T}^{-1} + \frac{\mathbf{T}^{-1}\mathbf{u} \otimes \mathbf{v}\mathbf{T}^{-1}}{1 - \mathbf{v} \cdot \mathbf{T}^{-1}\mathbf{u}}$.

Let $(\mathbf{T} + \mathbf{u} \otimes \mathbf{v})\mathbf{x} = \mathbf{y}$. In our attempt to find $\mathbf{x}$, it will be necessary to find the inverse of $\mathbf{T} + \mathbf{u} \otimes \mathbf{v}$. We proceed indirectly and operate the tensor $\mathbf{T}^{-1}$ on this vector euation and obtain,

$$\begin{aligned}&\mathbf{T}^{-1}(\mathbf{T} + \mathbf{u} \otimes \mathbf{v})\mathbf{x} = \mathbf{T}^{-1}\mathbf{y} \\ &= \mathbf{x} + (\mathbf{T}^{-1}\mathbf{u} \otimes \mathbf{v})\mathbf{x} = \mathbf{x} + \mathbf{T}^{-1}\mathbf{u}(\mathbf{v} \cdot \mathbf{x}) \\ &= \mathbf{x} + \mathbf{T}^{-1}\mathbf{u}(\mathbf{v} \cdot \mathbf{x}) = \mathbf{T}^{-1}\mathbf{y}\end{aligned}$$

Taking the dot product of both sides with $\mathbf{v}$,

$$\mathbf{v} \cdot \mathbf{x} + \mathbf{v} \cdot \mathbf{T}^{-1}\mathbf{u}(\mathbf{v} \cdot \mathbf{x}) = \mathbf{v} \cdot \mathbf{T}^{-1}\mathbf{y}$$

so that the scalar ,

$$\mathbf{v}\cdot\mathbf{x} = \frac{(\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{y})}{(1+\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{u})}$$

But we can write that,

$$\begin{aligned}\mathbf{x} &= (\mathbf{T}+\mathbf{u}\otimes\mathbf{v})^{-1}\mathbf{y}\\ &= \mathbf{T}^{-1}\mathbf{y} - \mathbf{T}^{-1}\mathbf{u}(\mathbf{v}\cdot\mathbf{x})\end{aligned}$$

Substituting for $\mathbf{v}\cdot\mathbf{x}$, we have,

$$\begin{aligned}(\mathbf{T}+\mathbf{u}\otimes\mathbf{v})^{-1}\mathbf{y} &= \mathbf{T}^{-1}\mathbf{y} - \frac{\mathbf{T}^{-1}\mathbf{u}(\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{y})}{1+\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{u}}\\ &= \mathbf{T}^{-1}\mathbf{y} - \frac{(\mathbf{T}^{-1}(\mathbf{u}\otimes\mathbf{v})\mathbf{T}^{-1})\mathbf{y}}{1+\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{u}}\end{aligned}$$

so that,

$$(\mathbf{T}+\mathbf{u}\otimes\mathbf{v})^{-1} = \mathbf{T}^{-1} - \frac{\mathbf{T}^{-1}(\mathbf{u}\otimes\mathbf{v})\mathbf{T}^{-1}}{1+\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{u}}$$

Provided, as we have been given, that $\mathbf{v}\cdot\mathbf{T}^{-1}\mathbf{u} \neq 1$.

82. For the invertible tensor $\mathbf{T}$ and the tensors $\mathbf{F}$, $\mathbf{V}$ and $\mathbf{U}$, show that

$$(\mathbf{T}+\mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}\left(\mathbf{F}^{-1}+\mathbf{VT}^{-1}\mathbf{U}\right)^{-1}\mathbf{VT}^{-1}$$

First consider the matrix $\begin{pmatrix}\mathbf{T} & -\mathbf{U}\\ \mathbf{V} & \mathbf{F}^{-1}\end{pmatrix}$. Its inverse is obtained by solving the matrix equation,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ \mathbf{V} & \mathbf{F}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$$

which yields,

$$\mathbf{A}\,\mathbf{T} + \mathbf{B}\mathbf{V} = \mathbf{1}$$
$$-\mathbf{A}\mathbf{U} + \mathbf{B}\mathbf{F}^{-1} = \mathbf{0} \;\Rightarrow \mathbf{B} = \mathbf{A}\mathbf{U}\mathbf{F}$$

so that,

$$\mathbf{A}\,\mathbf{T} + \mathbf{A}\mathbf{U}\mathbf{F}\mathbf{V} = \mathbf{A}(\mathbf{T} + \mathbf{U}\mathbf{F}\mathbf{V}) = \mathbf{1}$$
$$\Rightarrow \mathbf{A} = (\mathbf{T} + \mathbf{U}\mathbf{F}\mathbf{V})^{-1}$$

But $\mathbf{A} = \mathbf{T}^{-1} - \mathbf{B}\mathbf{V}\mathbf{T}^{-1}$ substituting in the second equation,
from which we can now write that $(\mathbf{T}^{-1} - \mathbf{B}\mathbf{V}\mathbf{T}^{-1})\mathbf{U} = \mathbf{B}\mathbf{F}^{-1}$ so that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{V}\mathbf{T}^{-1}\mathbf{U})^{-1}$$
$$\mathbf{A} = \mathbf{T}^{-1} - \mathbf{B}\mathbf{V}\mathbf{T}^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{V}\mathbf{T}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{T}^{-1}$$

Finally $\mathbf{A} = (\mathbf{T} + \mathbf{U}\mathbf{F}\mathbf{V})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{V}\mathbf{T}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{T}^{-1}$ as required

In the special case when $\mathbf{F}$ is the unit tensor, we have,

$$(\mathbf{T} + \mathbf{U}\mathbf{V})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{1} + \mathbf{V}\mathbf{T}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{T}^{-1}$$

83. Determinant is not a linear operation. For any two tensors $\mathbf{S}$ and $\mathbf{T}$, use the component representation of the cofactor to show that the determinant of the sum

$$\det(\mathbf{S} + \mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{c}\mathbf{S}^{T}) + \mathrm{tr}(\mathbf{S}^{c}\mathbf{T}^{T}) + \det(\mathbf{T})$$

$$\det(\mathbf{S} + \mathbf{T}) = e^{ijk}\left(S_i^1 + T_i^1\right)\left(S_j^2 + T_j^2\right)\left(S_k^3 + T_k^3\right)$$

$$= e^{ijk}\left[S_i^1 S_j^2 S_k^3 + \left(T_i^1 S_j^2 S_k^3 + S_i^1 T_j^2 S_k^3 + S_i^1 S_j^2 T_k^3\right) + T_i^1 T_j^2 T_k^3 + \left(T_i^1 T_j^2 S_k^3 + S_i^1 T_j^2 T_k^3 + T_i^1 S_j^2 T_k^3\right)\right]$$

Now, we can write,

$$\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{c}} = \left(S_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}\right)^{\mathrm{T}}\left(\frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}_l \otimes \mathbf{g}^i\right)$$

$$= \frac{1}{2}\delta_{ijk}^{lmn} S_{\alpha\beta} T_m^j T_n^k \left(\mathbf{g}^{\beta} \otimes \mathbf{g}^{\alpha}\right)\left(\mathbf{g}_l \otimes \mathbf{g}^i\right)$$

$$= \frac{1}{2}\delta_{ijk}^{lmn} S_{\alpha\beta} T_m^j T_n^k \delta_l^{\alpha}\left(\mathbf{g}^{\beta} \otimes \mathbf{g}^i\right) = \frac{1}{2}\delta_{ijk}^{lmn} S_{l\beta} T_m^j T_n^k \left(\mathbf{g}^{\beta} \otimes \mathbf{g}^i\right)$$

So that $\mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{c}}) = \frac{1}{2}\delta_{ijk}^{lmn} S_{l\beta} T_m^j T_n^k = e^{ijk}\left(T_i^1 T_j^2 S_k^3 + S_i^1 T_j^2 T_k^3 + T_i^1 S_j^2 T_k^3\right)$ after expansion. Similarly, $\mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{c}})$ expands to $e^{ijk}\left(T_i^1 S_j^2 S_k^3 + S_i^1 T_j^2 S_k^3 + S_i^1 S_j^2 T_k^3\right)$ from which the result immediately follows.

84. Given that $\mathbf{T}^{\mathrm{c}}$ is the cofactor of the tensor $\mathbf{T}$, show that $I_1(\mathbf{T}^{\mathrm{c}}) = I_2(\mathbf{T})$ that is, that the trace of the cofactor is the second principal invariant of the original tensor:

$$\mathrm{tr}(\mathbf{T}^{\mathrm{c}}) = \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}_l \cdot \mathbf{g}^i = I_1(\mathbf{T}^{\mathrm{c}})$$

$$= \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \delta_l^i = \frac{1}{2}\delta_{ijk}^{imn} T_m^j T_n^k$$

$$= \frac{1}{2}\left(\delta_j^m \delta_k^n - \delta_k^m \delta_j^n\right) T_m^j T_n^k$$

$$= \frac{1}{2}\left(T_j^j T_k^k - T_k^j T_j^k\right) = I_2(\mathbf{T})$$

85. For a scalar α show that $\det \alpha\mathbf{A} = \alpha^3 \det \mathbf{A}$

Given that $\det \mathbf{A} = \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, then

$$\det \alpha\mathbf{A} = \frac{[\alpha\mathbf{Aa}, \alpha\mathbf{Ab}, \alpha\mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \alpha^3 \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \alpha^3 \det \mathbf{A}$$

86. Define the inner product of tensors $\mathbf{T}$ and $\mathbf{S}$ as $\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^T\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^T)$ show that $I_1(\mathbf{T}) = \mathbf{T}:\mathbf{I}$

$$\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^T\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^T)$$

Let $\mathbf{S} = \mathbf{I}$;

$$\mathbf{T}:\mathbf{I} = \mathrm{tr}(\mathbf{T}^T\mathbf{I}) = \mathrm{tr}(\mathbf{T}\mathbf{I})$$
$$= \mathrm{tr}(\mathbf{T}) = I_1(\mathbf{T})$$

87. Given any scalar $k > 0$, for the scalar-valued tensor function, $f(\mathbf{S}) = \mathrm{tr}(\mathbf{S}^k)$, show that, $\frac{d}{d\mathbf{S}} f(\mathbf{S}) = k(\mathbf{S}^{k-1})^T$.

When $k = 1$,

$$
\begin{aligned}
Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \operatorname{tr}(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \operatorname{tr}(\mathbf{1}\, d\boldsymbol{S}) \\
&= \mathbf{I} : d\boldsymbol{S}
\end{aligned}
$$

So that,

$$
\frac{d}{d\boldsymbol{S}} \operatorname{tr}(\boldsymbol{S}) = \mathbf{I}.
$$

When $k = 2$, The Gateaux differential in this case,

$$
\begin{aligned}
Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \frac{d}{d\alpha} \operatorname{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})^2\}\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha} \operatorname{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\}\Big|_{\alpha=0} = \operatorname{tr}\left[\frac{d}{d\alpha}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\right]\Big|_{\alpha=0} \\
&= \operatorname{tr}[d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S}) + (\boldsymbol{S} + \alpha d\boldsymbol{S}) d\boldsymbol{S}]|_{\alpha=0} \\
&= \operatorname{tr}[d\boldsymbol{S}\, \boldsymbol{S} + \boldsymbol{S}\, d\boldsymbol{S}] = 2(\boldsymbol{S})^{\mathrm{T}} : d\boldsymbol{S}
\end{aligned}
$$

When $k = 3$,

$$
Df(\boldsymbol{S}, d\boldsymbol{S}) = \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \frac{d}{d\alpha} \operatorname{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})^3\}\Big|_{\alpha=0}
$$

$$= \frac{d}{d\alpha}\mathrm{tr}\{(\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S})\}\Big|_{\alpha=0}$$

$$= \mathrm{tr}\left[\frac{d}{d\alpha}(\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S})\right]\Big|_{\alpha=0}$$

$$= \mathrm{tr}[d\boldsymbol{S}(\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S}) + (\boldsymbol{S}+\alpha d\boldsymbol{S})d\boldsymbol{S}(\boldsymbol{S}+\alpha d\boldsymbol{S})$$

$$+ (\boldsymbol{S}+\alpha d\boldsymbol{S})(\boldsymbol{S}+\alpha d\boldsymbol{S})d\boldsymbol{S}]|_{\alpha=0}$$

$$= \mathrm{tr}[d\boldsymbol{S}\,\boldsymbol{S}\,\boldsymbol{S} + \boldsymbol{S}\,d\boldsymbol{S}\,\boldsymbol{S} + \boldsymbol{S}\,\boldsymbol{S}\,d\boldsymbol{S}] = 3(\boldsymbol{S}^2)^{\mathrm{T}}\colon d\boldsymbol{S}$$

It easily follows by induction that, $\frac{d}{d\boldsymbol{S}}\, f(\boldsymbol{S}) = k(\boldsymbol{S}^{k-1})^T$.

88. Find the derivative of the second principal invariant of the tensor $\boldsymbol{S}$

$$\frac{d}{d\boldsymbol{S}}\, I_2(\boldsymbol{S}) = \frac{1}{2}\frac{d}{d\boldsymbol{S}}[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)]$$

$$= \frac{1}{2}[2\mathrm{tr}(\boldsymbol{S})\boldsymbol{1} - 2\,\boldsymbol{S}^{\mathrm{T}}]$$

$$= \mathrm{tr}(\boldsymbol{S})\boldsymbol{1} - \boldsymbol{S}^{\mathrm{T}}$$

89. Find the derivative of the third principal invariant of the tensor $\boldsymbol{S}$

By Cayley-Hamilton, in terms of traces only,

$$I_3(\boldsymbol{S}) = \frac{1}{6}[\mathrm{tr}^3(\boldsymbol{S}) - 3\mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + 2\mathrm{tr}(\boldsymbol{S}^3)]$$

$$\frac{d}{d\boldsymbol{S}}\, I_3(\boldsymbol{S}) = \frac{1}{6}\frac{d}{d\boldsymbol{S}}[\mathrm{tr}^3(\boldsymbol{S}) - 3\mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + 2\mathrm{tr}(\boldsymbol{S}^3)]$$
$$= \frac{1}{6}[3\mathrm{tr}^2(\boldsymbol{S})\mathbf{I} - 3\mathrm{tr}(\boldsymbol{S}^2)\mathbf{I} - 3\mathrm{tr}(\boldsymbol{S})2\boldsymbol{S}^T + 2\times 3(\boldsymbol{S}^2)^T]$$
$$= I_2\mathbf{I} - I_1(\boldsymbol{S})\boldsymbol{S}^T + \boldsymbol{S}^{2T}$$

90. By the representation theorem, every real isotropic tensor function is a function of its principal invariants. Show that for every isotropic function $f(\mathbf{S})$ the derivative $\frac{\partial f(\mathbf{S})}{\partial \mathbf{S}}$ can be represented as a quadratic function of $\mathbf{S}^{\mathrm{T}}$.

By the representation theorem,

$$f(\mathbf{S}) = \phi\big(I_1(\mathbf{S}), I_2(\mathbf{S}), I_3(\mathbf{S})\big)$$

consequently,

$$\frac{\partial f(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial f(\mathbf{S})}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{S}}$$
$$= \frac{\partial f(\mathbf{S})}{\partial I_1}\mathbf{I} + \frac{\partial f(\mathbf{S})}{\partial I_2}(\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}) + \frac{\partial f(\mathbf{S})}{\partial I_3}(I_2\mathbf{I} - I_1(\mathbf{S})\mathbf{S}^T + \mathbf{S}^{2T})$$

$$= \left(\frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2} I_1(\mathbf{S}) + \frac{\partial f(\mathbf{S})}{\partial I_3} I_2(\mathbf{S})\right)\mathbf{I} - \left(\frac{\partial f(\mathbf{S})}{\partial I_2} + \frac{\partial f(\mathbf{S})}{\partial I_3} I_1(\mathbf{S})\right)\mathbf{S}^{\mathrm{T}}$$

$$+ \frac{\partial f(\mathbf{S})}{\partial I_3}\mathbf{S}^{2T}$$

$$= \alpha_0 \mathbf{I} + \alpha_1 \mathbf{S}^{\mathrm{T}} + \alpha_2 \mathbf{S}^{2T}$$

where $\alpha_0 = \frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2} I_1(\mathbf{S}) + \frac{\partial f(\mathbf{S})}{\partial I_3} I_2(\mathbf{S}), \alpha_1 = -\frac{\partial f(\mathbf{S})}{\partial I_2} - \frac{\partial f(\mathbf{S})}{\partial I_3} I_1(\mathbf{S})$ and $\alpha_2 = \frac{\partial f(\mathbf{S})}{\partial I_3}$ is the desired quadratic representation of the derivative.

91. Given arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the tensor $\mathbf{Q}$ is said to be orthogonal if $(\mathbf{Q}\boldsymbol{a}) \cdot (\mathbf{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ show that the inverse of $\mathbf{Q}$ is its transpose. and that $\mathbf{Q}$ is the cofactor of itself.

By definition of the transpose, we have that,

$$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathbf{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$

Clearly, $\mathbf{Q}^{\mathbf{T}}\mathbf{Q} = 1$ which makes the transpose the same as the inverse tensor.

A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse is equal to its transpose.

92. An orthogonal tensor $\mathbf{Q}$ is said to be "proper orthogonal" if its determinant $|\mathbf{Q}| = +1$. Show that a proper orthogonal tensor is the cofactor of itself. Show also that its first two invariants are equal.

$$\operatorname{cof}\mathbf{Q} = \det\mathbf{Q}\ \mathbf{Q}^{-\mathrm{T}} = +1\ (\mathbf{Q}^{\mathrm{T}})^{-\mathbf{1}} = 1\ (\mathbf{Q}^{-1})^{-\mathbf{1}} = \mathbf{Q}$$

$$I_2(\mathbf{Q}) = I_1(\mathbf{Q}^{\mathrm{c}})$$

The second principal invariant for any vector is equal to the first principal invariant of its co-factor. But we find here that $\mathbf{Q} = \mathbf{Q}^{\mathrm{c}}$. It follows that the first two invariants of a proper orthogonal tensor are equal. The third invariant, $I_3(\mathbf{Q}) = \det(\mathbf{Q})$=1. All essential information on an orthogonal tensor is known once we know its trace!

93. For any invertible tensor $\mathbf{S}$ show that $\det(\mathbf{S}^{\mathrm{c}}) = (\det\mathbf{S})^2$

First note that the determinant of the product of a tensor C with a scalar α is,

$$\det\alpha\mathbf{C} = \epsilon^{ijk}\left(\alpha C_i^1\right)\left(\alpha C_j^2\right)\left(\alpha C_k^3\right) = \alpha^3\det\mathbf{C}$$

The inverse of tensor $\mathbf{S}$,

$$\mathbf{S}^{-1} = (\det\mathbf{S})^{-1}(\mathbf{S}^{\mathrm{cT}})$$

Let the scalar $\alpha = \det\mathbf{S}$. We can see clearly that,

$$\mathbf{S}^{\mathrm{c}} = \alpha\mathbf{S}^{-T}$$

Taking the determinant of this equation, we have,

$$\det(\mathbf{S}^{\mathrm{c}}) = \alpha^3\det(\mathbf{S}^{-T}) = \alpha^3\det(\mathbf{S}^{-1})$$

as the transpose operation has no effect on the value of a determinant. Noting that the determinant of an inverse is the inverse of the determinant, we have,

$$\det(\mathbf{S}^{\mathrm{c}}) = \alpha^3 \det(\mathbf{S}^{-1}) = \frac{\alpha^3}{\alpha} = (\det \mathbf{S})^2$$

94. For any invertible tensor $\mathbf{S}$ and a scalar α show that show that the cofactor of the product of α and $\mathbf{S}$ equals $\alpha^2 \times$ the cofactor of $\mathbf{S}$, that is, $(\alpha\mathbf{S})^{\mathrm{c}} = \alpha^2\mathbf{S}^{\mathrm{c}}$

$$\begin{aligned}(\alpha\mathbf{S})^{\mathrm{c}} &= (\det(\alpha\mathbf{S}))(\alpha\mathbf{S})^{-\mathrm{T}} \\ &= (\alpha^3 \det(\mathbf{S}))\alpha^{-1}\boldsymbol{S}^{-\mathrm{T}} \\ &= (\alpha^2 \det(\mathbf{S}))\mathbf{S}^{-\mathrm{T}} \\ &= \alpha^2\mathbf{S}^{\mathrm{c}}\end{aligned}$$

95. For any invertible tensor $\mathbf{S}$ show that $(\mathbf{S}^{-1})^{\mathrm{c}} = (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$

$$\begin{aligned}(\mathbf{S}^{-1})^{\mathrm{c}} &= \det(\mathbf{S}^{-1})\,(\mathbf{S}^{-1})^{-T} \\ &= (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\end{aligned}$$

96. For any invertible tensor $\mathbf{S}$ show that $\mathbf{S}^{-\mathrm{c}} = (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$, that is, the inverse of the cofactor is the transpose divided by the determinant.

$$\mathbf{S}^{\mathrm{C}} = \det(\mathbf{S})\,\mathbf{S}^{-T}$$

Consequently,

$$\mathbf{S}^{-\mathrm{c}} = (\det \mathbf{S})^{-1}(\boldsymbol{S}^{-T})^{-1} = (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$$

97. For an invertible tensor show that the cofactor of the cofactor is the product of the original tensor and its determinant $\mathbf{S}^{\mathrm{cc}} = (\det \mathbf{S})\mathbf{S}$

$$\mathbf{S}^{\mathrm{c}} = \det(\mathbf{S})\, \mathbf{S}^{-T}$$

So that,

$$\begin{aligned}
\mathbf{S}^{\mathrm{cc}} &= (\det \mathbf{S}^{\mathrm{c}})(\mathbf{S}^{\mathrm{c}})^{-T} \\
&= (\det \mathbf{S})^2[(\mathbf{S}^{\mathrm{c}})^{-1}]^T = (\det \mathbf{S})^2[(\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}]^T \\
&= (\det \mathbf{S})^2(\det \mathbf{S})^{-1}\mathbf{S} \\
&= (\det \mathbf{S})\mathbf{S}
\end{aligned}$$

as required.

98. Show that for any invertible tensor $\mathbf{S}$ and any vector $\mathbf{u}$, $[(\mathbf{Su})\times] = \mathbf{S}^{\mathrm{c}}(\mathbf{u}\times)\mathbf{S}^{-1}$ where $\mathbf{S}^{\mathrm{c}}$ and $\mathbf{S}^{-1}$ are the cofactor and inverse of $\mathbf{S}$ respectively.

By definition,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

We are to prove that,

$$[(\mathbf{Su})\times] = \mathbf{S}^{\mathrm{c}}(\mathbf{u}\times)\mathbf{S}^{-1} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}(\mathbf{u}\times)\mathbf{S}^{-1}$$

or that,

$$\mathbf{S}^{\mathrm{T}}[(\mathbf{Su})\times] = (\mathbf{u}\times)(\det\mathbf{S})\mathbf{S}^{-1} = (\mathbf{u}\times)(\mathbf{S}^{\mathrm{c}})^{\mathrm{T}}$$

On the RHS, the contravariant ij component of $\boldsymbol{u}\times$ is

$$(\mathbf{u}\times)^{ij} = \epsilon^{i\alpha j}u_\alpha$$

which is exactly the same as writing, $(\mathbf{u}\times) = \epsilon^{i\alpha l}u_\alpha\mathbf{g}_i\otimes\mathbf{g}_l$ in the invariant form.

We now turn to the LHS;

$$[(\mathbf{Su})\times] = \epsilon^{l\alpha k}(\mathbf{Su})_\alpha\mathbf{g}_l\otimes\mathbf{g}_k = \epsilon^{l\alpha k}S^j_\alpha u_j\mathbf{g}_l\otimes\mathbf{g}_k$$

Now, $\mathbf{S} = S^{i.}_{.\beta}\mathbf{g}_i\otimes\mathbf{g}^\beta$ so that its transpose, $\mathbf{S}^{\mathrm{T}} = S^i_\beta\mathbf{g}^\beta\otimes\mathbf{g}_{\mathrm{i}} = S^\beta_i\mathbf{g}^i\otimes\mathbf{g}_\beta$ so that

$$\begin{aligned}\mathbf{S}^{\mathrm{T}}[(\mathbf{Su})\times] &= \epsilon^{l\alpha k}S^\beta_i S^j_\alpha u_j\mathbf{g}^i\otimes\mathbf{g}_\beta\cdot\mathbf{g}_l\otimes\mathbf{g}_k\\ &= \epsilon^{l\alpha k}S_{il}S^j_\alpha u_j\mathbf{g}^i\otimes\mathbf{g}_k = \epsilon^{l\alpha k}S^i_l S^j_\alpha u_j\mathbf{g}_i\otimes\mathbf{g}_k\\ &= \epsilon^{\alpha\beta k}u_j S^i_\alpha S^j_\beta\mathbf{g}_i\otimes\mathbf{g}_k = (\mathbf{u}\times)(\mathbf{S}^{\mathrm{c}})^{\mathrm{T}}\end{aligned}$$

99. Show that $[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$

The LHS in component invariant form can be written as:

$$[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \epsilon^{ijk}(\mathbf{S}^{\mathrm{c}}\mathbf{u})_j\mathbf{g}_i\otimes\mathbf{g}_k$$

where $(\mathbf{S}^{\mathrm{c}})^\beta_j = \frac{1}{2}\epsilon_{jab}\epsilon^{\beta cd}S^a_c S^b_d$ so that

$$(\mathbf{S}^{\mathrm{c}}\mathbf{u})_j = (\mathbf{S}^{\mathrm{c}})^\beta_j u_\beta = \frac{1}{2}\epsilon_{jab}\epsilon^{\beta cd}u_\beta S^a_c S^b_d$$

Consequently,

$$
\begin{aligned}
[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] &= \frac{1}{2}\epsilon^{ijk}\epsilon_{jab}\epsilon^{\beta cd}u_\beta S^a_c S^b_d \mathbf{g}_i \otimes \mathbf{g}_k \\
&= \frac{1}{2}\epsilon^{\beta cd}\left(\delta^k_a\delta^i_b - \delta^k_b\delta^i_a\right)u_\beta S^a_c S^b_d \mathbf{g}_i \otimes \mathbf{g}_k \\
&= \frac{1}{2}\epsilon^{\beta cd}u_\beta\left(S^k_c S^i_d - S^i_c S^k_d\right)\mathbf{g}_i \otimes \mathbf{g}_k \\
&= \epsilon^{\beta cd}u_\beta S^k_c S^i_d \mathbf{g}_i \otimes \mathbf{g}_k
\end{aligned}
$$

On the RHS, $(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}} = \epsilon^{\alpha\beta\gamma}u_\beta S^k_\gamma \mathbf{g}_i \otimes \mathbf{g}_k$. We can therefore write,

$$
\mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}} = \epsilon^{\alpha\beta\gamma}u_\beta S^i_\alpha S^k_\gamma \mathbf{g}_i \otimes \mathbf{g}_k
$$

Which on a closer look is exactly the same as the LHS so that, $[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$ as required.

100. For a tensor $\mathbf{S}$, given that $[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times] = \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\big((\operatorname{cof}\mathbf{S})\mathbf{u}\times\big)\mathbf{v} = \mathbf{S}(\mathbf{u}\times\mathbf{S}^{\mathrm{T}}\mathbf{v})$

The product of the given equation with the vector $\mathbf{v}$ immediately yields,

$$
\begin{aligned}
[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times]\mathbf{v} &= \mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}\mathbf{v} \\
\Rightarrow \big((\operatorname{cof}\mathbf{S})\mathbf{u}\times\big)\mathbf{v} &= \mathbf{S}(\mathbf{u}\times\mathbf{S}^{\mathrm{T}}\mathbf{v})
\end{aligned}
$$

101. Given that $\mathbf{\Omega}$ is a skew tensor with the corresponding axial vector $\boldsymbol{\omega}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\mathbf{\Omega u} \times \mathbf{\Omega v} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$ and, hence conclude that $\mathbf{\Omega}^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.

$$\mathbf{\Omega u} \times \mathbf{\Omega v} = (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) = (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v})$$
$$= [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\omega}]\mathbf{v} = [\boldsymbol{\omega} \cdot (\mathbf{u} \times \mathbf{v})]\boldsymbol{\omega} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$$

But by definition, the cofactor must satisfy,

$$\mathbf{\Omega u} \times \mathbf{\Omega v} = \mathbf{\Omega}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

which compared with the previous equation yields the desired result that

$$\mathbf{\Omega}^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}).$$

102. Show, using indicial notation, that the cofactor of a tensor can be written as $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^{\mathrm{T}}$ even if $\mathbf{S}$ is not invertible. I_1, I_2 are the first two invariants of $\mathbf{S}$.

The above equation can be written more explicitly as,

$$\mathbf{S}^c = \left(\mathbf{S}^2 - \mathrm{tr}(\mathbf{S})\mathbf{S} + \left[\frac{1}{2}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)]\right]\mathbf{1}\right)^{\mathrm{T}}$$

In the invariant component form, this is easily seen to be,

$$\boldsymbol{S}^{\mathrm{c}} = \left(S^i_\eta S^\eta_j - S^\alpha_\alpha S^i_j + \frac{1}{2}\left(S^\alpha_\alpha S^\beta_\beta - S^\alpha_\beta S^\beta_\alpha\right)\delta^i_j\right)\mathbf{g}^j \otimes \mathbf{g}_i$$

But we know that the cofactor can be obtained directly from the equation,

$$(\mathbf{S}^{\mathrm{c}}) = \frac{1}{2}\epsilon^{i\beta\gamma}\epsilon_{j\lambda\eta}S^{\lambda}_{\beta}S^{\eta}_{\gamma}\mathbf{g}_i \otimes \mathbf{g}^j = \frac{1}{2}\begin{bmatrix} \delta^i_j & \delta^i_\lambda & \delta^i_\eta \\ \delta^\beta_j & \delta^\beta_\lambda & \delta^\beta_\eta \\ \delta^\gamma_j & \delta^\gamma_\lambda & \delta^\gamma_\eta \end{bmatrix} S^{\lambda}_{\beta}S^{\eta}_{\gamma}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left(\delta^i_j \begin{bmatrix} \delta^\beta_\lambda & \delta^\beta_\eta \\ \delta^\gamma_\lambda & \delta^\gamma_\eta \end{bmatrix} - \delta^i_\lambda \begin{bmatrix} \delta^\beta_j & \delta^\beta_\eta \\ \delta^\gamma_j & \delta^\gamma_\eta \end{bmatrix} + \delta^i_\eta \begin{bmatrix} \delta^\beta_j & \delta^\beta_\lambda \\ \delta^\gamma_j & \delta^\gamma_\lambda \end{bmatrix}\right) S^{\lambda}_{\beta}S^{\eta}_{\gamma}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta^i_j\left(\delta^\beta_\lambda\delta^\gamma_\eta - \delta^\beta_\eta\delta^\gamma_\lambda\right) - \delta^i_\lambda\left(\delta^\beta_j\delta^\gamma_\eta - \delta^\beta_\eta\delta^\gamma_j\right) + \delta^i_\eta\left(\delta^\beta_j\delta^\gamma_\lambda - \delta^\beta_\lambda\delta^\gamma_j\right)\right] S^{\lambda}_{\beta}S^{\eta}_{\gamma}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta^i_j\left(S^{\alpha}_{\alpha}S^{\beta}_{\beta} - S^{\lambda}_{\eta}S^{\eta}_{\lambda}\right) - 2S^i_j S^{\alpha}_{\alpha} + 2S^i_\eta S^\eta_j\right]\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= [I_2(\mathbf{S})\mathbf{1} - I_1(\mathbf{S})\mathbf{S} + \mathbf{S}^2]^{\mathrm{T}}$$

103. Show, using direct notation, that the cofactor of a tensor can be written as $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^{\mathrm{T}}$ even if $\mathbf{S}$ is not invertible. I_1, I_2 are the first two invariants of $\mathbf{S}$.

For any three linearly independent vectors, the trace of a tensor $\mathbf{T}$

$$\operatorname{tr}\mathbf{T} \equiv I_1(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

Replacing $\mathbf{g}_1$ by $\mathbf{T}\mathbf{g}_1$ in the above equation, we have,

$$\operatorname{tr}\mathbf{T}\,[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]$$

Or, upon rearrangement,

$$[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3] = \operatorname{tr}\mathbf{T}\,[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

But, the second Invariant,

$$\begin{aligned} I_2(\mathbf{T}) &= \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\ &= \frac{\operatorname{tr}\mathbf{T}\,[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\ &= \frac{\operatorname{tr}\mathbf{T}\,[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + \mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3)}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\ &= \frac{[(\operatorname{tr}\mathbf{T})\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{c}\mathrm{T}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \end{aligned}$$

so that,

$$[(I_2(\mathbf{T})\mathbf{1})\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = [(\operatorname{tr}\mathbf{T})\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{c}\mathrm{T}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

From which we can write that

$$I_2(\mathbf{T})\mathbf{1} = (\operatorname{tr}\mathbf{T})\mathbf{T} - \mathbf{T}^2 + \mathbf{T}^{\mathrm{c}\mathrm{T}}$$

or,

$$\mathbf{T}^{\mathrm{c}} = (\mathbf{T}^2 - I_1(\mathbf{T})\mathbf{T} + I_2(\mathbf{T})\mathbf{1})^{\mathrm{T}}$$

104. Given a Euclidean Vector Space $\mathscr{E}$, a tensor $\mathbf{Q}$ is said to be rotation if in addition to satisfying $(\mathbf{Q}\boldsymbol{a}) \cdot (\mathbf{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}\ \forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{E}$, its determinant $(\det \mathbf{Q}) = +1$. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$ if $\mathbf{Q}$ is a rotation. That is, that the cofactor of $\mathbf{Q}$ is $\mathbf{Q}$ itself

We can write that

$$\mathbf{T}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$$

where

$$\mathbf{T} = \mathrm{cof}(\mathbf{Q}) = \det(\mathbf{Q})\, \mathbf{Q}^{-\mathrm{T}}$$

Now that $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = 1$, and because it is orthogonal, its inverse is its transpose:

$$\mathbf{Q}^{-\mathrm{T}} = (\mathbf{Q}^{-1})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q}$$

This implies that $\mathbf{T} = \mathbf{Q}$ and consequently,

$$\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$$

105. For a proper orthogonal tensor $\mathbf{Q}$, show that the eigenvalue equation always yields an eigenvalue of $+1$. This means that there is always a solution for the equation, $\mathbf{Q}\mathbf{u} = \mathbf{u}$.

For any invertible tensor, note that the cofactor is defined as,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

For a proper orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = 1$, and its inverse is its transpose. It therefore follows that,

$$\mathbf{Q}^{\mathbf{c}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

It is easily shown that $\mathrm{tr}\, \mathbf{Q}^{\mathrm{c}} = I_2(\mathbf{Q})$. It follows from the above that, $I_2(\mathbf{Q}) = I_1(\mathbf{Q})$. For the general eigenvalue equation, $\mathbf{Qu} = \lambda\mathbf{u}$, the characteristic equation is,

$$\mathbf{d}\mathrm{et}\,(\mathbf{Q} - \boldsymbol{\lambda}\mathbf{1}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

where $Q_1, Q_2 (= Q_1)$ and $Q_3 (= 1)$ are the first second and third invariants of the orthogonal tensor respectively. In this particular case, the characteristic equation becomes,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

Which is obviously satisfied by $\lambda = 1$. Hence there is always a solution to $\mathbf{Qu} = \mathbf{u}$.

106. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times)$ is the vector cross of $\mathbf{e}$. Show that $\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e} \times)$

We first observe that,

$$\mathbf{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)[\mathbf{e} \times (\mathbf{e} \otimes \mathbf{e})]$$

The last term vanishes immediately on account of the fact that $\mathbf{e} \otimes \mathbf{e}$ is a symmetric tensor. (The contraction of a symmetric and an antisymmetric tensor always vanishes). Consequently, we have,

$$\mathbf{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} = \mathbf{e} \otimes \mathbf{e}$$

which again means that $\mathbf{Q}(\theta)$ has no effect on $\mathbf{e} \otimes \mathbf{e}$ so that,

$$\begin{aligned}\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) &= \cos(\theta)\mathbf{1} + \big(1 - \cos(\theta)\big)\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times) - \mathbf{e} \otimes \mathbf{e} \\ &= \cos(\theta)\,(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e} \times)\end{aligned}$$

as required.

107. For an arbitrary unit vector $\mathbf{e}$, show that the skew tensor, $\mathbf{W} = (\mathbf{e} \times)$ is such that $\mathbf{W}^2 \equiv (\mathbf{e} \times)(\mathbf{e} \times) = (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}$

$$\begin{aligned}(\mathbf{e} \times)(\mathbf{e} \times) &= \big(\epsilon^{ijk} e_j \mathbf{g}_i \otimes \mathbf{g}_k\big)\big(\epsilon_{\alpha\beta\gamma} e^\beta \mathbf{g}^\alpha \otimes \mathbf{g}^\gamma\big) \\ &= \delta^{ijk}_{\alpha\beta\gamma} e_j e^\beta \mathbf{g}_i \otimes \mathbf{g}^\gamma \delta^\alpha_k \\ &= \delta^{ijk}_{k\beta\gamma} e_j e^\beta \mathbf{g}_i \otimes \mathbf{g}^\gamma \\ &= \left(\delta^i_\beta \delta^j_\gamma - \delta^j_\beta \delta^i_\gamma\right) e_j e^\beta \mathbf{g}_i \otimes \mathbf{g}^\gamma \\ &= e_\gamma e^\beta \mathbf{g}_\beta \otimes \mathbf{g}^\gamma - e_\beta e^\beta \mathbf{g}_i \otimes \mathbf{g}^i \\ &= e_\gamma e^\beta \mathbf{g}_\beta \otimes \mathbf{g}^\gamma - (\mathbf{e} \cdot \mathbf{e})\mathbf{g}_i \otimes \mathbf{g}^i \\ &= (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}\end{aligned}$$

upon noting that the dot product of the unit vector with itself is unity.

108. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times) \equiv \mathbf{W}$ is the vector cross of $\mathbf{e}$. Show that for, $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)$ [Note that $\mathbf{e} \otimes \mathbf{e} = \mathbf{W}^2 + \mathbf{I}$]

Using the noted result,

$$\begin{aligned}\mathbf{Q}(\theta) &= \cos\theta\,\mathbf{I} + (1 - \cos\theta)\mathbf{e} \otimes \mathbf{e} + \sin\theta\,(\mathbf{e} \times)\\ &= \cos\theta\,\mathbf{I} + (1 - \cos\theta)(\mathbf{W}^2 + \mathbf{I}) + \mathbf{W}\sin\theta\\ &= \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)\end{aligned}$$

109. Use the fact that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)$ where $\mathbf{W} \equiv (\mathbf{e} \times)$ - the vector cross of the unit tensor, rotates every vector about the axis of $\mathbf{e}$ by the angle θ to find the tensor that rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_3\}$.

Clearly, the rotation axis here is the unit vector $\mathbf{e}_3$ and the angle of rotation is $\frac{\pi}{2}$.

Consequently, since $\mathbf{e}_3 = \{0,0,1\}$,

$$\mathbf{W} \equiv (\mathbf{e}_3 \times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{W}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Q}\left(\frac{\pi}{2}\right) = \mathbf{I} + \mathbf{W}\sin\frac{\pi}{2} + \mathbf{W}^2\left(1 - \cos\frac{\pi}{2}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This same tensor can be found directly by recognizing that the tensor, $\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ so that the tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

110. Given that $\mathbf{e}_1 = \{1,0,0\}, \mathbf{e}_2 = \{0,1,0\}, \mathbf{e}_3 = \{0,0,1\}, \mathbf{e}_4 = \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \mathbf{e}_5 = \left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \mathbf{e}_6 = \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}$, Find the tensor that transforms from $\{\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_3\}$ to $\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$.

Tensor, $\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. The tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_4 \otimes \mathbf{e}_2 + \mathbf{e}_5 \otimes \mathbf{e}_1 - \mathbf{e}_6 \otimes \mathbf{e}_3$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

111. Find the rotation tensor around an axis parallel to the unit vector, $\mathbf{e} = \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$ through an angle $\frac{\pi}{3}$.

The skew tensor $(\mathbf{e}\times) = \mathbf{W} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$.

$$(\mathbf{e}\times)^2 = \mathbf{W}^2 = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{Q}\left(\frac{\pi}{6}\right) = \mathbf{I} + \mathbf{W}\sin\frac{\pi}{6} + \mathbf{W}^2\left(1-\cos\frac{\pi}{6}\right)$$

$$= \begin{pmatrix} 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & -\frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \\ \frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) \\ -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) & -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & 1+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.888354 & -0.430577 & 0.159465 \\ 0.385919 & 0.888354 & 0.248782 \\ -0.248782 & -0.159465 & 0.955341 \end{pmatrix}$$

The inverse of this tensor is its transpose and its determinant is unity. Clearly, it is the rotation tensor we seek.

112. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the rotation tensor around the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis through an angle θ.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

Along the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis, $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2}$. Consequently, $\mathbf{w} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$.

$$\mathbf{W}\left(\frac{\pi}{4},\frac{\pi}{2}\right) = (\mathbf{w}\times) = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \mathbf{W}^2\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

And the rotation tensor for this axis is,

$$\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$$

$$= \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{2}(1-\cos\theta) & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{2}(1+\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{pmatrix}.$$

113. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the general rotation through an angle θ.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

$$\mathbf{Q}(\alpha,\beta,\theta) = \mathbf{I} + \mathbf{W}(\alpha,\beta)\sin\theta + \mathbf{W}^2(\alpha,\beta)(1-\cos\theta) =$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 1:

$$\{(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\beta))+1,$$
$$\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))-\cos(\beta)\sin(\theta),$$
$$\sin(\alpha)\sin(\beta)\sin(\theta)+\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 2:

$$\{\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))+\cos(\beta)\sin(\theta),$$

$$(1-\cos(\theta))\left(-\cos^2(\alpha)\sin^2(\beta)-\cos^2(\beta)\right)+1,$$
$$\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\cos(\alpha)\sin(\beta)\sin(\theta)\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 3

$$\{\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\sin(\alpha)\sin(\beta)\sin(\theta),$$
$$\sin(\alpha)\,\sin(\beta)\,\cos(\beta)\,(1-\cos(\theta))+\cos(\alpha)\,\sin(\beta)\,\sin(\theta)\,,$$
$$(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\alpha)\sin^2(\beta))+1\}$$

114. Given that the skew tensor $(\mathbf{e}\times)\equiv\mathbf{W}$, and that $\mathbf{Q}(\theta)\equiv\ \mathbf{I}+\mathbf{W}\sin\theta+\mathbf{W}^2(1-\cos\theta)$ is the rotation along the axis $\mathbf{e}$ through the angle θ, Find out if the set $\{\mathbf{I},\mathbf{W},\mathbf{W}^2\}$ is linearly independent.

First note that $\mathbf{W}$ is antisymmetric but $\mathbf{W}^2=(\mathbf{e}\otimes\mathbf{e})-\mathbf{I}$ is the linear combination of two symmetric tensors, and therefore symmetric. Assume that $\{\mathbf{I},\mathbf{W},\mathbf{W}^2\}$ to be linearly dependent. It means we can find α,β and γ not all equal to zero such that

$$\alpha\mathbf{I}+\beta\mathbf{W}+\gamma\mathbf{W}^2=0$$

Since α,β and γ are not all equal to zero, we assume in particular that $\beta\neq 0$. Consequently, we can write,

$$\mathbf{W}=-\frac{\alpha}{\beta}\mathbf{I}-\frac{\gamma}{\beta}\mathbf{W}^2$$

In which we have expressed the anti-symmetric tensor $\mathbf{W}$ as a linear combination of two symmetric tensors! A contradiction! We can conclude that the set $\{\mathbf{I},\mathbf{W},\mathbf{W}^2\}$ is

linearly independent.

115. Given that every rotation tensor $\mathbf{Q}$ can be expressed in terms of the skew tensor $\mathbf{W}(\equiv \mathbf{e}\times)$ as a function of the rotation angle θ: $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ and that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent set of tensors, show that, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^{\mathrm{T}}\}$ is also a linearly independent set.

Assume that the tensor set, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^{\mathrm{T}}\}$ is linearly dependent. It means we can find α, β and γ not all equal to zero such that

$$\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^{\mathrm{T}} = 0$$

Since $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$, we substitute and obtain,

$$\begin{aligned}
&\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^{\mathrm{T}} = \\
&= \alpha\mathbf{I} + \beta\big(\mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) + \gamma\big(\mathbf{I} - \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) \\
&\qquad = (\alpha + \beta + \gamma)\mathbf{I} + (\beta - \gamma)\,\mathbf{W}\sin\theta + (\beta + \gamma)\mathbf{W}^2(1-\cos\theta) \\
&\qquad = a\mathbf{I} + b\mathbf{W} + c\mathbf{W}^2 = 0
\end{aligned}$$

if we write $(\alpha + \beta + \gamma) = a$, $(\beta - \gamma)\sin\theta = b$ and $(\beta + \gamma)(1-\cos\theta) = c$ thereby contradicting the well-known fact that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is a linearly independent set.

116. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times)$ is the vector cross of $\mathbf{e}$. Given that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$, and that, for any scalar α, $\mathbf{Q}(\theta)(\alpha\mathbf{e}) = \alpha\mathbf{e}$, What is the physical meaning of $\mathbf{Q}(\theta)$?

> $\mathbf{Q}(\theta)$ is a rotation about the vector $\mathbf{e}$ counterclockwise through an angle θ. It therefore does not alter the magnitude or direction of any vector in the direction of $\mathbf{e}$; for any other vector, it has no effect on the magnitude but affects direction.

117. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times)$ is the vector cross of $\mathbf{e}$. Show that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$. What is the physical meaning of $\mathbf{Q}(\theta)$?

> Let the scalar $x \equiv \mathbf{e} \cdot \mathbf{u}$ be the projection of $\mathbf{u}$ onto the unit vector $\mathbf{e}$. The square of the magnitude of $\mathbf{v}$ is $|\mathbf{v}|^2$

$$
\begin{aligned}
= \mathbf{v} \cdot \mathbf{v} &= \big(\cos\theta(\mathbf{1u}) + (1 - \cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\big) \\
&\quad \cdot \big(\cos\theta(\mathbf{1u}) + (1 - \cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\big) \cdot \\
&= \big(\mathbf{u}\cos\theta + (1 - \cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)^2 \\
&= (\mathbf{u}\cos\theta) \cdot \big(\mathbf{u}\cos\theta + (1 - \cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)
\end{aligned}
$$

$$+x\mathbf{e}\cdot\big(\mathbf{u}\cos\theta+(1-\cos\theta)x\mathbf{e}+\sin\theta(\mathbf{e}\times\mathbf{u})\big)(1-\cos\theta)$$
$$+(\mathbf{e}\times\mathbf{u})\cdot\big(\mathbf{u}\cos\theta+(1-\cos\theta)x\mathbf{e}+\sin\theta(\mathbf{e}\times\mathbf{u})\big)\sin\theta$$
$$=\ \mathbf{u}^2\cos^2\theta+2(\cos\theta-\cos^2\theta)x^2+\ 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta+(1-\cos\theta)^2x^2$$
$$+\ 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta+\sin^2\theta\,(\mathbf{e}\times\mathbf{u})^2$$
$$=\mathbf{u}^2\cos^2\theta+2(\cos\theta-\cos^2\theta)x^2+\ 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta+(1-\cos\theta)^2x^2$$
$$+\ 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta+\sin^2\theta\,(\mathbf{u}^2-x^2)$$
$$=\mathbf{u}^2(\cos^2\theta+\sin^2\theta)+x^2[2(\cos\theta-\cos^2\theta)+(1-\cos\theta)^2-\sin^2\theta]$$
$$=\mathbf{u}^2$$

As the term in square brackets vanish when expanded.

118. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta)=\cos(\theta)\mathbf{1}+(1-\cos(\theta))\mathbf{e}\otimes\mathbf{e}+\sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that for arbitrary $0<\alpha,\beta\leq 2\pi$, that $\mathbf{Q}(\alpha+\beta)=\mathbf{Q}(\alpha)\mathbf{Q}(\beta)$.

It is convenient to write $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ in terms of their i,j components; we assume that the unit vector $\mathbf{e}=(x_1,x_2,x_3)$:

$$[\mathbf{Q}(\alpha)]_{ij}=\cos\alpha\,\delta_{ij}+(1-\cos\alpha)x_ix_j-\sin\alpha\,\epsilon_{ijk}x_k$$

Consequently, we can write for the product $\mathbf{Q}(\alpha)\mathbf{Q}(\beta)$,

$$[\mathbf{Q}(\alpha)\mathbf{Q}(\beta)]_{ij}=[\mathbf{Q}(\alpha)]_{ik}[\mathbf{Q}(\beta)]_{kj}=$$

$$
\begin{aligned}
&= [\cos\alpha\,\delta_{ik} + (1-\cos\alpha)x_i x_k - \sin\alpha\,\epsilon_{ikl}x_l]\big[\cos\beta\,\delta_{kj} + (1-\cos\beta)x_k x_j \\
&\qquad - \sin\beta\,\epsilon_{kjn}x_n\big] \\
&= \cos\alpha\cos\beta\,\delta_{ik}\delta_{kj} + \cos\alpha\,(1-\cos\beta)\delta_{ik}x_k x_j - \cos\alpha\sin\beta\,\delta_{ik}\epsilon_{kjn}x_n \\
&\qquad + (1-\cos\alpha)\cos\beta\,x_i x_k\delta_{kj} + (1-\cos\alpha)(1-\cos\beta)x_i x_k^2 x_j \\
&\qquad - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ikl}x_l\delta_{kj} \\
&\qquad - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l \\
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j \\
&\qquad + (1-\cos\alpha)(1-\cos\beta)x_i x_j - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} \\
&\qquad - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l \\
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j \\
&\qquad + (1-\cos\alpha)(1-\cos\beta)x_i x_j - \boxed{(1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn}} \\
&\qquad - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \boxed{\sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j} \\
&\qquad + \sin\alpha\sin\beta\,\big(\delta_{lj}\delta_{in} - \delta_{ln}\delta_{ji}\big)x_n x_l \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\delta_{ij} + [1 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]x_i x_j \\
&\qquad - [(\cos\alpha\sin\beta - \sin\alpha\cos\beta)]\epsilon_{ijn}x_n \\
&= [\mathbf{Q}(\alpha+\beta)]_{ij}
\end{aligned}
$$

With the boxed terms vanishing on account of antisymmetric contraction with symmetry.

119. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times)$ is the vector cross of $\mathbf{e}$. Show that . $\mathbf{Q}(\theta)$ is a periodic tensor function with period 2π. [Hint: $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$

Since $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$ we can write that $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi)$. But a direct substitution shows that, $\mathbf{Q}(0) = \mathbf{Q}(2\pi) = \mathbf{1}$. We therefore have that,
$\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi) = \mathbf{Q}(\alpha)$
which completes the proof. The above results show that $\mathbf{Q}(\alpha)$ is a rotation along the unit vector $\mathbf{e}$ through an angle α.

120. Given that $\mathbf{Q}$ is an orthogonal tensor, show that the principal invariants of a tensor S satisfy $I_k(\mathbf{QSQ}^{\mathrm{T}}) = I_k(\mathbf{S}), k = 1,2,\text{or } 3$, that is, Rotations and orthogonal transformations do not change the Invariants.

$$
\begin{aligned}
I_1(\mathbf{QSQ}^{\mathrm{T}}) &= \mathrm{tr}(\mathbf{QSQ}^{\mathrm{T}}) \\
&= \mathrm{tr}(\mathbf{Q}^{\mathrm{T}}\mathbf{QS}) = \mathrm{tr}(\mathbf{S}) \\
&= I_1(\mathbf{S}) \\
I_2(\mathbf{QSQ}^{\mathrm{T}}) &= \frac{1}{2}[\mathrm{tr}^2(\mathbf{QSQ}^{\mathrm{T}}) - \mathrm{tr}(\mathbf{QSQ}^{\mathrm{T}}\mathbf{QSQ}^{\mathrm{T}})]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{QS^2Q^T})] \\
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{Q^TQS^2})] \\
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S^2})] = I_2(\mathbf{S}) \\
I_3(\mathbf{QSQ^T}) &= \det(\mathbf{QSQ^T}) \\
&= \det(\mathbf{Q^TQS}) = \det(\mathbf{S}) \\
&= I_3(\mathbf{S})
\end{aligned}
$$

Hence $I_k(\mathbf{QSQ^T}) = I_k(\boldsymbol{S}),\ \ k = 1,2,\ \text{or } 3$