1 HOMEWORK 2.1

1. For any tensor $\mathbf{S}$, show that, $(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha = \mathbf{S}$
2. Given vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, establish the identities:
   a) $(\mathbf{u}\,\times)(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$
   b) $(\mathbf{u} \otimes\,)(\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$
3. Show that that if the tensor $\mathbf{T}$ is invertible, for any vector $\mathbf{k}$, $\mathbf{T}\mathbf{k} = \mathbf{o}$ automatically means that $\mathbf{k} = \mathbf{o}$.
4. Show that if the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are independent and $\mathbf{T}$ is invertible, then the vectors $\mathbf{T}\mathbf{u}$, $\mathbf{T}\mathbf{v}$ and $\mathbf{T}\mathbf{w}$ are also independent.

Due Nov 2, 2017

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TENSOR ALGEBRA

TENSORS AS LINEAR MAPPINGS

4 SECOND ORDER TENSOR

A second Order Tensor $\boldsymbol{T}$ is a linear mapping from a vector space to itself. Given $\boldsymbol{u} \in \mathcal{V}$ the mapping,

$$\boldsymbol{T}\colon \mathcal{V} \to \mathcal{V}$$

states that $\exists\, \boldsymbol{w} \in \mathcal{V}$ such that,

$$\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{w}.$$

Every other definition of a second order tensor can be derived from this simple definition. The tensor character of an object can be established by observing its action on a vector.

5 LINEARITY

- The mapping is linear. This means that if we have two runs of the process, we first input $\boldsymbol{u}$ and later input $\mathbf{v}$. The outcomes $\boldsymbol{T}(\boldsymbol{u})$ and $\boldsymbol{T}(\mathbf{v})$, added would have been the same as if we had added the inputs $\boldsymbol{u}$ and $\mathbf{v}$ first and supplied the sum of the vectors as input. More compactly, this means,

$$\boldsymbol{T}(\boldsymbol{u}+\mathbf{v}) = \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\mathbf{v})$$

6 LINEARITY

Linearity further means that, for any scalar α and tensor $\boldsymbol{T}$

$$\boldsymbol{T}(\alpha\boldsymbol{u}) = \alpha\boldsymbol{T}(\boldsymbol{u})$$

The two properties can be added so that, given $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$, then

$$\boldsymbol{T}(\alpha\boldsymbol{u} + \beta\mathbf{v}) = \alpha\boldsymbol{T}(\boldsymbol{u}) + \beta\boldsymbol{T}(\mathbf{v})$$

The sum of two tensors is the tensor that will give an output which will be the sum of the outputs of the two tensors when each is given that input.

7 VECTOR SPACE

In general, $\alpha, \beta \in \mathscr{R}$, $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$ and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\alpha \boldsymbol{S}\boldsymbol{u} + \beta \boldsymbol{T}\boldsymbol{u} = (\alpha \boldsymbol{S} + \beta \boldsymbol{T})\boldsymbol{u}$$

With the definition above, the set of tensors constitute a vector space with its rules of addition and multiplication by a scalar. It will become obvious later that it also constitutes a Euclidean vector space with its own rule of the inner product.

8 SPECIAL TENSORS

Notation.

It is customary to write the tensor mapping without the parentheses. Hence, we can write,

$$\boldsymbol{T}\boldsymbol{u} \equiv \boldsymbol{T}(\boldsymbol{u})$$

For the mapping by the tensor $\boldsymbol{T}$ on the vector variable and dispense with the parentheses unless when needed.

9 ZERO TENSOR OR ANNIHILATOR

The annihilator $\boldsymbol{O}$ is defined as the tensor that maps all vectors to the zero vector, $\boldsymbol{o}$:

$$\boldsymbol{O}u = \boldsymbol{o}, \qquad \forall \boldsymbol{u} \in \mathscr{V}$$

10 THE IDENTITY

The identity tensor $\mathbf{1}$ is the tensor that leaves every vector unaltered. $\forall \boldsymbol{u} \in \mathscr{V}$,

$$\mathbf{1}\boldsymbol{u} = \boldsymbol{u}$$

Furthermore, $\forall \alpha \in \mathscr{R}$, the tensor, $\alpha\mathbf{1}$ is called a spherical tensor.

The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\boldsymbol{T} \in \mathscr{T}$ and $\boldsymbol{u} \in \mathscr{V}$, the mapping $\boldsymbol{w} \equiv \boldsymbol{T}\boldsymbol{u}$ produces a vector.

THE INVERSE

Consider a linear mapping that, operating on $\boldsymbol{w}$, produces our original argument, $\boldsymbol{u}$, if we can find it:

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{u}$$

As a linear mapping, operating on a vector, clearly, $\boldsymbol{Y}$ is a tensor. It is called the inverse of $\boldsymbol{T}$ because,

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{Y}\boldsymbol{T}\boldsymbol{u} = \boldsymbol{u}$$

So that the composition $\boldsymbol{Y}\boldsymbol{T} = \mathbf{1}$, the identity mapping. For this reason, we write,

$$\boldsymbol{Y} = \boldsymbol{T}^{-1}$$

12 INVERSE

It is easy to show that if $\boldsymbol{YT} = \mathbf{1}$, then $\boldsymbol{TY} = \boldsymbol{YT} = \mathbf{1}$.

- ***HW: Show this.***

The set of invertible sets is closed under composition. It is also closed under inversion. It forms a group with the identity tensor as the group's neutral element

13 TRANSPOSITION OF TENSORS

Given $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot \left(\boldsymbol{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of A.

A tensor indistinguishable from its transpose is said to be symmetric.

14 INVARIANTS

There are certain mappings from the space of tensors to the real space. Such mappings are called Invariants of the Tensor. Three of these, called Principal invariants play key roles in the application of tensors to design and analysis. We shall define them shortly.

The definition given here is free of any association with a coordinate system. It is a good practice to derive any other definitions from these fundamental ones:

15 THE TRACE

If we write

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

- where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors.

For any second order tensor $\boldsymbol{T}$, and linearly independent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the linear mapping $I_1\colon \mathscr{T} \to \mathscr{R}$

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{T}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \boldsymbol{T}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Is independent of the choice of the basis vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is called the First Principal Invariant of $\boldsymbol{T}$ or Trace of $\boldsymbol{T} \equiv \mathrm{tr}(\boldsymbol{T}) \equiv I_1(\boldsymbol{T})$

16 THE TRACE

Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary independent vectors let us choose the Cartesian Basis vectors, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

For any second order tensor $\boldsymbol{T}$,

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = [\boldsymbol{T}\mathbf{i}, \mathbf{j}, \mathbf{k}] + [\mathbf{i}, \mathbf{Tj}, \mathbf{k}] + [\mathbf{i}, \mathbf{j}, \mathbf{Tk}]$$

Since $[\mathbf{i}, \mathbf{j}, \mathbf{k}] = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1.$

17 THE TRACE

The trace is a linear mapping. It is easily shown that $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\text{tr}(\alpha\boldsymbol{S} + \beta\boldsymbol{T}) = \alpha\text{tr}(\boldsymbol{S}) + \beta\text{tr}(\boldsymbol{T})$$

HW. Show this by appealing to the linearity of the vector space.

While the trace of a tensor is linear, the other two principal invariants are nonlinear. We now proceed to define them

18 SQUARE OF THE TRACE

The second principal invariant $I_2(\boldsymbol{S})$ is related to the trace. In fact, you may come across books that define it so. However, the most common definition is that

$$I_2(\boldsymbol{S}) = \frac{1}{2}\left[I_1^2(\boldsymbol{S}) - I_1(\boldsymbol{S}^2)\right]$$

Independently of the trace, we can also define the second principal invariant as,

19 SECOND PRINCIPAL INVARIANT

The Second Principal Invariant, $I_2(\boldsymbol{T})$, using the same notation as above is

$$\frac{[(\boldsymbol{Ta}),(\boldsymbol{Tb}),\boldsymbol{c}]+[\boldsymbol{a},(\boldsymbol{Tb}),(\boldsymbol{Tc})]+[(\boldsymbol{Ta}),\boldsymbol{b},(\boldsymbol{Tc})]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

This quantity remains unchanged for any arbitrary selection of linearly independent vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

THE DETERMINANT

The third mapping from tensors to the real space underlying the tensor is the determinant of the tensor. While you may be familiar with that operation and can easily extract a determinant from a matrix, it is important to understand the definition for a tensor that is independent of the component expression. The latter remains relevant even when we have not expressed the tensor in terms of its components in a particular coordinate system.

21

THE DETERMINANT

As before, For any second order tensor $\boldsymbol{T}$, and any linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

- The determinant of the tensor $\boldsymbol{T}$,

$$\det(\boldsymbol{T}) = \frac{[(\boldsymbol{Ta}), (\boldsymbol{Tb}), (\boldsymbol{Tc})]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

(In the special case when the chosen vectors are orthonormal, the denominator is unity)

22 OTHER PRINCIPAL INVARIANTS

- It is good to note that there are other principal invariants that can be defined. The ones we defined here are the ones you are most likely to find in other texts.
- An invariant is a scalar derived from a tensor that remains unchanged in any coordinate system. Mathematically, it is a mapping from the tensor space to the real space. Or simply **a scalar valued function of the tensor**.

23 DEVIATORIC TENSORS

- When the trace of a tensor is zero, the tensor is said to be traceless. A traceless tensor is also called a deviatoric tensor.
- Given any tensor $\mathbf{S}$, A deviatoric tensor may be created from $\mathbf{S}$ by the following process:

$$\mathbf{S}_0 \equiv \operatorname{dev}\mathbf{S} \equiv \mathbf{S} - \frac{1}{3}(\operatorname{tr}\mathbf{S})\mathbf{1} = \mathbf{S} - s\mathbf{1}$$

So that $s = \frac{1}{3}(\operatorname{tr}\mathbf{S})$; $s\mathbf{1}$ is called the spherical part, and $\mathbf{S}_0$ as defined here is called the deviatoric part of $\mathbf{S}$.

Every tensor thus admits the decomposition,

$$\mathbf{S} = \mathbf{S}_0 + s\mathbf{1}$$

24 SYMMETRIC, ANTISYMMETRIC PARTS

Every second order tensor can be split into its symmetric and antisymmetric parts:

$$\frac{1}{2}(\mathbf{S}+\mathrm{S}^{\mathrm{T}})+\frac{1}{2}(\mathbf{S}-\mathbf{S}^{\mathrm{T}})\equiv \operatorname{sym}\mathrm{S}+\operatorname{skw}\mathrm{S}$$

This decomposition is unique. The component representation of these two parts will be given shortly.

25 INNER PRODUCT OF TENSORS

The trace provides a simple way to define the inner product of two second-order tensors. Given $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

The trace,

$$\operatorname{tr}\left(\boldsymbol{S}^T \boldsymbol{T}\right) = \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Is a scalar, independent of the coordinate system chosen to represent the tensors. This is defined as the inner or scalar product of the tensors $\boldsymbol{S}$ and $\boldsymbol{T}$. That is,

$$\boldsymbol{S}:\boldsymbol{T} \equiv \operatorname{tr}\left(\boldsymbol{S}^T \boldsymbol{T}\right) = \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

26 THE TENSOR PRODUCT

A product mapping from two vector spaces to $\mathscr{T}$ is defined as the tensor product. It has the following properties:

$$"\otimes": \mathscr{V} \times \mathscr{V} \to \mathscr{T}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

It is an ordered pair of vectors. It acts on any other vector by creating a new vector in the direction of its first vector as shown above. This product of two vectors is called a tensor product or a simple dyad.

27 DYAD PROPERTIES

The tensor product is linear in its two factors.

Based on the obvious fact that for any tensor $\boldsymbol{T}$ and

$$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$$

$$\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{T}\boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = [(\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}]\boldsymbol{w}$$

It is clear that $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}$

Furthermore, the contraction,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes \left(\boldsymbol{T}^T \boldsymbol{v}\right)$$

A fact that can be established by operating each side on the same vector.

28 TRANSPOSE OF A DYAD

For $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot \left(\boldsymbol{A}^{\mathrm{T}} \mathbf{v}\right) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of $\boldsymbol{A}$. Now let $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ a dyad.

$$\begin{aligned} \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w}) &= \\ &= \mathbf{v} \cdot [(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w}] = \mathbf{v} \cdot [\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{w})] \\ &= (\mathbf{v} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{b})(\mathbf{v} \cdot \boldsymbol{a}) \\ &= \boldsymbol{w} \cdot (\boldsymbol{b} \otimes \boldsymbol{a})\mathbf{v} \end{aligned}$$

So that $(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}$

Showing that the transpose of a dyad is simply a reversal of its factors.

29 COMPOSITION WITH TENSORS

Operate on the vector $\boldsymbol{z}$ and let $\boldsymbol{Tz} = \boldsymbol{w}$. On the LHS,

$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{Tz} = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$. On the RHS, we have:

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T \boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}\left((\boldsymbol{T}^T \boldsymbol{v}) \cdot \boldsymbol{z}\right) = \boldsymbol{u}\left(\boldsymbol{z} \cdot (\boldsymbol{T}^T \boldsymbol{v})\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\boldsymbol{u} \otimes \left(\boldsymbol{z} \cdot (\boldsymbol{T}^T \boldsymbol{v})\right) = \boldsymbol{u} \otimes (\boldsymbol{v} \cdot (\boldsymbol{Tz}))$$

follows from the definition of transposition. Hence,

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T \boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

30 DYAD ON DYAD COMPOSITION

For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \mathscr{V}$, We can show that the dyad composition,

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{u} \otimes \boldsymbol{x})(\boldsymbol{v} \cdot \boldsymbol{w})$$

Again, the proof is to show that both sides produce the same result when they act on the same vector. Let $\boldsymbol{y} \in \mathscr{V}$, then the LHS on $\boldsymbol{y}$ yields:

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x})\boldsymbol{y} = (\boldsymbol{u} \otimes \boldsymbol{v})[\boldsymbol{w}(\boldsymbol{x} \cdot \boldsymbol{y})] = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w})(\boldsymbol{x} \cdot \boldsymbol{y})$$

Which is obviously the result from the RHS also.

This therefore makes it straightforward to contract dyads by breaking and joining as seen above.

31 TRACE OF A DYAD

Show that the trace of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v}$.

Given any three independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, (No loss of generality in letting the three independent vectors be the curvilinear basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$). Using the above definition of trace, we can write that,

32 TRACE OF A DYAD

$$\mathrm{tr}(\mathbf{u}\otimes\mathbf{v}) = \frac{[\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_1\},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{e}_2\},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_2,\{(\mathbf{u}\otimes\mathbf{v})\mathbf{e}_3\}]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,]}$$

$$= \frac{1}{e_{123}}[\{v_1\mathbf{u}\},\mathbf{e}_2,\mathbf{e}_3] + [\mathbf{e}_1,\{v_2\mathbf{u}\},\mathbf{e}_3] + [\mathbf{e}_1,\mathbf{e}_2,\{\boldsymbol{v}_3\mathbf{u}\}]$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{23i}\mathbf{e}_i) + (e_{31i}\mathbf{e}_i)\cdot(v_2\,\mathbf{u}) + (e_{12i}\mathbf{e}_i)\cdot(v_3\mathbf{u})\}$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u})\cdot(e_{231}\mathbf{e}_1) + (e_{312}e_2)\cdot(v_2\mathbf{u}) + (e_{123}e_3)\cdot(v_3\mathbf{u})\} = \mathbf{u}\cdot\mathbf{v}$$

33 OTHER INVARIANTS OF A DYAD

- It is easy to show that for a tensor product

$$\boldsymbol{D} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V}$$

$$\boldsymbol{I}_2(\boldsymbol{D}) = \boldsymbol{I}_3(\boldsymbol{D}) = 0$$

HW. Show that this is so.

We proved earlier that $\boldsymbol{I}_1(\boldsymbol{D}) = \boldsymbol{u} \cdot \boldsymbol{v}$

Furthermore, if $\boldsymbol{T} \in \mathscr{T}$, then,

$$\mathrm{tr}(\boldsymbol{T}\boldsymbol{u} \otimes \boldsymbol{v}) = \mathrm{tr}(\boldsymbol{w} \otimes \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{v}$$

34 COMPONENT REPRESENTATION

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

- The coefficient T_{ij} can be found by,

$$T_{ij} = \mathbf{T}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$$
$$= \mathrm{tr}\left(\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right)$$
$$= \mathrm{tr}\left((\mathbf{T}\mathbf{e}_j) \otimes \mathbf{e}_i\right)$$
$$= \left(\mathbf{T}\mathbf{e}_j\right) \cdot \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$$

- For the identity tensor, it is easy to show that,

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Showing that the Kronecker deltas are actually the coefficients of the identity tensor.

35 SYMMETRY

For tensor $\boldsymbol{T}$ in component form,

$$\boldsymbol{T} = \boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

The transpose,

$$\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T} = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$$
$$= T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

If the tensor is symmetrical,

$$\boldsymbol{T} = \boldsymbol{T}^{\mathrm{T}}$$

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \boldsymbol{T}^{\mathrm{T}} = T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that symmetry implies that,

$$T_{ij} = T_{ji}$$

36 ANTISYMMETRY

- A tensor is antisymmetric if its transpose is its negative. In product bases that are either covariant or contravariant, antisymmetry, like symmetry can be expressed in terms of the components:

If $\boldsymbol{T}$ is antisymmetric, then,

$$\boldsymbol{T} = -\boldsymbol{T}^{\mathrm{T}}$$

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -\boldsymbol{T}^{\mathrm{T}} = -T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

So that symmetry implies that,

$$T_{ij} = -T_{ji}$$

Antisymmetric tensors are also said to be skew-symmetric.

37 SYMMETRIC & SKEW PARTS OF TENSORS

For any tensor $\mathbf{T}$, define the symmetric and skew parts

$\text{sym}\,\mathbf{T} \equiv \frac{1}{2}(\mathbf{T}+\mathbf{T}^{\mathrm{T}})$, and $\text{skw}\,\mathbf{T} \equiv \frac{1}{2}(\mathbf{T}-\mathbf{T}^{\mathrm{T}})$. It is easy to show the following:

$\mathbf{T} = \text{sym}\,\mathbf{T} + \text{skw}\,\mathbf{T}$

$\text{skw}(\text{sym}\,\mathbf{T}) = \text{sym}(\text{skw}\,\mathbf{T}\ ) = 0$. We can also write that,

$$\text{sym}\,\mathbf{T} = \frac{1}{2}(T_{ij}+T_{ji})\mathbf{e}_i \otimes \mathbf{e}_j$$

and

$$\text{skw}\,\mathbf{T} = \frac{1}{2}(T_{ij}-T_{ji})\mathbf{e}_i \otimes \mathbf{e}_j$$

38 COMPOSITION

Composition of tensors in component form follows the rule of the composition of dyads.

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\boldsymbol{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\begin{aligned}
\boldsymbol{TS} &= \left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right) \\
&= T_{ij}S_{\alpha\beta}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right) \\
&= T_{ij}S_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\delta_{j\alpha} \\
&= T_{ij}S_{j\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta \\
&= T_{i\alpha}S_{\alpha j}\mathbf{e}_i \otimes \mathbf{e}_j
\end{aligned}$$

39 ADDITION

- Addition of two tensors of the same order is the addition of their components provided they are refereed to the same product basis.

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\boldsymbol{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\boldsymbol{T} + \boldsymbol{S} = (T_{ij} + S_{ij})\mathbf{e}_i \otimes \mathbf{e}_j,$$

40 COMPONENT REPRESENTATION OF INVARIANTS

- Invoking the definition of the three principal invariants, we now find expressions for these in terms of the components of tensors in various product bases.
- First note that for $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, the triple product, $[\{\mathbf{Te}_1\}, \mathbf{e}_2, \mathbf{e}_3] =$ $\left[\left\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\right\}, \mathbf{e}_2, \mathbf{e}_3\right]$

$$= \left[\left\{T_{ij}\mathbf{e}_i\delta_{1j}\right\}, \mathbf{e}_2, \mathbf{e}_3\right] = T_{i1}[\mathbf{e}_i, \mathbf{e}_2, \mathbf{e}_3] = T_{i1}e_{i23}$$

41 THE TRACE

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\operatorname{tr}(\boldsymbol{T}) = \frac{[\mathbf{Te}_1, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

$$= \frac{[\{(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, (T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3]}{e_{123}}$$

$$= T_{i1}e_{i23} + T_{i2}e_{1i3} + T_{i3}e_{12i}$$

$$= T_{11} + T_{22} + T_{33}$$

$$= T_{ii}$$

42 SECOND INVARIANT

The Trace of the Tensor $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{[\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

Which, in a similar way to the above, can be shown to be,

$$\mathrm{I}_2(\boldsymbol{T}) = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji})$$

Which is half the square of the trace minus trace of the square of the tensor $\mathbf{T}$

43 DETERMINANT

The third invariant,

$$\frac{[(\boldsymbol{T}\mathbf{e}_{\mathbf{1}}),(\boldsymbol{T}\mathbf{e}_2),(\boldsymbol{T}\mathbf{e}_3)]}{[\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3]}=e_{ijk}T_{i1}T_{j2}T_{k3}$$

$$=\det(\boldsymbol{T})$$

44 THE VECTOR CROSS

Given a vector $\boldsymbol{u} = u_i \mathbf{e}_i$, the tensor

$$(\boldsymbol{u}\times) \equiv \epsilon_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

is called a vector cross. The following relation is easily established between a the vector cross and its associated vector:

$$\forall \mathbf{v} \in \mathscr{V}, (\boldsymbol{u}\times)\mathbf{v} = \boldsymbol{u} \times \mathbf{v}$$

The vector cross is *traceless* and *antisymmetric*. *(HW. Show this)*

Traceless tensors are also called deviatoric or deviator tensors.

45 EXAMPLES

Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $(\mathbf{u}\times):(\mathbf{v}\times) = 2\mathbf{u}\cdot\mathbf{v}$. Hence show that $||\mathbf{u}\times|| = \sqrt{2}||\mathbf{u}||$

46 ORTHOGONAL TENSORS

Given a Euclidean Vector Space $\mathscr{E}$, a tensor $\boldsymbol{Q}$ is said to be orthogonal if, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{E}$,

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$$

Specifically, we can allow $\boldsymbol{a} = \boldsymbol{b}$, so that

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{a}$$

Or

$$\|\boldsymbol{Q}\boldsymbol{a}\| = \|\boldsymbol{a}\|$$

In which case the mapping leaves the magnitude unaltered.

47 ORTHOGONAL TENSORS

Let $\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{a}$

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

By definition of the transpose, we have that,

$$\boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{q} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}$$

Clearly, $\boldsymbol{Q}^T\boldsymbol{Q} = \mathbf{1}$

A condition necessary and sufficient for a tensor $\boldsymbol{Q}$ *to be orthogonal is that* $\boldsymbol{Q}$ *be invertible and its inverse equal to its transpose.*

48 ORTHOGONAL

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\boldsymbol{Q}^T\boldsymbol{Q}\right) = \left(\det \boldsymbol{Q}^T\right)(\det \boldsymbol{Q}) = (\det \boldsymbol{Q})^2 = 1$$

Which clearly shows that

$$(\det \boldsymbol{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "*proper orthogonal*".

49 ROTATION & REFLECTION

A rotation is a proper orthogonal tensor while a reflection is not.

50 ROTATION

- **Let $\boldsymbol{Q}$ be a rotation. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$**

This question is the same as showing that the cofactor of $\boldsymbol{Q}$ is $\boldsymbol{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\boldsymbol{T}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

where

$$\mathbf{T} = \mathrm{cof}(\boldsymbol{Q}) = \det(\boldsymbol{Q})\, \boldsymbol{Q}^{-\mathrm{T}}$$

Now that $\boldsymbol{Q}$ is a rotation, $\det(\boldsymbol{Q}) = 1$, and

$$\boldsymbol{Q}^{-\mathrm{T}} = (\boldsymbol{Q}^{-1})^{T} = (\boldsymbol{Q}^{\mathrm{T}})^{T} = \boldsymbol{Q}$$

This implies that $\boldsymbol{T} = \boldsymbol{Q}$ and consequently,

$$\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$