1. Given that $\forall\, \mathbf{v} \in \mathscr{V}, \boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\boldsymbol{a} = \mathbf{b}$

We are given that $\forall\, \mathbf{v} \in \mathscr{V}, \boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$ this implies,

$$\boldsymbol{a} \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} = (\boldsymbol{a} - \mathbf{b}) \cdot \mathbf{v} = 0$$

Define the vector $\mathbf{c} \equiv \boldsymbol{a} - \mathbf{b}$. The equation becomes,

$$\mathbf{c} \cdot \mathbf{v} = \|\mathbf{c}\|\|\mathbf{v}\| \cos\theta = 0.$$

Because $\mathbf{v}$ can be any vector, it does not have to be perpendicular to $\mathbf{c}$ and we can rule out the trivial case of its being the zero vector. This leaves us with the only choice that $\|\mathbf{c}\| = 0$. And, the only vector that has zero magnitude is the zero vector. So that,

$\mathbf{c} \equiv \boldsymbol{a} - \mathbf{b} = \mathbf{o}$, or $\boldsymbol{a} = \mathbf{b}$.

2. Given that $\forall\, \mathbf{v} \in \mathscr{V}, , \boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$, show that $\boldsymbol{a} = \mathbf{b}$

We are given that $\forall\, \mathbf{v} \in \mathscr{V}, , \boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$,

Now take a dot product with $\boldsymbol{a}$, we have that,

$$\boldsymbol{a} \cdot \mathbf{b} \times \mathbf{v} = (\boldsymbol{a} \times \mathbf{b}) \cdot \mathbf{v} = 0 = \mathbf{o} \cdot \mathbf{v}$$

for all $\mathbf{v}$ proving from that $\boldsymbol{a} \times \mathbf{b} = \mathbf{o}$. This shows that $\boldsymbol{a}$ and $\mathbf{b}$ are collinear. We can therefore write that $\mathbf{b} = \alpha\boldsymbol{a}$

Hence, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} = \alpha\boldsymbol{a} \times \mathbf{v}$ where α is a scalar. So that

$$(\boldsymbol{a} \times \mathbf{v})(1 - \alpha) = \mathbf{o} \Rightarrow 1 = \alpha$$

showing that $\boldsymbol{a} = \mathbf{b}$ as was required.

3. Given that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, find the values of scalars α and β in the equation, $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \boldsymbol{\alpha}\mathbf{u} + \boldsymbol{\beta}\mathbf{v}$

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= e_{\alpha i\beta}\left(e_{ijk}u_j v_k\right)\left(w_\beta \mathbf{e}_\alpha\right) \\ &= \left(\delta_{\beta j}\delta_{\alpha k} - \delta_{\alpha j}\delta_{\beta k}\right)u_j v_k w_\beta \mathbf{e}_\alpha \\ &= u_\beta v_\alpha w_\beta\, \mathbf{e}_\alpha - u_\alpha v_\beta w_\beta\, \mathbf{e}_\alpha \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}\end{aligned}$$

So that $(\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{v}$

Clearly, $\alpha = -(\mathbf{v} \cdot \mathbf{w})$ and $\beta = (\mathbf{u} \cdot \mathbf{w})$

4. Given that $\mathbf{n}$ is a unit vector, use the fact that $\mathbf{n} \cdot \mathbf{u}$ is the projection of the vector $\mathbf{u}$ in the direction of $\mathbf{n}$ to represent $\mathbf{u}$ as $(\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ or $(\mathbf{n} \otimes \mathbf{n})\mathbf{u} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$.

By simple vector addition, we can represent $\mathbf{u}$ as $(\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n}$.

Since $\mathbf{n}$ is a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. Therefore,

$$\begin{aligned}\mathbf{u} &= (\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{u})\mathbf{n} + (\mathbf{n} \cdot \mathbf{n})\mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n}\end{aligned}$$

$$= (\mathbf{n}\cdot\mathbf{u})\mathbf{n} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$$
$$= (\mathbf{n}\otimes\mathbf{n})\mathbf{u} + \mathbf{n}\times(\mathbf{u}\times\mathbf{n})$$

5. Simplify the following by employing the substitution properties of the Kronecker delta $(a) e_{ijk}\delta_{kn}, (b) e_{ijk}\delta_{is}\delta_{jm}\ (c) e_{ijk}\delta_{is}\delta_{jm}\ \ (d)\ a_{ij}\delta_{in}\ (e)\delta_{ij}\delta_{jn}\ (f)\delta_{ij}\delta_{jn}\delta_{ni}$

   $(a)\ e_{ijn}\ (b) e_{smk}\ (c) e_{smk}\ (d) a_{nj} (e)\delta_{in}\ (f)\delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

6. Show that the sum of triple products, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} = \mathbf{o}$

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$
$$(\mathbf{v}\times\mathbf{w})\times\mathbf{u} = (\mathbf{v}\cdot\mathbf{u})\mathbf{w} - (\mathbf{w}\cdot\mathbf{u})\mathbf{v}$$
$$(\mathbf{w}\times\mathbf{u})\times\mathbf{v} = (\mathbf{w}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

   Adding the three, we find that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} = \mathbf{o}$

7. Given that, $I_{ij} = \iiint_V \left(x^m x^m \delta_{ij} - x^i x^j\right) \rho(x^1, x^2, x^3) dx^1 dx^2 dx^3$ is the moment of inertia along thebaxis $i - j$ where $x = x^1, y = x^2, z = x^3$ and $\rho(x^1, x^2, x^3)$ is scalar density of the material find all the components of the tensor.

$$I_{11} = \iiint_V (y^2 + z^2)\,\rho(x,y,z)dxdydz, \quad I_{21} = I_{12} = \iiint_V xy\,\rho(x,y,z)dxdydz,$$

$$I_{22} = \iiint_V (z^2 + x^2)\,\rho(x,y,z)dxdydz, \quad I_{32} = I_{23} = \iiint_V yz\,\rho(x,y,z)dxdydz,$$
$$I_{31} = I_{13} = \iiint_V xy\,\rho(x,y,z)dxdydz, \quad I_{33} = \iiint_V (x^2 + y^2)\,\rho(x,y,z)dxdydz$$

You will see from the property values in your Fusion 360 models that the inertia tensor is symmetric.

8. Show that the Cylindrical Polar basis vectors,

$$\boldsymbol{e}_r(r,\phi,z) = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}$$
$$\boldsymbol{e}_\phi(r,\phi,z) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}$$
$$\boldsymbol{e}_z(r,\phi,z) = \boldsymbol{k}$$

constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

$$\|\boldsymbol{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$
$$\left\|\boldsymbol{e}_\phi\right\|^2 = \sin^2\phi + \cos^2\phi = 1$$
$$\|\boldsymbol{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs:

$$\boldsymbol{e}_r \cdot \boldsymbol{e}_\phi = -\cos\phi \sin\phi + \cos\phi \sin\phi = 0$$

$$\boldsymbol{e}_\phi \cdot \boldsymbol{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$

$$\boldsymbol{e}_z \cdot \boldsymbol{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$

So that they are pairwise orthogonal.

9. Show that the Spherical Polar basis vectors

$$\boldsymbol{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$$

$$\boldsymbol{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}$$

$$\boldsymbol{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}.$$

Constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

Follow the same procedure as in the above question and obtain the similar result for the spherical polar case.

10. Find the derivatives of all the basis vectors in Q8 and Q9.

11. Given that the position vector in spherical coordinates is given by $\boldsymbol{R} = \rho \boldsymbol{e}_\rho(\theta, \phi)$, where $\boldsymbol{e}_\rho = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$ show that the set $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \theta}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\}$ forms a basis set of orthogonal vectors. This is called the natural basis for the coordinate system. Normalize them to form an orthonormal (physical) basis.

$$\frac{\partial \boldsymbol{R}}{\partial \rho} = \boldsymbol{e}_\rho,$$
$$\frac{\partial \boldsymbol{R}}{\partial \theta} = \rho\frac{\partial \boldsymbol{e}_\rho}{\partial \theta} = \rho\frac{\partial}{\partial \theta}(\sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})$$
$$= \rho(\cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}) = \rho\boldsymbol{e}_\theta.$$
$$\frac{\partial \boldsymbol{R}}{\partial \phi} = \rho\frac{\partial \boldsymbol{e}_\rho}{\partial \phi} = \rho\frac{\partial}{\partial \phi}(\sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})$$
$$= \rho(-\sin\theta\sin\phi\,\boldsymbol{i} + \sin\theta\cos\phi\,\boldsymbol{j})$$
$$= \rho\sin\theta\,\boldsymbol{e}_\phi.$$

From these, we can see that $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \theta}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\} = \left\{\boldsymbol{e}_\rho, \rho\boldsymbol{e}_\theta, \rho\sin\theta\,\boldsymbol{e}_\phi\right\}$. Obviously, the magnitudes are $\{1, \rho, \rho\sin\theta\}$ respectively. Consequently, this basis set can be normalized to $\left\{\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\right\}$

12. Given that the position vector in cylindrical polar coordinates is given by $\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$, where $\boldsymbol{e}_r = \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}$, and $\boldsymbol{e}_z = \boldsymbol{k}$ show that the set $\left\{\frac{\partial \boldsymbol{R}}{\partial r}, \frac{\partial \boldsymbol{R}}{\partial \phi}, \frac{\partial \boldsymbol{R}}{\partial z}\right\}$ forms a basis set of orthogonal vectors. This is called the natural basis for the coordinate system. Normalize them to form an orthonormal basis (the physical basis).

$$\frac{\partial \boldsymbol{R}}{\partial r} = \boldsymbol{e}_r,$$
$$\frac{\partial \boldsymbol{R}}{\partial \phi} = r\frac{\partial \boldsymbol{e}_r}{\partial \phi} = r\frac{\partial}{\partial \phi}(\cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}) = r(-\sin\phi\, \boldsymbol{i} + \cos\phi\, \boldsymbol{j}) = r\boldsymbol{e}_\phi$$
$$\frac{\partial \boldsymbol{R}}{\partial z} = \boldsymbol{e}_z$$

From these, we can see that $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \phi}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\} = \{\boldsymbol{e}_r, r\boldsymbol{e}_\phi, \boldsymbol{e}_z\}$. Obviously, the magnitudes are $\{1, r, 1\}$ respectively. Consequently, this basis set can be normalized to $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$.

13. For Cartesian Coordinates, show that the natural basis coincides with the physical basis. [**Hint:** Obtain the natural basis from the set, $\left\{\frac{\partial \boldsymbol{R}}{\partial x}, \frac{\partial \mathbf{R}}{\partial y}, \frac{\partial \boldsymbol{R}}{\partial z}\right\}$. The physical basis is the normalized natural basis.]

$$\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$\frac{\partial \boldsymbol{R}}{\partial x} = \mathbf{i}, \frac{\partial \boldsymbol{R}}{\partial y} = \mathbf{j}$, and $\frac{\partial \boldsymbol{R}}{\partial z} = \mathbf{k}$. This shows that the natural basis (computed from the derivatives of the position vector) are also the physical bases $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ we have used all along.

14. Show that the contraction of a symmetric object with an antisymmetric object equals zero. For example given that $a_{mn}, m, n = 1,2,3$ is antisymmetric, Show that $a_{mn}x^m x^n = 0$.

(a) It is easily seen that $x^m x^n = x^n x^m$ hence symmetric. If a_{mn} is anti-symmetric, then the contraction $a_{mn}x^m x^n$ must necessarily vanish.

(b) Given that $a_{mn}x^m x^n = 0$ for arbitrary values of $x^n, n = 1,2,3$ then we can write,

$$a_{mn}x^m x^n = -a_{mn}x^m x^n$$

because zero is also a negative of itself. Swapping the roles of x^m and x^n on the RHS of the above, we can write,

$$\begin{aligned} a_{mn}x^m x^n &= -a_{mn}x^m x^n \\ &= -a_{mn}x^n x^m \\ &= -a_{nm}x^n x^m \end{aligned}$$

after swapping the roles of the two dummy indices. We therefore consolidate on the LHS by writing,

$$a_{mn}x^m x^n + a_{nm}x^n x^m = 0$$

$$(a_{mn} + a_{nm})x^m x^n = 0$$

Notice that the quantity in the parenthesis is always symmetric. And also note the contraction of two symmetric tensors can only vanish if one or both tensors vanish. Here, $x^m x^n$ is a product of arbitrary tensors. We are left with the fact that

$$a_{mn} + a_{nm} = 0$$

or,

$$a_{mn} = -a_{nm}$$

which is the definition of anti-symmetry.

15. Noting that $e_{ijk}\sigma_{jk} = 0$ observe that e_{ijk} is perfectly antisymmetric. What does this tell about σ_{jk}?

It tells that σ_{ij} is symmetric. The contraction of a symmetric tensor with an anti-symmetric is zero. Since we know that e_{ijk} is anti-symmetric, the given result of the contraction shows that σ_{ij} is symmetric.

16. Given that A_{mn} and B_{mn} are symmetric, Let $A_{mn}\ x^m x^n = B_{mn}\ x^m x^n$ for arbitrary values of $x^i, i = 1,2,3$, show that $A_{mn} = B_{mn}$ for all values of $\boldsymbol{m}, \boldsymbol{n}$

We can place the RHS of the equation with the LHS to obtain,

$$(A_{mn} - B_{mn})x^m x^n = 0.$$

By the arguments in Number Next, it is clear than in this case,

$$(A_{mn} - B_{mn}) = 0$$

This proves the desired result that

$$A_{mn} = B_{mn}.$$

17. Given that $a_{ij} = B_i B_j$, where B_1, B_2 and $B_3$are constants Calculate the determinant $|a_{ij}|$

The determinant

$$|A| = e_{ij} a_{1i} a_{2j} = e_{ij} B_1 B_i B_2 B_j = B_1 B_2 (e_{ij} B_i B_j) = 0$$

The last equality results again from the fact that the contraction of a symmetric object with an anti-symmetric object results in zero.

18. If A_{ij} is symmetric and B_{ij} is antisymmetric, find the value of $C = A_{ij} B_{ij}$

We are given that,

$$\begin{aligned} C &= A_{ij} B_{ij} \\ &= A_{ji} B_{ij} \quad \text{since } A_{ij} \text{is symmetric} = -A_{ji}\, B_{ji} \quad \text{since } B_{ij} \text{is anti symmetric} \\ &= -A_{ij} B_{ij} \quad \text{Interchanging the roles of } i \text{ and } j \\ &= 0 \end{aligned}$$

Because zero is the only scalar that can be negative of itself. An interchange of dummy indices is a valid step. Note we could not do that for a free index! This result is an important one. The contraction of a symmetric quantity with an anti-symmetric one results in zero.

19. Show that the second-order system T_{ij} can be expressed as the sum of a symmetric system and an anti-symmetric system. Find an expression for these.

We desire to find symmetric and anti-symmetric tensors A_{ij}, B_{ij} respectively such that

$$T_{ij} = A_{ij} + B_{ij}.$$

Let us in fact assume that this is so and see if we can find A_{ij}, B_{ij} with these properties satisfying the equation above. We begin by transposing the equation:

$$T_{ji} = A_{ji} + B_{ji}$$

We now add the two equations to obtain,

$$T_{ij} + T_{ji} = A_{ij} + A_{ji} + B_{ij} + B_{ji} = 2A_{ij}$$

as the last two terms cancel themselves out on account of anti-symmetry while the first two add on account of symmetry. We can therefore see that we have a unique value for A_{ij} that is,

$$A_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$$

In the same way, a subtraction instead of addition would have led to,

$$T_{ij} - T_{ji} = A_{ij} - A_{ji} + B_{ij} - B_{ji} = 2B_{ij}$$

$$B_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$$

This is a general rule that any second order indexed quantity can be made the sum of two parts: One symmetric the other, anti-symmetric.

20. Show that the decomposition of a tensor into the symmetric and anti-symmetric parts is unique.

$$\mathbf{S} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{S} - \mathbf{S}^{\mathrm{T}}) = \operatorname{sym}\mathbf{S} + \operatorname{skw}\mathbf{S}$$

Suppose there is another decomposition into symmetric and antisymmetric parts similar to the above so that $\exists\mathbf{B}$ such that $\mathbf{S} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}})$. Now take the inner product of the two expressions for the tensor $\mathbf{S}$ and a symmetric tensor $\mathbf{D}$

$$\begin{aligned}\mathbf{S}:\mathbf{D} &= (\operatorname{sym}\mathbf{S} + \operatorname{skw}\mathbf{S}):\mathbf{D} \\ &= (\operatorname{sym}\mathbf{S}):\mathbf{D} \\ &= \left(\frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}) + \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}})\right):\mathbf{D} \\ &= \left(\frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}})\right):\mathbf{D}\end{aligned}$$

since the inner products of symmetric and anti-symmetric tensors vanish. The above equation leads to,

$$(\text{sym}\,\mathbf{S}):\mathbf{D}-\left(\frac{1}{2}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})\right):\mathbf{D}=\left(\text{sym}\,\mathbf{S}-\frac{1}{2}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})\right):\mathbf{D}=\mathbf{0}$$

Since $\mathbf{D}$ is arbitrary, it is clear that,

$$\text{sym}\,\mathbf{S}-\frac{1}{2}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})=\mathbf{0}$$

or, $\text{sym}\,\mathbf{S}\equiv\frac{1}{2}(\mathbf{S}+\mathbf{S}^{\mathrm{T}})=\frac{1}{2}(\mathbf{B}+\mathbf{B}^{\mathrm{T}})$ showing uniqueness of the symmetrical part. The uniqueness of the anti-symmetrical part is arrived at by taking the inner product with an antisymmetric tensor at the beginning.

21. The angle $0\le\theta\le\pi$ between two skew lines in space is defined as the angle between their direction vectors when these vectors are placed at the origin. Show that for two lines with direction numbers a_i and $b_i, i = 1;\ 2;\ 3$ the cosine of the angle between these lines satisfies

$$\cos\theta=\frac{a_ib_i}{\sqrt{(a_ia_i)}\sqrt{(b_ib_i)}}$$

First note that the dot product of the two line vectors is,

$$\mathbf{a}\cdot\mathbf{b}=a_ib_i=|a||b|\cos\theta=\sqrt{(a_ia_i)}\sqrt{(b_ib_i)}\cos\theta$$

From where it is obvious that,

$$\cos\theta = \frac{a_i b_i}{\sqrt{(a_i a_i)}\sqrt{(b_i b_i)}}$$

as required.

22. Let $\lambda = A_{ij}x_i x_j$ where $A_{ij} = A_{ji}$. **Calculate** $(a)\frac{\partial\lambda}{\partial x_m}, (b)\frac{\partial^2\lambda}{\partial x_m \partial x_k}$

$$\lambda = A_{ij}x_i x_j = A_{mk}x_m x_k$$

$$\frac{\partial\lambda}{\partial x_m} = A_{mk}x_k + A_{lk}x_l \frac{\partial x_k}{\partial x_m}$$

$$= A_{mk}x_k + A_{lk}x_l\delta_{km} = A_{mk}x_k + A_{lk}x_l\delta_{km}$$

$= A_{mk}x_k + A_{lm}x_l = 2A_{mk}x_k$, and furthermore,

$$\frac{\partial^2\lambda}{\partial x_m \partial x_k} = 2A_{mk}.$$

Remember that in the above we made use of the liberty to alter the dummy indices to conform to the requirements of the derivative. The substitutionary attribute of the Kronecker delta has been used to advantage here.

23. If $A_{ij} = A_i B_j \neq 0 \; \forall \; i, j$ values and $A_{ij} = A_{ji}$ for $i, j = 1, 2, \dots, N$ Show that $A_{ij} = \lambda B_i B_j$ where λ is constant. Find λ.

We are given that

$$A_{ij} = A_i B_j = A_j B_i$$

Since the composition is symmetrical. Let us contract this quantity with B_j. We obtain,

$$A_i B_j B_j = A_j B_i B_j \text{ so that,}$$

$$A_i = B_i \left(\frac{A_k B_k}{B_j B_j} \right) = B_i \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \right) = \lambda B_i$$

It is correct to divide the equation by $B_j B_j$ since it is a scalar sum and hence a scalar. Division of indexed objects is not defined and hence not permissible. Yet we can write the last equation out in full component form as,

$$A_1 = \lambda B_1, A_2 = \lambda B_2, \dots, A_N = \lambda B_N$$

It is obvious from the above that,

$$\lambda = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \dots = \frac{A_N}{B_N}$$

24. Let $x_i = a_{ij}\bar{x}_j \;\; i,j = 1,2,3.$ denote a change of variables from a barred system of coordinates to an unbarred system and assume that $A_i = a_{ij}A_j$ where a_{ij} are constants. $\bar{A}_{\boldsymbol{i}}$ is a function of the $\bar{x}_j$ variables and A_j is a function of the x_i variables. Calculate $\frac{\partial \bar{A}_i}{\partial \bar{x}_m}$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}_m} = \frac{\partial}{\partial \bar{x}_m}\left(a_{ij}A_j\right) \text{ recall that } a_{ij} \text{is a constant, so that,}$$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}_m} = a_{ij}\frac{\partial A_j}{\partial \bar{x}_m} = a_{ij}\frac{\partial A_j}{\partial x_k}\frac{\partial x_k}{\partial \bar{x}_m} = a_{ij}a_{km}\frac{\partial A_j}{\partial x_k}$$

The last equality coming from the fact that, $x_k = a_{km}\bar{x}_m$ so that $\frac{\partial x_k}{\partial \bar{x}_m} = a_{km}$. Note that the dummy indices may change in any convenient way. The free indices of this expression are i and m.

(2)

25. Show that $e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$

Expanding row wise, we have,

$$\begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$

$$= \delta_{1i}(\delta_{2j}\delta_{3k} - \delta_{2k}\delta_{3j}) + \delta_{1j}(\delta_{2k}\delta_{3i} - \delta_{2i}\delta_{3k}) + \delta_{1k}(\delta_{2i}\delta_{3j} - \delta_{2j}\delta_{3i})$$

Case 1:, $i = j$: (parentheses indicating a temporary suspension of summation convention)

$$\begin{aligned} e_{(i)(i)k} &= 0 \\ &= \delta_{1(i)}(\delta_{2(i)}\delta_{3k} - \delta_{2k}\delta_{3(i)}) + \delta_{1(i)}(\delta_{2k}\delta_{3(i)} - \delta_{2(i)}\delta_{3k}) \\ &\quad + \delta_{1k}(\delta_{2(i)}\delta_{3(i)} - \delta_{2(i)}\delta_{3(i)}) \\ &= \delta_{1(i)}\delta_{2(i)}\delta_{3k} - \delta_{1(i)}\delta_{2k}\delta_{3(i)} + \delta_{1(i)}\delta_{2k}\delta_{3(i)} - \delta_{1(i)}\delta_{2(i)}\delta_{3k} + \delta_{1k}\delta_{2(i)}\delta_{3(i)} \\ &\quad - \delta_{1k}\delta_{2(i)}\delta_{3(i)} \\ &= 0 \end{aligned}$$

Case 2, $j = k$:

$$\begin{aligned} e_{i(j)(j)} &= 0 \\ &= \delta_{1(i)}(\delta_{2(j)}\delta_{3k} - \delta_{2j}\delta_{3(j)}) + \delta_{1(j)}(\delta_{2j}\delta_{3(i)} - \delta_{2(i)}\delta_{3j}) \\ &\quad + \delta_{1j}(\delta_{2(i)}\delta_{3(j)} - \delta_{2(j)}\delta_{3(i)}) \\ &= 0 \end{aligned}$$

Case 3, $k = i$

$$
\begin{aligned}
e_{(i)j(k)} &= 0 \\
&= \delta_{1(i)}\left(\delta_{2(j)}\delta_{3i} - \delta_{2i}\delta_{3(j)}\right) + \delta_{1(j)}\left(\delta_{2i}\delta_{3(i)} - \delta_{2(i)}\delta_{3i}\right) \\
&\quad + \delta_{1i}\left(\delta_{2(i)}\delta_{3(j)} - \delta_{2(j)}\delta_{3(i)}\right) \\
&= \delta_{1(i)}\delta_{2(j)}\delta_{3i} - \delta_{1(i)}\delta_{2i}\delta_{3(j)} + 0 + \delta_{1i}\delta_{2(i)}\delta_{3(j)} - \delta_{1i}\delta_{2(j)}\delta_{3(i)} \\
&= 0
\end{aligned}
$$

Case 4, $i = 1, j = 2, k = 3$

$$
\begin{aligned}
e_{123} &= 1 \\
&= \delta_{11}(\delta_{22}\delta_{33} - \delta_{23}\delta_{32}) + \delta_{12}(\delta_{23}\delta_{31} - \delta_{21}\delta_{33}) + \delta_{13}(\delta_{21}\delta_{32} - \delta_{22}\delta_{31}) \\
&= (1 - 0) + 0(0 - 0) + 0(0 - 0) = 1
\end{aligned}
$$

In all cases, there is equality, Hence,

$$
e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}
$$

26. Show that $e_{ijk}e_{rst} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$

Since we know that $e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$, and that $e^{rst} = \begin{vmatrix} \delta_{1r} & \delta_{1s} & \delta_{1t} \\ \delta_{2r} & \delta_{2s} & \delta_{2t} \\ \delta_{3r} & \delta_{3s} & \delta_{3t} \end{vmatrix}$

Transposing the latter and taking products,

That

$$e_{ijk}e_{rst} = \begin{vmatrix} \delta_{1i} & \delta_{2i} & \delta_{3i} \\ \delta_{1j} & \delta_{2j} & \delta_{3j} \\ \delta_{1k} & \delta_{2k} & \delta_{3k} \end{vmatrix} \begin{vmatrix} \delta_{1r} & \delta_{1s} & \delta_{1t} \\ \delta_{2r} & \delta_{2s} & \delta_{2t} \\ \delta_{3r} & \delta_{3s} & \delta_{3t} \end{vmatrix} = \begin{vmatrix} \delta_{il}\delta_{lr} & \delta_{il}\delta_{ls} & \delta_{il}\delta_{lt} \\ \delta_{jl}\delta_{lr} & \delta_{jl}\delta_{ls} & \delta_{jl}\delta_{lt} \\ \delta_{kl}\delta_{lr} & \delta_{kl}\delta_{ls} & \delta_{kl}\delta_{lt} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Since a transposition does not alter the value of a determinant.

27. Use a symbolics processor to calculate the conjugate metric tensor given that, $\mathbf{g}_1 \cdot \mathbf{g}_2 = \mathbf{g}_2 \cdot \mathbf{g}_1 = \cos\vartheta_{12}$, $\mathbf{g}_2 \cdot \mathbf{g}_3 = \mathbf{g}_3 \cdot \mathbf{g}_3 = \cos\vartheta_{23}$ and $\mathbf{g}_3 \cdot \mathbf{g}_1 = \mathbf{g}_1 \cdot \mathbf{g}_3 = \cos\vartheta_{13}$

The matrix

$$[g_{ij}] = \begin{bmatrix} 1 & \cos\vartheta_{12} & \cos\vartheta_{13} \\ \cos\vartheta_{12} & 1 & \cos\vartheta_{23} \\ \cos\vartheta_{13} & \cos\vartheta_{23} & 1 \end{bmatrix}$$

Now the conjugate metric tensor has a matrix that is the inverse of the above. First consider the fact that the determinant of $[g_{ij}]$ obtained by direct evaluation, is:

$$\Delta \equiv |g_{ij}| = 1 - \cos^2\vartheta_{12} - \cos^2\vartheta_{23} - \cos^2\vartheta_{13} + 2\cos\vartheta_{12}\cos\vartheta_{13}\cos\vartheta_{23}$$

The Inverse matrix (using a symbolic algebra processor) is therefore,

$$[g^{ij}] = \frac{1}{\Delta}\begin{bmatrix} 1 - \mathrm{c}^2\vartheta_{23} & -\mathrm{c}\vartheta_{12} + \mathrm{c}\vartheta_{13}\,\mathrm{c}\vartheta_{23} & -\mathrm{c}\vartheta_{13} + \mathrm{c}\vartheta_{12}\,\mathrm{c}\vartheta_{23} \\ -\mathrm{c}\vartheta_{12} + \mathrm{c}\vartheta_{13}\,\mathrm{c}\vartheta_{23} & 1 - \mathrm{c}^2\vartheta_{13} & -\mathrm{c}\vartheta_{23} + \mathrm{c}\vartheta_{12}\,\mathrm{c}\vartheta_{13} \\ -\mathrm{c}\vartheta_{13} + \mathrm{c}\vartheta_{12}\,\mathrm{c}\vartheta_{23} & -\mathrm{c}\vartheta_{23} + \mathrm{c}\vartheta_{12}\,\mathrm{c}\vartheta_{13} & 1 - \mathrm{c}^2\vartheta_{12} \end{bmatrix}$$

Where for simplicity we have written, $\mathrm{c}\vartheta \equiv \cos\vartheta$

Notice that the metric tensor is always symmetric. Why must this be so? This symmetry is obvious from its definition: $g_{ij} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j}$ since the dot product is commutative.