SSG 516 Continuum Mechanics

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Purpose of the Course

* Available to beginning graduate students in Engineering
* Provides a background to several other courses such as Elasticity, Plasticity, Fluid Mechanics, Heat Transfer, Fracture Mechanics, Rheology, Dynamics, Acoustics, etc. These courses are taught in several of our departments. This present course may be viewed as an advanced introduction to the modern approach to these courses
* It is taught so that related graduate courses can build on this modern approach.

What you will need

* The slides here are quite extensive. They are meant to assist the serious learner. They are NO SUBSTITUTES for the course text which must be read and followed concurrently.
* Preparation by reading ahead is **ABSOLUTELY** necessary to follow this course
* Assignments are given at the end of each class and they are due (No excuses) **exactly** five days later.
* Late submission carry zero grade.

Software

* The software for the Course is Mathematica 9 by Wolfram Research. Each student is entitled to a licensed copy. Find out from the LG Laboratory
* It your duty to learn to use it. Students will find some examples too laborious to execute by manual computation. It is a good idea to start learning Mathematica ahead of your need of it.
* For later courses, commercial FEA Simulations package such as ANSYS, COMSOL or NASTRAN will be needed. Student editions of some of these are available. We have COMSOL in the LG Laboratory

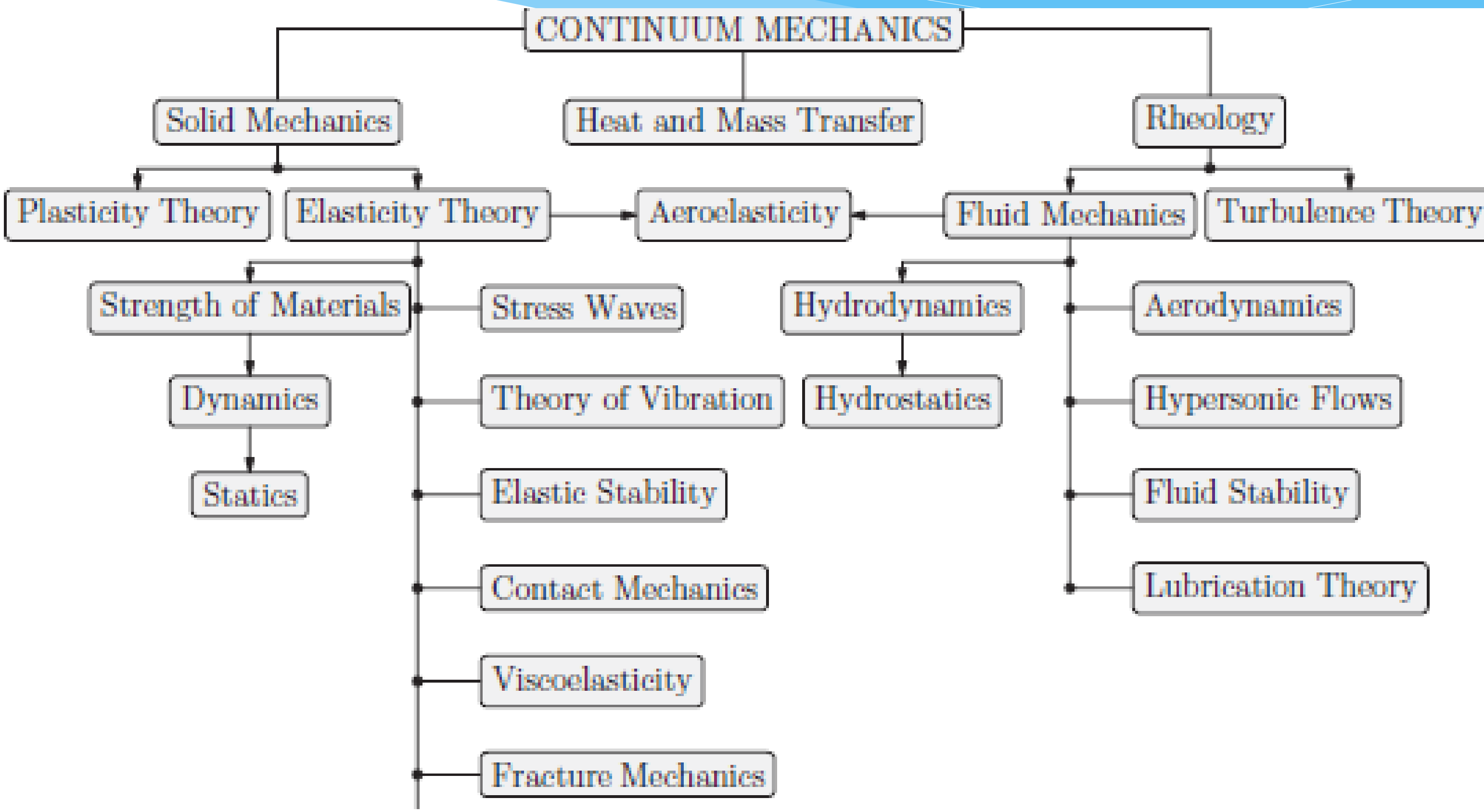
Linear Spaces

Introduction

Unified Theory

* Continuum Mechanics can be thought of as the grand unifying theory of engineering science.
* Many of the courses taught in an engineering curriculum are closely related and can be obtained as special cases of the general framework of continuum mechanics.
* This fact is easily lost on most undergraduate and even some graduate students.

Continuum Mechanics



Physical Quantities

Continuum Mechanics views matter as continuously distributed in space. The physical quantities we are interested in can be

- **Scalars** or **reals**, such as time, energy, power,
- **Vectors**, for example, position vectors, velocities, or forces,
- **Tensors**: deformation gradient, strain and stress measures.

Physical Quantities

- Since we can also interpret scalars as zeroth-order tensors, and vectors as 1st-order tensors, all continuum mechanical quantities can generally be considered as tensors of different orders.
- It is therefore clear that a thorough understanding of Tensors is essential to continuum mechanics. This is NOT always an easy requirement;
- The notational representation of tensors is often inconsistent as different authors take the liberty to express themselves in several ways.

Physical Quantities

- There are two major divisions common in the literature: Invariant or direct notation and the component notation.
- Each has its own advantages and shortcomings. It is possible for a reader that is versatile in one to be handicapped in reading literature written from the other viewpoint. In fact, it has been alleged that
- "*Continuum Mechanics may appear as a fortress surrounded by the walls of tensor notation*" It is our hope that the course helps every serious learner overcome these difficulties

Real Numbers & Tuples

* The set of real numbers is denoted by $\mathscr{R}$
* Let $\mathscr{R}^n$ be the set of n-tuples so that when $n = 2$, $\mathscr{R}^2$ we have the set of pairs of real numbers. For example, such can be used for the x and y coordinates of points on a Cartesian coordinate system.

Vector Space

A real vector space $\mathscr{V}$ is a set of elements (called vectors) such that,

1. **Addition operation** is **defined** and it is **commutative** and **associative** under $\mathscr{V}$: that is, $\boldsymbol{u}+\mathbf{v}\in\mathscr{V}$, $\boldsymbol{u}+\mathbf{v}=\mathbf{v}+\boldsymbol{u}, \boldsymbol{u}+(\mathbf{v}+\boldsymbol{w})=(\boldsymbol{u}+\mathbf{v})+\boldsymbol{w}, \forall\, \boldsymbol{u},\mathbf{v},\boldsymbol{w}\in\mathscr{V}$. Furthermore, $\mathscr{V}$ is **closed** under addition: That is, given that $\boldsymbol{u},\mathbf{v}\in\mathscr{V}$, then $\boldsymbol{w}=\boldsymbol{u}+\mathbf{v}=\mathbf{v}+\boldsymbol{u},\Rightarrow \boldsymbol{w}\in\mathscr{V}$.
2. $\mathscr{V}$ **contains** a **zero element** $\boldsymbol{o}$ such that $\boldsymbol{u}+\boldsymbol{o}=\boldsymbol{u}\ \forall\ \boldsymbol{u}\in\mathscr{V}$. For every $\boldsymbol{u}\in\mathscr{V}$, $\exists -\boldsymbol{u}: \boldsymbol{u}+(-\boldsymbol{u})=\boldsymbol{o}$.
3. **Multiplication by a scalar**. For $\alpha,\beta\in\mathscr{R}$ and $\boldsymbol{u},\boldsymbol{v}\in\mathscr{V}$, $\alpha\boldsymbol{u}\in\mathscr{V}$, $1\boldsymbol{u}=\boldsymbol{u}, \alpha(\beta\boldsymbol{u})=(\alpha\beta)\boldsymbol{u}, (\alpha+\beta)\boldsymbol{u}=\alpha\boldsymbol{u}+\beta\boldsymbol{u},\ \alpha(\boldsymbol{u}+\mathbf{v})=\alpha\boldsymbol{u}+\alpha\mathbf{v}$.

Euclidean Vector Space

- An **Inner-Product** (also called a **Euclidean Vector**) **Space** $\mathscr{E}$ is a real vector space that defines the scalar product: for each pair $\boldsymbol{u}, \mathbf{v} \in \mathscr{E}$, $\exists\, l \in \mathscr{R}$ such that, $l = \boldsymbol{u} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{u}$. Further, $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$, the zero value occurring only when $\boldsymbol{u} = 0$. It is called "Euclidean" because the laws of Euclidean geometry hold in such a space.
- The inner product also called a dot product, is the mapping

$$" \cdot ": \mathscr{V} \times \mathscr{V} \to \mathscr{R}$$

from the product space to the real space.

Magnitude & Direction

* We were previously told that a vector is something that has magnitude and direction. We often represent such objects as directed lines. Do such objects conform to our present definition?
* To answer, we only need to see if the three conditions we previously stipulated are met:

Dimensionality

From our definition of the Euclidean space, it is easy to see that,

$$\mathbf{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \cdots + \alpha_n \boldsymbol{u}_n$$

such that, $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathscr{R}$ and $\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n \in \mathscr{E}$, is also a vector. The subset

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \cdots, \boldsymbol{u}_n\} \subset \mathscr{E}$

is said to be ***linearly independent*** or ***free*** if for any choice of the subset $\{\alpha_1, \alpha_2, \cdots, \alpha_n\} \subset \mathscr{R}$ other than $\{0,0, \cdots, 0\}$, $\mathbf{v} \neq 0$. If it is possible to find linearly independent vector systems of order n where n is a finite integer, but there is no free system of order $n+1$, then the dimension of $\mathscr{E}$ is n. In other words, the dimension of a space is the highest number of linearly independent members it can have.

Basis

* Any linearly independent subset of $\mathscr{E}$ is said to form ***a basis*** for $\mathscr{E}$ in the sense that any other vector in $\mathscr{E}$ can be expressed as a linear combination of members of that subset. In particular our familiar Cartesian vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ is a famous such subset in three dimensional Euclidean space.
* From the above definition, it is clear that a basis is not necessarily unique.

Directed Line

1. **Addition operation** for a directed line segment is defined by the parallelogram law for addition.
2. $\mathscr{V}$ **contains** a **zero element *o*** in such a case is simply a point with zero length..
3. **Multiplication by a scalar** α. Has the meaning that

$0 < \alpha \leq 1 \rightarrow$ Line is shrunk along the same direction by α

$\alpha > 1 \rightarrow$ Elongation by α

Negative value is same as the above with a change of direction.

Other Vectors

* Now we have confirmed that our original notion of a vector is accommodated. It is not all that possess magnitude and direction that can be members of a vector space.
* A book has a size and a direction but because we cannot define addition, multiplication by a scalar as we have done for the directed line segment, it is not a vector.

Other Vectors

Complex Numbers. The set $\mathscr{C}$ of complex numbers is a real vector space or equivalently, a vector space over $\mathscr{R}$.

2-D Coordinate Space. Another real vector space is the set of all pairs of $x_i \in \mathscr{R}$ forms a 2-dimensional vector space over $\mathscr{R}$ is the two dimensional coordinate space you have been graphing on! It satisfies each of the requirements:

Set of Pairs

$\boldsymbol{x} = \{x_1, x_2\}, x_1, x_2 \in \mathscr{R}, \boldsymbol{y} = \{y_1, y_2\}, y_1, y_2 \in \mathscr{R}.$

* **Addition** is easily defined as $\boldsymbol{x} + \boldsymbol{y} = \{x_1 + y_1, x_2 + y_2\}$ clearly $\boldsymbol{x} + \boldsymbol{y} \in \mathscr{R}^2$ since $x_1 + y_1, x_2 + y_2 \in \mathscr{R}.$ Addition operation creates other members for the vector space – Hence closure exists for the operation.
* **Multiplication by a scalar:** $\alpha \boldsymbol{x} = \{\alpha x_1, \alpha x_2\}, \forall \alpha \in \mathscr{R}.$
* **Zero element:** $\boldsymbol{o} = \{0,0\}$. Additive Inverse: $-\boldsymbol{x} = \{-x_1, -x_2\}, x_1, x_2 \in \mathscr{R}$
  Type equation here.A standard basis for this : $\boldsymbol{e}_1 = \{1,0\}, \boldsymbol{e}_2 = \{0,1\}$
  Type equation here.Any other member can be expressed in terms of this basis.

N-tuples

$\boldsymbol{n}$ **−D Coordinate Space.** For any positive number n, we may create n −tuples such that, $\boldsymbol{x} = \{x_1, x_2, \dots, x_n\}$ where $x_1, x_2, \dots, x_n \in \mathscr{R}$ are members of $\mathscr{R}^n$ – a real vector space over the $\mathscr{R}$. $\boldsymbol{y} = \{y_1, y_2, \dots, y_n\}$, $y_1, y_2, \dots, y_n \in \mathscr{R}$. $\boldsymbol{x} + \boldsymbol{y} = \{x_1 + y_1, x_2 + y_2, \dots, x_n + y_n\}$,. $\boldsymbol{x} + \boldsymbol{y} \in \mathscr{R}^n$ since $x_1 + y_1, x_2 + y_2, \dots, x_n + y_n \in \mathscr{R}$

N-tuples

- **Addition operation** creates other members for the vector space – Hence closure exists for the operation.
- **Multiplication by a scalar:** $\alpha \boldsymbol{x} = \{\alpha x_1, \alpha x_2, \dots, \alpha x_n\}$, $\forall \alpha \in R$.
- **Zero element** $\boldsymbol{o} = \{0,0,\dots,0\}$. **Additive Inverse** $-\boldsymbol{x} = \{-x_1, -x_2, \dots, -x_n\}, x_1, x_2, \dots, x_n \in \mathscr{R}$
  There is also a standard basis which is easily proved to be linearly independent: $\boldsymbol{e}_1 = \{1,0,\dots,0\}, \boldsymbol{e}_2 = \{0,1,\dots,0\}, \dots, \boldsymbol{e}_n = \{0,0,\dots,0\}$

Matrices

Let $\mathscr{R}^{m\times n}$ denote the set of matrices with entries that are real numbers (same thing as saying members of the real space $\mathscr{R}$, Then, $\mathscr{R}^{m\times n}$ is a real vector space. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry by the same scalar). The zero vector here is just the zero matrix. The dimension of this space is mn. For example, in $\mathscr{R}^{3\times 3}$ we can choose basis in the form,

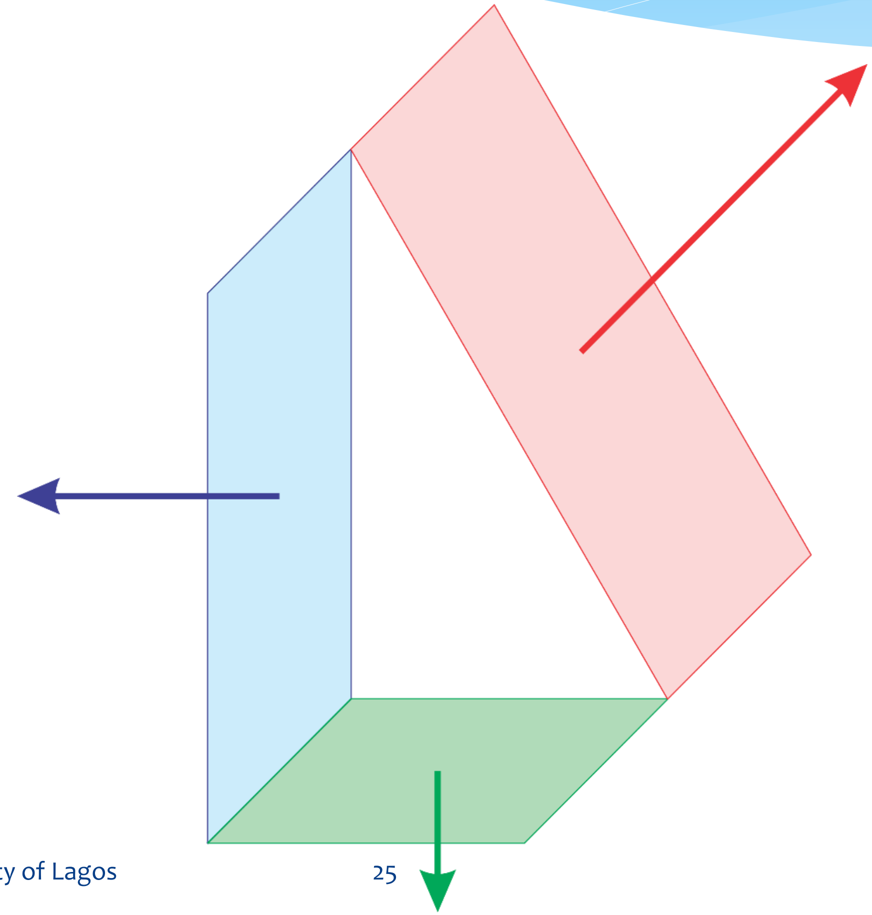
* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Polynomials

Forming polynomials with a single variable x to order n when n is a real number creates a vector space. It is left as an exercise to demonstrate that this satisfies all the three rules of what a vector space is.

Area Vector

* Is a 2D area embedded in a Euclidean Point Space a vector? Does it satisfy the conditions? Discuss.



Euclidean Vector Space

- An **Inner-Product** (also called a **Euclidean Vector**) **Space** $\mathscr{E}$ is a real vector space that defines the scalar product: for each pair $\boldsymbol{u}, \mathbf{v} \in \mathscr{E}$, $\exists\, l \in \mathscr{R}$ such that, $l = \boldsymbol{u} \cdot \mathbf{v} = \mathbf{v} \cdot \boldsymbol{u}$. Further, $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$, the zero value occurring only when $\boldsymbol{u} = 0$. It is called "Euclidean" because the laws of Euclidean geometry hold in such a space.
- The inner product also called a dot product, is the mapping

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from the product space to the real space.

Co-vectors

- A mapping from a vector space is also called a functional; a term that is more appropriate when we are looking at a function space.
- A linear functional $\mathbf{v}*: \mathscr{V} \to \mathscr{R}$ on the vector space $\mathscr{V}$ is called a covector or a dual vector. For a finite dimensional vector space, the set of all covectors forms the dual space $\mathscr{V}*$ of $\mathscr{V}$. If $\mathscr{V}$ is an Inner Product Space, then there is no distinction between the vector space and its dual.

Magnitude & Direction Again

Magnitude The norm, length or magnitude of $\boldsymbol{u}$, denoted $||\boldsymbol{u}||$ is defined as the positive square root of $\boldsymbol{u} \cdot \boldsymbol{u} = ||\boldsymbol{u}||^2$. When $||\boldsymbol{u}|| = 1$, $\boldsymbol{u}$ is said to be a unit vector. When $\boldsymbol{u} \cdot \mathbf{v} = 0$, $\boldsymbol{u}$ and $\mathbf{v}$ are said to be orthogonal.

Direction Furthermore, for any two vectors $\boldsymbol{u}$ and $\mathbf{v}$, the angle between them is defined as,

$$\cos^{-1}\left(\frac{\boldsymbol{u} \cdot \mathbf{v}}{||\boldsymbol{u}||\,||\mathbf{v}||}\right)$$

The scalar **distance** d between two vectors $\boldsymbol{u}$ and $\mathbf{v}$

$$d = ||\boldsymbol{u} - \mathbf{v}||$$

3-D Euclidean Space

* A 3-D Euclidean space is a Normed space because the inner product induces a norm on every member.
* It is also a metric space because we can find distances and angles and therefore measure areas and volumes
* Furthermore, in this space, we can define the cross product, a mapping from the product space

$$" \times ": \mathscr{V} \times \mathscr{V} \rightarrow \mathscr{V}$$

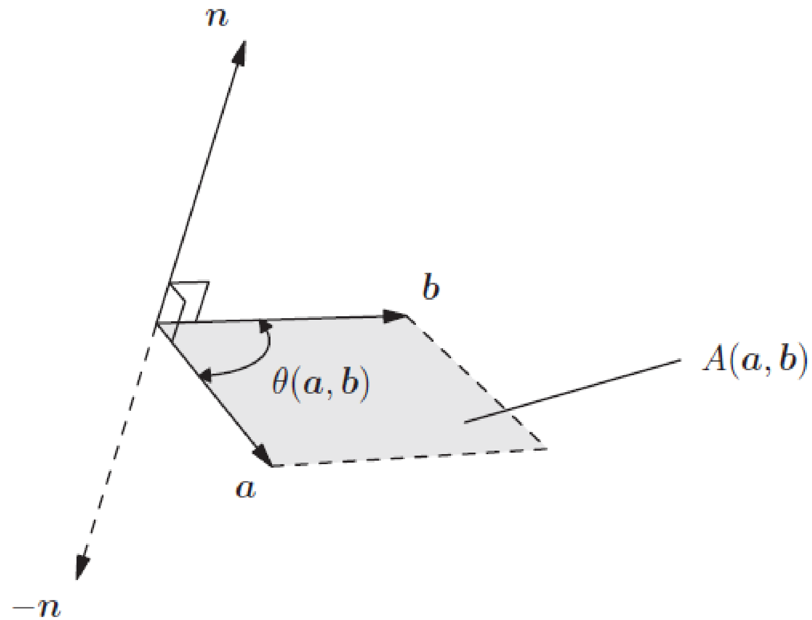
Which takes two vectors and produces a vector.

Cross Product

Without any further ado, our definition of cross product is exactly the same as what you already know from elementary texts. We simply repeat a few of these for emphasis:

1. The magnitude
   $$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\sin\theta \qquad (0 \le \theta \le \pi)$$
   of the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is the area $A(\boldsymbol{a}, \boldsymbol{b})$ spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. This is the area of the parallelogram defined by these vectors. This area is non-zero only when the two vectors are linearly independent.
2. θ is the angle between the two vectors.
3. The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$

Cross Product



The area $A(\boldsymbol{a}, \boldsymbol{b})$ spanned by the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The unit vector $\boldsymbol{n}$ in the direction of the cross product can be obtained from the quotient, $\frac{\boldsymbol{a}\times\boldsymbol{b}}{\|\boldsymbol{a}\times\boldsymbol{b}\|}$.

Cross Product

The cross product is bilinear and anti-commutative:

Given $\alpha \in \mathscr{R}, \forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathscr{V},$

$$(\alpha\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \alpha(\boldsymbol{a} \times \boldsymbol{c}) + \boldsymbol{b} \times \boldsymbol{c}$$

$$\boldsymbol{a} \times (\alpha\boldsymbol{b} + \boldsymbol{c}) = \alpha(\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{a} \times \boldsymbol{c}$$

So that there is linearity in both arguments.

Furthermore, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

Tripple Products

The trilinear mapping,

$$[\ ,\ ,]\ :\ \mathscr{V} \times \mathscr{V} \times \mathscr{V} \to \mathscr{R}$$

From the product set $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to real space is defined by:

$$[\,\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \equiv \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$$

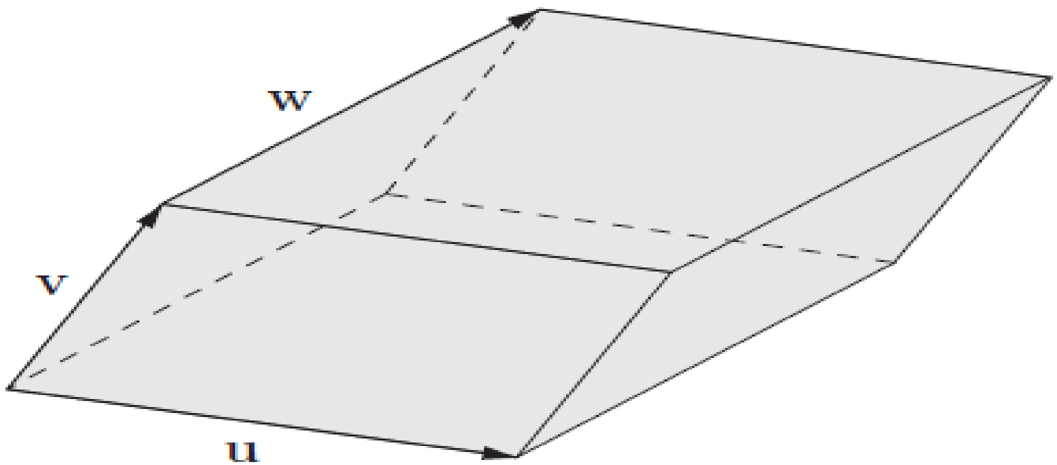
Has the following properties:

1. $[\,\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]=[\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}]=[\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}]=-[\boldsymbol{b},\boldsymbol{a},\boldsymbol{c}]=-[\boldsymbol{c},\boldsymbol{b},\boldsymbol{a}]=-[\boldsymbol{a},\boldsymbol{c},\boldsymbol{b}]$

HW: Prove this

1. Vanishes when ***a, b*** and ***c*** are linearly dependent.
2. It is the volume of the parallelepiped defined by ***a, b*** and ***c***

Tripple product



Parallelepiped defined by **u, v** and **w**

Summation Convention

* We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we deal with higher ranked objects, such notational conveniences become even more important. We shall often deal with coordinate transformations.

Summation Convention

- When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

Summation Convention

- Consider transformation equations such as,

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$
$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

- We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^{n} a_{1j}x_j$$
$$y_2 = \sum_{j=1}^{n} a_{2j}x_j$$
$$y_3 = \sum_{j=1}^{n} a_{3j}x_j$$

Summation Convention

* In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j$$
$$y_2 = a_{2j}x_j$$
$$y_3 = a_{3j}x_j$$

* Finally, we observe that $y_1, y_2,$ and y_3 can be represented as we have been doing by $y_i, \ i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \qquad i = 1,2,3$$

Summation Convention

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij}x_j$$

to represent compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

Summation Convention

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

for $i = 1, k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1, k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1, k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2, k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2, k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2, k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3, k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3, k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3, k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

Summation Convention

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}$$

In this same way, we could have also proved that,

$$a_{ij} a_{kj} = b_{ik}$$

* Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

Vector Components in ONB

Consider an orthogonal normal basis

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \qquad or\ \mathbf{e}_i, i = 1,2,3$$

The fact that $\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0$, and also that the norm on each, that is, $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$ make the computation of the components on any vector referred to this bases easy.

This is the main attraction of the ONB, also called Cartesian Coordinates.

Vector Components in ONB

Given a known vector $\mathbf{F}$, (known because its magnitude and direction are given) we can find its components in the ONB in these simple steps:

$$\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$$

Where $F_1, F_2\ and\ F_3$, the scalar quantities to be determined are called the components of $\mathbf{F}$ in the ONB $\mathbf{e}_i, i = 1,2,3$

Take the scalar product of the above equation with $\mathbf{e}_1$, we have,

Vector Components in ONB

$$\begin{aligned}\mathbf{F}\cdot\mathbf{e}_1 &= F_1\mathbf{e}_1\cdot\mathbf{e}_1 + F_2\mathbf{e}_2\cdot\mathbf{e}_1 + F_3\mathbf{e}_3\cdot\mathbf{e}_1\\ &= F_1\mathbf{e}_1\cdot\mathbf{e}_1 + \boxed{F_2\boldsymbol{e}_2\cdot\boldsymbol{e}_1} + \boxed{F_3\mathbf{e}_3\cdot\mathbf{e}_1}\end{aligned}$$

The boxed items vanish on account of orthogonality while the first term simply becomes F_1 on account of normality. We therefore find that $F_1 = \mathbf{F}\cdot\mathbf{e}_1$. We can repeat this process by taking the dot product with the other basis vectors and find also that, $F_2 = \mathbf{F}\cdot\mathbf{e}_2$ and that $F_3 = \mathbf{F}\cdot\mathbf{e}_3$.

In index notation, we could write the nine orthonormality equations as,

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$$

Vector Components in ONB

Furthermore, the result of calculating the components, that

$$\begin{aligned}\mathbf{F} &= F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 \\ &= (\mathbf{F}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{F}\cdot\mathbf{e}_2)\mathbf{e}_2 + (\mathbf{F}\cdot\mathbf{e}_3)\mathbf{e}_3\end{aligned}$$

Can also be written as,

$$\mathbf{F} = F_i\mathbf{e}_i = \left(\mathbf{F}\cdot\mathbf{e}_j\right)\mathbf{e}_j$$

Where the repetition of the indices indicate summation.

In the above computation, everything was easy because the orthogonality and normality conditions made some terms vanish and the other terms were products with unity. It is this ease of computation that makes the ONB very attractive.

Vector Components in ONB

When we relax the conditions on our basis vectors, these nice properties disappear. Yet, there is an extension of this situation, which, if understood deeply, will make things relatively easy also.

Cartesian Vector Components

It is convenient for us to represent the vectors of the Cartesian system of coordinates as $\mathbf{e}_i, i = 1,2,3$ which is just a shorthand for writing $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ instead of calling these unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

This change may look trivial at first, but when combined with Einstein's convention we will introduce, allows us the benefit of reducing the number of terms in expressions in a unique way necessary to express tensor terms.

Vector Components

Clearly, addition and linearity of the vector space ⇒

$$\mathbf{v} + \mathbf{w} = (v_i + w_i)\mathbf{e}_i$$

Multiplication by scalar rule implies that if $\alpha \in \mathscr{R}, \forall \mathbf{v} \in \mathscr{V},$

$$\alpha\,\mathbf{v} = (\alpha v_i)\mathbf{e}_i$$

Kronecker Delta

Kronecker Delta: δ_{ij} has the following properties:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$
$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$
$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

As is obvious, these are obtained by allowing the indices to attain all possible values in the range. The Kronecker delta is defined by the fact that when the indices explicit values are equal, it has the value of unity. Otherwise, it is zero. The above nine equations can be written more compactly as,

$$\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$$

Levi Civita Symbol

- The Levi-Civita Symbol: e_{ijk}
- $e_{111} = 0, e_{112} = 0, e_{113} = 0, e_{121} = 0, e_{122} = 0, e_{123} = 1, e_{131} = 0, e_{132} = -1, e_{133} = 0$
  $e_{211} = 0, e_{212} = 0, e_{213} = -1, e_{221} = 0, e_{222} = 0, e_{223} = 0, e_{231} = 1, e_{232} = 0, e_{233} = 0$
  $e_{311} = 0, e_{312} = 1, e_{313} = 0, e_{321} = -1, e_{322} = 0, e_{323} = 0, e_{331} = 0, e_{332} = 0, e_{333} = 0$

Levi Civita Symbol

* While the above equations might look arbitrary at first, a closer look shows there is a simple logic to it all. In fact, note that whenever the value of an index is repeated, the symbol has a value of zero. Furthermore, we can see that once the indices are an even arrangement (permutation) of 1,2, and 3, the symbols have the value of 1, When we have an odd arrangement, the value is -1. Again, we desire to avoid writing twenty seven equations to express this simple fact. Hence we use the index notation to define the Levi-Civita symbol as follows:
* $e_{ijk} = \begin{cases} 1 \text{ if } i,j \text{ and } k \text{ are an even permutation of } 1{,}2 \text{ and } 3 \\ -1 \text{ if } i,j \text{ and } k \text{ are an odd permutation of } 1{,}2 \text{ and } 3 \\ 0 \;\; \text{In all other cases} \end{cases}$

Cross Product of Basis Vectors

It is not difficult to prove that

$$\mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = e_{ijk}$$

* This relationship immediately implies that,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k.$$

Show that the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $a_i b_j e_{ijk} \mathbf{e}_k$ **where** a_i, b_j **are the components of $\boldsymbol{a}$ and $\boldsymbol{b}$** in the Cartesian system.

Express vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as contravariant components: $\boldsymbol{a} = a_i \mathbf{e}_i$, and $\boldsymbol{b} = b_j \mathbf{e}_j$. Using the above result, we can write that,

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_i \mathbf{e}_i) \times \left(b_j \mathbf{e}_j\right) \\ &= a_i b_j (\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j e_{ijk} \mathbf{e}_k. \end{aligned}$$

The last expression coming from the fact that $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k$.

Given that,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$$

Show that $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$

Expanding the equation, we have:

$$e_{rsk}e_{ijk} = \delta_{ki}\begin{vmatrix} \delta_{rj} & \delta_{rk} \\ \delta_{sj} & \delta_{sk} \end{vmatrix} - \delta_{kj}\begin{vmatrix} \delta_{ri} & \delta_{rk} \\ \delta_{si} & \delta_{sk} \end{vmatrix} + 3\begin{vmatrix} \delta_{ri} & \delta_{rj} \\ \delta_{si} & \delta_{sj} \end{vmatrix}$$

$$= \delta_{ki}\left(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}\right) - \delta_{kj}(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk})$$

$$+ 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)$$

$$= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right)$$

$$= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$$

Show that $e_{rjk}e_{ijk} = 2\delta_{ri}$

Contracting one more index, we have:

$$e_{rjk}e_{ijk} = \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} = 3\delta_{ri} - \delta_{ri} = 2\delta_{ri}$$

These results are useful in several situations.

Show that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

$$\mathbf{u} \times \mathbf{v} = e_{ijk} u_i v_j \mathbf{e}_k$$

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_\alpha \mathbf{e}_\alpha \cdot \left(e_{ijk} u_i v_j \mathbf{e}_k\right)$$

$$= u_\alpha \left(e_{ijk} u_i v_j\right) \delta_{\alpha k}$$

$$= \epsilon^{ijk} u_i v_j u_k$$

$$= 0$$

On account of the symmetry and antisymmetry in i and k.

Show that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= e_{ijk} u_i v_j \mathbf{e}_k \\ &= -e_{jik} u_i v_j \mathbf{e}_k = -e_{ijk} u_j v_i \mathbf{e}_k \\ &= -\mathbf{v} \times \mathbf{u}\end{aligned}$$

Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Let $\mathbf{z} = \mathbf{v} \times \mathbf{w} = e_{ijk} v_i w_j \mathbf{e}_k$

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \times \mathbf{z} = e_{\alpha\beta\gamma} u_\alpha z_\beta \mathbf{e}_\gamma \\
&= e_{\alpha\beta\gamma}\ u_\alpha z_\beta \mathbf{e}_\gamma \\
&= e_{ij\beta} e_{\gamma\alpha\beta} u_\alpha v_i w_j \mathbf{e}_\gamma \\
&= \left(\delta_{i\gamma}\delta_{j\alpha} - \delta_{i\alpha}\delta_{j\gamma}\right) u_\alpha v_i w_j \mathbf{e}_\gamma \\
&= u_j v_\gamma w_j \mathbf{e}_\gamma - u_i v_i w_\gamma \mathbf{e}_\gamma \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}
\end{aligned}$$