Homework 2.1

1. For any tensor $\mathbf{S}$, show that, $(\mathbf{S}\mathbf{e}_\alpha) \otimes \mathbf{e}_\alpha = \mathbf{S}$
2. Gurtin 2.6.1
3. Show that that if the tensor $\mathbf{T}$ is invertible, for any vector $\mathbf{k}$, $\mathbf{T}\mathbf{k} = \mathbf{o}$ automatically means that $\mathbf{k} = \mathbf{o}$.
4. Show that if the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are independent and $\mathbf{T}$ is invertible, then the vectors $\mathbf{T}\mathbf{u}, \mathbf{T}\mathbf{v}$ and $\mathbf{T}\mathbf{w}$ are also independent.
5. Show that $\mathbf{w} \times (\mathbf{w} \otimes \mathbf{w}) = \mathbf{0}$ and that $(\mathbf{w} \times)(\mathbf{w} \times) = \mathbf{w} \otimes \mathbf{w} - \|\mathbf{w}\|^2 \mathbf{1}$
6. Gurtin 2.8.5
7. Gurtin 2.9.1
8. Gurtin 2.9.2
9. Gurtin 2.9.4

Due July 29, 2016

Homework 2.2

11. Gurtin 2.11.1 d&e
12. Gurtin 2.11.3
13. Gurtin 2.11.4
14. Gurtin 2.11.5
15. Let $\mathbf{Q}$ be a rotation. For any pair of independent vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Qu}) \times (\mathbf{Qv})$
16. For a proper orthogonal tensor $\mathbf{Q}$, show that the eigenvalue equation always yields an eigenvalue of +1.
17. For an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (\mathbf{1} - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the skew tensor whose ij component is $\epsilon_{jik} e_k$, show that $\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) = \cos(\boldsymbol{\theta})(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\,\times)$.
18. For an arbitrary unit vector $\mathbf{e}$ and the tensor, $\mathbf{Q}(\theta)$ defined as above, Show for an arbitrary vector $\mathbf{u}$ that $\mathbf{v} = \mathbf{Q}(\theta)\mathbf{u}$ has the same magnitude as $\mathbf{u}$.

Due March 28, 2016

Homework 2.3

1. Gurtin 2.13.1
2. Gurtin 2.14.1
3. Gurtin 2.14.2
4. Gurtin 2.14.3
5. Gurtin 2.14.4
6. Gurtin 2.14.5
7. Gurtin 2.15 1-3a, 3b, 3c
8. Gurtin 2.16 1-8

Due August 4, 2016

Quiz

For a given a tensor $\mathbf{T}$ and its transpose $\mathbf{T}^{\mathrm{T}}$, Write out expressions for the

1. Symmetric Part
2. Skew Part
3. Spherical Part
4. Deviatoric Part.

What is the magnitude of $\mathbf{T}$?

Tensor Algebra

Tensors as Linear Mappings

July 22 to July 29, 2016

No	Topics	From Slide	Date
0	Home Work & Due dates & Quiz	1	
1	Definitions, Special Tensors	7	July 22
2	Scalar Functions or Invariants	17	
3	Inner Product, Euclidean Tensors	26	
4	The Tensor Product	29-39	
5	Tensor Basis & Component Representation	40	
6	The Vector Cross, Axial Vectors	60	
7	The Cofactor	68	July 25
8	Orthogonal Tensors	88	
9	Eigenvalue Problem, Spectral Decomposition & Cayley Hamilton	100	Weekend

Second Order Tensor

A second Order Tensor $\mathbf{T}$ is a linear mapping from a vector space to itself. Given $\mathbf{u} \in \mathscr{V}$ the mapping,

$$\mathbf{T}:\ \mathscr{V} \to \mathscr{V}$$

states that $\exists\ \mathbf{w} \in \mathscr{V}$ such that,

$$\mathbf{T}(\mathbf{u}) = \mathbf{w}.$$

Every other definition of a second order tensor can be derived from this simple definition. The tensor character of an object can be established by observing its action on a vector.

Linearity

* The mapping is linear. This means that if we have two runs of the process, we first input $\boldsymbol{u}$ and later input $\mathbf{v}$. The outcomes $\mathbf{T}(\mathbf{u})$ and $\mathbf{T}(\mathbf{v})$, added would have been the same as if we had added the inputs $\boldsymbol{u}$ and $\mathbf{v}$ first and supplied the sum of the vectors as input. More compactly, this means,

$$\mathbf{T}(\mathbf{u}+\mathbf{v}) = \mathbf{T}(\mathbf{u}) + \mathbf{T}(\mathbf{v})$$

Linearity

Linearity further means that, for any scalar α and tensor $\boldsymbol{T}$

$$\mathbf{T}(\alpha\mathbf{u}) = \alpha\mathbf{T}(\mathbf{u})$$

The two properties can be added so that, given $\alpha, \beta \in \mathscr{R}$, and $\mathbf{u}, \mathbf{v} \in \mathscr{V}$, then

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{T}(\mathbf{u}) + \beta\mathbf{T}(\mathbf{v})$$

Since we can think of a tensor as a process that takes an input and produces an output, two tensors are equal only if they produce the same outputs when supplied with the same input. The sum of two tensors is the tensor that will give an output which will be the sum of the outputs of the two tensors when each is given that input.

Vector Space

In general, $\alpha, \beta \in \mathscr{R}$, $\mathbf{u}, \mathbf{v} \in \mathscr{V}$ and $\mathbf{S}, \mathbf{T} \in \mathscr{T}$

$$\alpha\mathbf{S}\mathbf{u} + \beta\mathbf{T}\mathbf{u} = (\alpha\mathbf{S} + \beta\mathbf{T})\mathbf{u}$$

With the definition above, the set of tensors constitute a vector space with its rules of addition and multiplication by a scalar. It will become obvious later that it also constitutes a Euclidean vector space with its own rule of the inner product.

Special Tensors

Notation.

It is customary to write the tensor mapping without the parentheses. Hence, we can write,

$$\mathbf{Tu} \equiv \mathbf{T}(\mathbf{u})$$

For the mapping by the tensor $\mathbf{T}$ on the vector variable and dispense with the parentheses unless when needed.

Zero Tensor or Annihilator

The annihilator $\mathbf{O}$ is defined as the tensor that maps all vectors to the zero vector, $\mathbf{o}$:

$$\mathbf{O}\mathbf{u} = \mathbf{o}, \qquad \forall \mathbf{u} \in \mathscr{V}$$

The Identity

The identity tensor $\mathbf{I}$ is the tensor that leaves every vector unaltered. $\forall \boldsymbol{u} \in \mathscr{V}$,

$$\mathbf{Iu} = \mathbf{u}$$

Furthermore, $\forall \alpha \in \mathscr{R}$, the tensor, $\alpha\mathbf{I}$ is called a spherical tensor.

The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\mathbf{T} \in \mathscr{T}$ and $\mathbf{u} \in \mathscr{V}$, the mapping $\mathbf{w} \equiv \mathbf{Tu}$ produces a vector.

The Inverse

Consider a linear mapping that, operating on $\mathbf{w}$, produces our original argument, $\mathbf{u}$, if we can find it:

$$\mathbf{Yw} = \mathbf{u}$$

As a linear mapping, operating on a vector, clearly, $\mathbf{Y}$ is a tensor. It is called the inverse of $\mathbf{T}$ because,

$$\mathbf{Yw} = \mathbf{YTu} = \mathbf{u}$$

So that the composition $\mathbf{YT} = \mathbf{I}$, the identity mapping. For this reason, we write,

$$\mathbf{Y} = \mathbf{T}^{-1}$$

Inverse

It is easy to show that if $\mathbf{YT} = \mathbf{I}$, then $\mathbf{TY} = \mathbf{YT} = \mathbf{I}$.

* ***HW: Show this.***

The set of invertible sets is closed under composition. It is also closed under inversion. It forms a group with the identity tensor as the group's neutral element

Answer

Given that $\mathbf{YT} = \mathbf{I}$ we want to show that $\mathbf{TY} = \mathbf{YT} = \mathbf{I}.$

Consider $\mathbf{TYTu}$ where $\mathbf{u}$ is a vector. Since $\mathbf{YT} = \mathbf{I},$ it follows that $\mathbf{TYTu} = \mathbf{TIu} = \mathbf{Tu} \equiv \mathbf{v}$ where $\mathbf{v}$ is a vector. Clearly,

$$\mathbf{TYTu} = \mathbf{TYv} = \mathbf{v}$$

which immediately shows that $\mathbf{TY} = \mathbf{I}$ as required to be shown.

Transposition of Tensors

Given $\mathbf{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\mathbf{A}^{\mathrm{T}}$ satisfying

$$\mathbf{w} \cdot \left(\mathbf{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\mathbf{A}\mathbf{w})$$

Is called the transpose of $\mathbf{A}$.

A tensor indistinguishable from its transpose is said to be symmetric.

Invariants

There are certain mappings from the space of tensors to the real space. Such mappings are called Invariants of the Tensor. Three of these, called Principal invariants play key roles in the application of tensors to continuum mechanics. We shall define them shortly.

The definition given here is free of any association with a coordinate system. It is a good practice to derive any other definitions from these fundamental ones:

The Trace

If we write

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors.

For any second order tensor $\boldsymbol{T}$, and linearly independent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the linear mapping $I_1 \colon \mathscr{T} \to \mathscr{R}$

$$I_1(\mathbf{T}) \equiv \mathrm{tr}(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Is independent of the choice of the basis vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is called the First Principal Invariant of $\mathbf{T}$ or Trace of $\mathbf{T} \equiv \mathrm{tr}(\mathbf{T}) \equiv I_1(\mathbf{T})$

Invariance of the Trace

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{Ta},\boldsymbol{b},\boldsymbol{c}] + [\boldsymbol{a},\boldsymbol{Tb},\boldsymbol{c}] + [\boldsymbol{a},\boldsymbol{b},\boldsymbol{Tc}]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

Let us refer each vector to a covariant basis so that, $\boldsymbol{a} = a_i\mathbf{e}_i$, $\boldsymbol{b} = b_j\mathbf{e}_j$, and $\boldsymbol{c} = c_k\mathbf{e}_k$. Hence,

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}(a_i\mathbf{e}_i), b_j\mathbf{e}_j, c_k\mathbf{e}_k] + [a_i\mathbf{e}_i, \boldsymbol{T}(b_j\mathbf{e}_j), c_k\mathbf{e}_k] + [a_i\mathbf{e}_i, b_j\mathbf{e}_j, \boldsymbol{T}(c_k\mathbf{e}_k)]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

$$= \frac{a_i b_j c_k\left([\boldsymbol{T}\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k] + [\mathbf{e}_i, \boldsymbol{T}\mathbf{e}_j, \mathbf{e}_k] + [\mathbf{e}_i, \mathbf{e}_j, \boldsymbol{T}\mathbf{e}_k]\right)}{e_{ijk}a_i b_j c_k}$$

But once we are able to express tensors in component form, we will easily prove that $[\boldsymbol{T}\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k] + [\mathbf{e}_i, \boldsymbol{T}\mathbf{e}_j, \mathbf{e}_k] + [\mathbf{e}_i, \mathbf{e}_j, \boldsymbol{T}\mathbf{e}_k] = T_{\alpha\alpha}[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k]$. Using this result, we have that

$$I_1(\boldsymbol{T}) = \frac{a^i b^j c^k T_{\alpha\alpha}[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k]}{e_{ijk}a_i b_j c_k} = \frac{a^i b^j c^k T_{\alpha\alpha} e_{ijk}}{e_{ijk}a_i b_j c_k}$$

$$= \frac{e_{ijk}a^i b^j c^k}{e_{ijk}a^i b^j c^k} T_{\alpha\alpha} = T_{\alpha\alpha}$$

Which, in either case, is obviously independent of the choice of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

The Trace

The trace is a linear mapping. It is easily shown that $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\mathrm{tr}(\alpha \boldsymbol{S} + \beta \boldsymbol{T}) = \alpha \mathrm{tr}(\boldsymbol{S}) + \beta \mathrm{tr}(\boldsymbol{T})$$

HW. Show this by appealing to the linearity of the vector space.

While the trace of a tensor is linear, the other two principal invariants are nonlinear. We now proceed to define them

Square of the trace

The second principal invariant $I_2(\boldsymbol{S})$ is related to the trace. In fact, you may come across books that define it so. However, the most common definition is that

$$I_2(\boldsymbol{S}) = \frac{1}{2}\left[I_1^2(\boldsymbol{S}) - I_1(\boldsymbol{S}^2)\right]$$

Independently of the trace, we can also define the second principal invariant as,

Second Principal Invariant

The Second Principal Invariant, $I_2(\boldsymbol{T})$, using the same notation as above is

$$\frac{[(\boldsymbol{Ta}),(\boldsymbol{Tb}),\boldsymbol{c}]+[\boldsymbol{a},(\boldsymbol{Tb}),(\boldsymbol{Tc})]+[(\boldsymbol{Ta}),\boldsymbol{b},(\boldsymbol{Tc})]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

$$=\frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{T})-\mathrm{tr}(\boldsymbol{T}^2)\right]$$

that is half the square of trace minus the trace of the square of $\boldsymbol{T}$ which is the second principal invariant.

* This quantity remains unchanged for any arbitrary selection of basis vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

The Determinant

The third mapping from tensors to the real space underlying the tensor is the determinant of the tensor. While you may be familiar with that operation and can easily extract a determinant from a matrix, it is important to understand the definition for a tensor that is independent of the component expression. The latter remains relevant even when we have not expressed the tensor in terms of its components in a particular coordinate system.

The Determinant

As before, For any second order tensor $\boldsymbol{T}$, and any linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

- The determinant of the tensor $\boldsymbol{T}$,

$$\det(\boldsymbol{T}) = \frac{[(\boldsymbol{T}\boldsymbol{a}), (\boldsymbol{T}\boldsymbol{b}), (\boldsymbol{T}\boldsymbol{c})]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

(In the special case when the basis vectors are orthonormal, the denominator is unity)

Other Principal Invariants

* It is good to note that there are other principal invariants that can be defined. The ones we defined here are the ones you are most likely to find in other texts.
* An invariant is a scalar derived from a tensor that remains unchanged in any coordinate system. Mathematically, it is a mapping from the tensor space to the real space. Or simply **a scalar valued function of the tensor**.

Deviatoric Tensors

* When the trace of a tensor is zero, the tensor is said to be traceless. A traceless tensor is also called a deviatoric tensor.
* Given any tensor $\mathbf{S}$, A deviatoric tensor may be created from $\mathbf{S}$ by the following process:

$$\mathbf{S}_0 \equiv \text{dev}\,\mathbf{S} \equiv \mathbf{S} - \frac{1}{3}(\text{tr}\,\mathbf{S})\mathbf{I} = \mathbf{S} - s\mathbf{I}$$

So that $s = \frac{1}{3}(\text{tr}\,\mathbf{S})$; $s\mathbf{I}$ is called the spherical part, and $\mathbf{S}_0$ as defined here is called the deviatoric part of $\mathbf{S}$.

Every tensor thus admits the decomposition,

$$\mathbf{S} = \mathbf{S}_0 + s\mathbf{I}$$

Inner Product of Tensors

The trace provides a simple way to define the inner product of two second-order tensors. Given $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

The trace,

$$\mathrm{tr}(\boldsymbol{S}^T\boldsymbol{T}) = \mathrm{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Is a scalar, independent of the coordinate system chosen to represent the tensors. This is defined as the inner or scalar product of the tensors $\boldsymbol{S}$ and $\boldsymbol{T}$. That is,

$$\boldsymbol{S}:\boldsymbol{T} \equiv \mathrm{tr}(\boldsymbol{S}^T\boldsymbol{T}) = \mathrm{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Attributes of a Euclidean Space

The trace automatically induces the concept of the norm of a vector (This is not the determinant! Note!!) The square root of the scalar product of a tensor with itself is the norm, magnitude or length of the tensor:

$$\|\boldsymbol{T}\| = \sqrt{\mathrm{tr}(\boldsymbol{T}^T\boldsymbol{T})} = \sqrt{\boldsymbol{T}:\boldsymbol{T}}$$

Distance and angles

Furthermore, the distance between two tensors as well as the angle they contain are defined. The scalar distance $d(\boldsymbol{S},\boldsymbol{T})$between tensors $\boldsymbol{S}$ and $\boldsymbol{T}$:

$$d(\boldsymbol{S},\boldsymbol{T}) = \|\boldsymbol{S}-\boldsymbol{T}\| = \|\boldsymbol{T}-\boldsymbol{S}\|$$

And the angle $\theta(\boldsymbol{S},\boldsymbol{T})$,

$$\theta = \cos^{-1}\frac{\boldsymbol{S}:\boldsymbol{T}}{\|\boldsymbol{S}\|\,\|\boldsymbol{T}\|}$$

The Tensor Product

A product mapping from two vector spaces to $\mathscr{T}$ is defined as the tensor product. It has the following properties:

$$"\otimes": \mathscr{V} \times \mathscr{V} \to \mathscr{T}$$
$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

It is an ordered pair of vectors. It acts on any other vector by creating a new vector in the direction of its first vector as shown above. This product of two vectors is called a tensor product or a simple dyad.

Dyad Properties

It is very easily shown that the transposition of dyad is simply a reversal of its order. (Shown below).

The tensor product is linear in its two factors.

Based on the obvious fact that for any tensor $\boldsymbol{T}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$, $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{T}\boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = [(\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}]\boldsymbol{w}$

It is clear that

$$\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}$$

Show this neatly by operating either side on a vector

Furthermore, the contraction,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes (\boldsymbol{T}^T \boldsymbol{v})$$

A fact that can be established by operating each side on the same vector.

Composition with Tensors

Operate on the vector $\boldsymbol{z}$ and let $\boldsymbol{Tz} = \boldsymbol{w}$. On the LHS,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{Tz} = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

On the RHS, we have:

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T\boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}\left((\boldsymbol{T}^T\boldsymbol{v}) \cdot \boldsymbol{z}\right) = \boldsymbol{u}\left(\boldsymbol{z} \cdot (\boldsymbol{T}^T\boldsymbol{v})\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\boldsymbol{u}\left(\boldsymbol{z} \cdot (\boldsymbol{T}^T\boldsymbol{v})\right) = \boldsymbol{u}\left(\boldsymbol{v} \cdot (\boldsymbol{Tz})\right)$$

follows from the definition of transposition. Hence,

$$\left(\boldsymbol{u} \otimes (\boldsymbol{T}^T\boldsymbol{v})\right)\boldsymbol{z} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

Transpose of a Dyad

Recall that for $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot (\boldsymbol{A}^{\mathrm{T}}\mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of $\boldsymbol{A}$. Now let $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ a dyad.

$$\begin{aligned} \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w}) = & \\ & = \mathbf{v} \cdot [(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w}] = \mathbf{v} \cdot [\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{w})] \\ & = (\mathbf{v} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{b})(\mathbf{v} \cdot \boldsymbol{a}) \\ & = \boldsymbol{w} \cdot (\boldsymbol{b} \otimes \boldsymbol{a})\mathbf{v} \end{aligned}$$

So that $(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}$

Showing that the transpose of a dyad is simply a reversal of its factors.

If $\mathbf{n}$ is the unit normal to a given plane, show that the tensor $\mathbf{T} \equiv \mathbf{1} - \mathbf{n} \otimes \mathbf{n}$ is such that $\mathbf{Tu}$ is the projection of the vector $\mathbf{u}$ to the plane in question.

Consider the fact that

$$\mathbf{Tu} = \mathbf{1u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} = \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n}$$

The above vector equation shows that $\mathbf{Tu}$ is what remains after we have subtracted the projection $(\mathbf{n} \cdot \mathbf{u})\mathbf{n}$ onto the normal. Obviously, this is the projection to the plane itself. $\mathbf{T}$ as we shall see later is called a tensor projector.

Dyad on Dyad Composition

For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \mathscr{V}$, We can show that the dyad composition,

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{u} \otimes \boldsymbol{x})(\boldsymbol{v} \cdot \boldsymbol{w})$$

Again, the proof is to show that both sides produce the same result when they act on the same vector. Let $\boldsymbol{y} \in \mathscr{V}$, then the LHS on $\boldsymbol{y}$ yields:

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x})\boldsymbol{y} = (\boldsymbol{u} \otimes \boldsymbol{v})[\boldsymbol{w}(\boldsymbol{x} \cdot \boldsymbol{y})]$$
$$= \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w})(\boldsymbol{x} \cdot \boldsymbol{y})$$

Which is obviously the result from the RHS also.

This therefore makes it straightforward to contract dyads by breaking and joining as seen above.

Trace of a Dyad

Show that the trace of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v}$.

Given any three independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, (No loss of generality in letting the three independent vectors be the curvilinear basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$). Using the above definition of trace, we can write that,

Trace of a Dyad

$$\text{tr}(\mathbf{u} \otimes \mathbf{v})$$

$$= \frac{[\{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{e}_2\}, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{e}_3\}]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

$$= \frac{1}{e_{123}}\{[v_1\mathbf{u}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, v_2\mathbf{u}, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, v_3\mathbf{u}]\}$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u}) \cdot (e_{23i}\mathbf{e}_i) + (e_{31i}\mathbf{e}_i) \cdot (v_2\mathbf{u}) + (e_{12i}\mathbf{e}_i) \cdot (v_3\mathbf{u})\}$$

$$= \frac{1}{e_{123}}\{(v_1\mathbf{u}) \cdot (e_{231}\mathbf{e}_1) + (e_{312}\mathbf{e}_2) \cdot (v_2\mathbf{u}) + (e_{123}\,\mathbf{e}_3)$$

$$= u_i v_i = \mathbf{u} \cdot \mathbf{v}$$

Other Invariants of a Dyad

* It is easy to show that for a tensor product

$$\boldsymbol{D} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V}$$

$$\boldsymbol{I}_2(\boldsymbol{D}) = \boldsymbol{I}_3(\boldsymbol{D}) = 0$$

HW. Show that this is so.

We proved earlier that $\boldsymbol{I}_1(\boldsymbol{D}) = \boldsymbol{u} \cdot \boldsymbol{v}$

Furthermore, if $\boldsymbol{T} \in \mathscr{T}$, then,

$$\operatorname{tr}(\boldsymbol{T}\boldsymbol{u} \otimes \boldsymbol{v}) = \operatorname{tr}(\boldsymbol{w} \otimes \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{T}\boldsymbol{u} \cdot \boldsymbol{v}$$