Working to the Answer

- Consider a vector $\mathbf{F}$. Its components on the $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ coordinates are $\mathrm{F}_1 = \mathbf{F} \cdot \mathbf{e}_1,\ \mathrm{F}_2 = \mathbf{F} \cdot \mathbf{e}_2$ and $\mathrm{F}_1 = \mathbf{F} \cdot \mathbf{e}_3$. Instead of wasting too much space, we simply say that the component $\mathrm{F}_i = \mathbf{F} \cdot \mathbf{e}_i, \mathrm{i} = 1,2,3.$
- To get the components of a tensor is equally easy. But remember that instead of three equations, we are dealing now with nine equations.
- Consider a vector $\mathbf{T}$. Its components on the product bases $\mathbf{e}_i \otimes \mathbf{e}_j$ are $T_{ij} \equiv \mathbf{T}: \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \equiv \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$
- The following pages show why this is so. The analogy between the vector and the tensor such that dot product becomes double dot and every other thing remains the same is the easy way to remember this…
- When you get stuck, note that you have forgotten a simple rule

Tensor Bases & Component Representation

Given $\boldsymbol{T} \in \mathscr{T}$, for any Cartesian basis vectors $\mathbf{e}_i \in \mathscr{V}, i = 1,2,3$

$$\mathbf{t}_j \equiv \mathbf{T}\mathbf{e}_j \in \mathscr{V}, j = 1,2,3$$

by the law of tensor mapping. We proceed to find the components of $\mathbf{t}_j$ on this same basis. Its components, just like in any other vector are the scalars,

$$\left(\mathbf{t}_j\right)_\alpha = \mathbf{t}_j \cdot \mathbf{e}_\alpha$$

Specifically, these components are $\left(\left(\mathbf{t}_j\right)_1, \left(\mathbf{t}_j\right)_2, \left(\mathbf{t}_j\right)_3\right)$

Tensor Components

We can dispense with the parentheses and write that

$$T_{j\alpha} \equiv \left(\mathbf{t}_j\right)_\alpha = \mathbf{t}_j \cdot \mathbf{e}_\alpha$$

So that the vector

$$\mathbf{T}\mathbf{e}_j = \mathbf{t}_j = T_{j\alpha}\mathbf{e}_\alpha$$

The components T_{ij} can be found by taking the dot product of the above equation with $\mathbf{e}_i$:

$$\mathbf{e}_i \cdot \left(\mathbf{T}\mathbf{e}_j\right) = T_{j\alpha}(\mathbf{e}_i \cdot \mathbf{e}_\alpha) = T_{ij}$$

$$T_{ij} = \mathbf{e}_i \cdot \left(\mathbf{T}\mathbf{e}_j\right)$$

$$= \mathrm{tr}\left(\mathbf{T}\mathbf{e}_j \otimes \mathbf{e}_i\right) = \mathbf{T}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$$

Tensor Components

The component T_{ij} is simply the result of the inner product of the tensor $\boldsymbol{T}$ on the tensor product $\mathbf{e}_i \otimes \mathbf{e}_j$. These are the components of $\boldsymbol{T}$ on the product dual of this particular product base.

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

This is a general result and applies to all product bases:

It is straightforward to prove the results on the following table:

* $\mathbf{T}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = \operatorname{tr}\left(\mathbf{T}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right)$
$$= \operatorname{tr}\left(\mathbf{T}\mathbf{e}_j \otimes \mathbf{e}_i\right) = \mathbf{T}\mathbf{e}_j \cdot \mathbf{e}_i$$
$$= \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j$$
* $\mathbf{I}:\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = \operatorname{tr}\left(\mathbf{I}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)\right)$
$$= \operatorname{tr}\left(\mathbf{I}\mathbf{e}_j \otimes \mathbf{e}_i\right) = \mathbf{I}\mathbf{e}_j \cdot \mathbf{e}_i$$
$$= \mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j = \delta_{ij}$$
* The tensor $\mathbf{T}$ itself is represented as $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$

Symmetry

The transpose of $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is $\boldsymbol{T}^T = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$.

If $\boldsymbol{T}$ is symmetric, then,

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i = T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

Clearly, in this case,

$$T_{ij} = T_{ji}$$

AntiSymmetry

* A tensor is antisymmetric if its transpose is its negative. In product bases that are either covariant or contravariant, antisymmetry, like symmetry can be expressed in terms of the components:

The transpose of $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is $\boldsymbol{T}^T = T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$.

If $\boldsymbol{T}$ is antisymmetric, then,

$$T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i = -T_{ji}\mathbf{e}_i \otimes \mathbf{e}_j$$

Clearly, in this case,

$$T_{ij} = -T_{ji}$$

It is straightforward to establish the same for contravariant components. Antisymmetric tensors are also said to be skew-symmetric.

Symmetric & Skew Parts of Tensors

For any tensor $\mathbf{T}$, define the symmetric and skew parts $\text{sym}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{\text{T}}\right)$, and $\text{skw}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^{\text{T}}\right)$. It is easy to show the following:

$$\mathbf{T} = \text{sym}\,\mathbf{T} + \text{skw}\,\mathbf{T}$$

$$\text{skw}(\text{sym}\,\mathbf{T}) = \text{sym}(\text{skw}\,\mathbf{T}\,) = 0$$

We can also write that,

$$\text{sym}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

and

$$\text{skw}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

Composition

Composition of tensors in component form follows the rule of the composition of dyads.

$$\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

$$\boldsymbol{S} = S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

$$\begin{aligned}\boldsymbol{TS} &= \left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\\ &= T_{ij}S_{\alpha\beta}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\left(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\right)\\ &= T_{ij}S_{\alpha\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\delta_{j\alpha}\\ &= T_{ij}S_{j\beta}\mathbf{e}_i \otimes \mathbf{e}_\beta\\ &= T_{i\alpha}S_{\alpha j}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

Addition

* Addition of two tensors of the same order is the addition of their components provided they are refereed to the same product basis.

Component Representation of Invariants

- Invoking the definition of the three principal invariants, we now find expressions for these in terms of the components of tensors in various product bases.
- First note that for $\boldsymbol{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, the triple product, $[\{\mathbf{Te}_1\}, \mathbf{e}_2, \mathbf{e}_3] = \left[\left\{\left(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) \mathbf{e}_1\right\}, \mathbf{e}_2, \mathbf{e}_3\right]$
$= \left[\left\{T_{ij}\mathbf{e}_i\delta_{j1}\right\}, \mathbf{e}_2, \mathbf{e}_3\right] = [T_{i1}\mathbf{e}_i \cdot (\epsilon_{231}\mathbf{e}_1)] = T_{i1}\delta_{i1}\epsilon_{231}$
- Recall that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{e}_k$

The Trace

The Trace of the Tensor $\mathbf{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$

$$\text{tr}(\mathbf{T}) = \frac{[\{\mathbf{T}\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \{\mathbf{T}\mathbf{e}_2\}, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \{\mathbf{T}\mathbf{e}_3\}]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

$$= \frac{T_{i1}\delta_{i1}\epsilon_{231} + T_{i2}\delta_{i2}\epsilon_{312} + T_{i3}\delta_{i3}\epsilon_{123}}{\epsilon_{123}}$$

$$= T_{i1}\delta_{i1} + T_{i2}\delta_{i2} + T_{i3}\delta_{i3} = T_{11} + T_{22} + T_{33} = T_{ii}$$

Write the second tensor invariant in terms of components

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$I_2(\mathbf{T}) = \frac{[\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

The first of the numerator terms can be simplified as,

$$\begin{aligned}[\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] &= \left[(T_{i1}\mathbf{e}_i), \left(T_{j2}\mathbf{e}_j\right), \mathbf{e}_3\right] \\ &= T_{i1}T_{j2}\left[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_3\right]\end{aligned}$$

The other terms are similarly simplified. Clearly,

$$\begin{aligned}I_2(\mathbf{T}) &= \frac{T_{i1}T_{j2}\left[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_3\right] + T_{j2}T_{k3}\left[\mathbf{e}_1, \mathbf{e}_j, \mathbf{e}_k\right] + T_{i1}T_{k3}[\mathbf{e}_i, \mathbf{e}_2, \mathbf{e}_3]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \\ &= \frac{T_{i1}T_{j2}\epsilon_{ij3} + T_{j2}T_{k3}\epsilon_{1jk} + T_{i1}T_{k3}\epsilon_{i2k}}{\epsilon_{123}} \\ &= \frac{[(T_{11}T_{22} - T_{21}T_{12}) + (T_{22}T_{33} - T_{32}T_{23}) + (T_{33}T_{11} - T_{13}T_{31})]\epsilon_{123}}{\epsilon_{123}} \\ &= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)\end{aligned}$$

Determinant

Express the third tensor invariant in terms of its components.

Consider the three Cartesian basis vectors, $\mathbf{e}_i, i = 1,2,3$. For any tensor $\mathbf{T}$,

$$I_3(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} = \frac{\left[(T_{i1}\mathbf{e}_i), \left(T_{j2}\mathbf{e}_j\right), (T_{k3}\mathbf{e}_k)\right]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}$$

$$= T_{i1}T_{j2}T_{k3}\left[\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\right]$$

$$= \frac{T_{i1}T_{j2}T_{k3}e_{ijk}}{e_{123}} = T_{i1}T_{j2}T_{k3}e_{ijk} = \det \mathbf{T}$$

A direct expansion is the only way to see that this is true.

The Vector Cross

Given a vector $\boldsymbol{u} = u_i \mathbf{e}_i$, the tensor

$$(\boldsymbol{u}\times) \equiv \epsilon_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

is called a vector cross. The following relation is easily established between a the vector cross and its associated vector:

$$\forall \mathbf{v} \in \mathscr{V}, (\boldsymbol{u}\times)\mathbf{v} = \boldsymbol{u} \times \mathbf{v}$$

The vector cross is *traceless* and *antisymmetric*. *(HW. Show this). The converse is the axial vector.*

Traceless tensors are also called deviatoric or deviator tensors.

Axial Vector

- For any antisymmetric tensor $\mathbf{\Omega}$, $\exists \boldsymbol{\omega} \in \mathscr{V}$, such that
$$\mathbf{\Omega} = (\boldsymbol{\omega} \times)$$

$\boldsymbol{\omega}$ which can always be found, is called the axial vector to the skew tensor.

It can be proved that

$$\boldsymbol{\omega} = -\frac{1}{2} e_{ijk} \Omega_{jk} \mathbf{e}_i$$

(HW: Prove it by contracting both sides of $\Omega_{ij} = e_{i\alpha j}\omega_\alpha$ with $e_{ij\beta}$ while noting that $e_{ij\beta} e_{i\alpha j} = -2\delta_{\beta\alpha}$)

Examples

Gurtin 2.8.5 Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $(\mathbf{u}\times):(\mathbf{v}\times) = 2\mathbf{u}\cdot\mathbf{v}$. Hence show that $||\mathbf{u}\times|| = \sqrt{2}||\mathbf{u}||$

$(\mathbf{u}\times) = e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$, $(\mathbf{v}\times) = e_{lmn}\mathbf{e}_l \otimes \mathbf{e}_n$. Hence,

$$
\begin{aligned}
(\mathbf{u}\times):(\mathbf{v}\times) &= e_{ijk}e_{lmn}u_j v_m(\mathbf{e}_i \otimes \mathbf{e}_k):(\mathbf{e}_l \otimes \mathbf{e}_n)\\
&= e_{ijk}e_{lmn}u_j v_m(\mathbf{e}_i \cdot \mathbf{e}_l)(\mathbf{e}_k \cdot \mathbf{e}_n)\\
&= e_{ijk}e_{lmn}u_j v_m\delta_{il}\delta_{kn}\\
&= e_{ijk}e_{imk}u_j v_m\delta_{il}\delta_{kn}\\
&= 2\delta_{jm}u_j v_m = 2u_j v_j = 2\,\mathbf{u}\cdot\,\mathbf{v}
\end{aligned}
$$

The rest of the result follows by setting $\mathbf{u} = \mathbf{v}$

For vectors u, v and w, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times$.

The tensor $(\mathbf{u}\times) = -e_{lmn}u_n\mathbf{e}_l\otimes\mathbf{e}_m$

Similarly, $(\mathbf{v}\times) = -e_{\alpha\beta\gamma}\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$ and $(\mathbf{w}\times) = -e_{ijk}w_k\mathbf{e}_i\otimes\mathbf{e}_j$.

Clearly,

$$(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)$$

$$= -e_{lmn}e_{\alpha\beta\gamma}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)(\mathbf{e}_l\otimes\mathbf{e}_m)(\mathbf{e}_i\otimes\mathbf{e}_j)$$

$$= -e_{lmn}e_{\alpha\beta\gamma}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_j)\delta_{l\beta}\delta_{mi}$$

$$= -e_{lin}e_{\alpha l\gamma}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_j)$$

$$= -e_{lin}e_{l\gamma\alpha}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_j)$$

$$= -(\delta_{\alpha n}\delta_{\gamma i} - \delta_{\alpha i}\delta_{\gamma n})e_{ijk}u_n v_\gamma w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_j)$$

$$= -e_{ijk}u_\alpha v_i w_k(\mathbf{e}_\alpha\otimes\mathbf{e}_j) + e_{ijk}u_n v_n w_k(\mathbf{e}_i\otimes\mathbf{e}_j)$$

$$= [\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]$$

Show that $[\mathbf{u},\mathbf{v},\mathbf{w}] = -\mathrm{tr}[(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)]$

In the above we have shown that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = [\mathbf{v}\otimes(\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]$

Because the vector cross is traceless, the trace of $[(\mathbf{u}\cdot$

Cofactor Definition

We will define the cofactor of a tensor as,

$$\text{cofac}\,\mathbf{T} \equiv \mathbf{T}^{\text{c}} \equiv \mathbf{T}^{-\mathbf{T}} \det \mathbf{T}$$

and proceed to show that, for any pair of independent vectors $\mathbf{u}$ and $\mathbf{v}$ the cofactor satisfies,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\text{c}}(\mathbf{u} \times \mathbf{v})$$

We will further find an invariant component representation for the cofactor tensor. Lastly, in this section, we will find an important relationship between the trace of the cofactor and second invariant of the tensor itself: $\text{tr}(\mathbf{T}^{\text{c}}) = I_2(\mathbf{T})$

Transformed Basis

First note that if $\mathbf{T}$ is invertible, the independence of the vectors $\mathbf{u}$ and $\mathbf{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$. Consequently we can define the non-vanishing

$$\mathbf{n} \equiv \mathbf{Tu} \times \mathbf{Tv} \neq 0.$$

It follows that $\mathbf{n}$ must be on the perpendicular line to both $\mathbf{Tu}$ and $\mathbf{Tv}$. Therefore,

$$\mathbf{n} \cdot \mathbf{Tu} = \mathbf{n} \cdot \mathbf{Tv} = 0.$$

We can also take a transpose and write,

$$\mathbf{u} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{v} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = 0$$

Showing that the vector $\mathbf{T}^{\mathrm{T}}\mathbf{n}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It follows that $\exists\, \alpha \in \mathscr{R}$ such that

$$\mathbf{T}^{\mathrm{T}}\mathbf{n} = \alpha(\mathbf{u} \times \mathbf{v})$$

Cofactor Transformation

Therefore, $\mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v})$.

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ so that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent, then we can take a scalar product of the above equation and obtain,

$$\mathbf{w} \cdot \mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$$

The LHS is also $\mathbf{Tw} \cdot (\mathbf{Tu} \times \mathbf{Tv}) = \mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw}$. In the equation, $\mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw} = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$, it is clear that

$$\alpha = \det \mathbf{T}$$

We therefore have that, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v})$.

Cofactor Tensor

We therefore have that,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T} \, (\mathbf{u} \times \mathbf{v}).$$

This quantity, $\mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$ is the cofactor of $\mathbf{T}$. If we write,

$$\operatorname{cofac} \mathbf{T} \equiv \mathbf{T}^{\mathrm{c}} \equiv \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$$

we can see that the cofactor satisfies, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$

We now express the cofactor in its general components.

$$\begin{aligned}\mathbf{T}^{\mathrm{c}} &= (\mathbf{T}^{\mathrm{c}})_{\alpha i} \mathbf{e}_\alpha \otimes \mathbf{e}_i = (\mathbf{e}_\alpha \cdot \mathbf{T}^{\mathrm{c}} \mathbf{e}_i) \mathbf{e}_\alpha \otimes \mathbf{e}_i \\ &= \frac{1}{2} e_{ijk} \left[\mathbf{e}_\alpha \cdot \mathbf{T}^{\mathrm{c}} (\mathbf{e}_j \times \mathbf{e}_k) \right] \mathbf{e}_\alpha \otimes \mathbf{e}_i \\ &= \frac{1}{2} e_{ijk} \left[\mathbf{e}_\alpha \cdot (\mathbf{T}\mathbf{e}_j) \times (\mathbf{T}\mathbf{e}_k) \right] \mathbf{e}_\alpha \otimes \mathbf{e}_i .\end{aligned}$$

Cofactor Components

The scalar in brackets,

$$\mathbf{e}_\alpha \cdot (\mathbf{T}\mathbf{e}_j) \times (\mathbf{T}\mathbf{e}_k) = \mathbf{e}_\alpha \cdot e_{lmn}(\mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_j)(\mathbf{e}_n \cdot \mathbf{T}\mathbf{e}_k)\mathbf{e}_l$$
$$= \delta_{\alpha l} e_{lmn}(\mathbf{e}_m \cdot \mathbf{T}\mathbf{e}_j)(\mathbf{e}_n \cdot \mathbf{T}\mathbf{e}_k)$$
$$= \delta_{\alpha l} e_{lmn} T_{mj} T_{nk} = e_{\alpha mn} T_{mj} T_{nk}$$

Inserting this above, we therefore have, in invariant component form,

$$\mathbf{T}^{\mathrm{c}} = \frac{1}{2} e_{ijk} [\mathbf{e}_\alpha \cdot (\mathbf{T}\mathbf{e}_j) \times (\mathbf{T}\mathbf{e}_k)] \mathbf{e}_\alpha \otimes \mathbf{e}_i$$
$$= \frac{1}{2} e_{ijk} e_{\alpha mn} T_{mj} T_{nk} \mathbf{e}_\alpha \otimes \mathbf{e}_i$$

Trace of the Cofactor

For any invertible tensor, show that the trace of the cofactor is the second principal invariant of the original tensor: $I_1(\mathbf{T}^c) = I_2(\mathbf{T})$

$$\begin{aligned}
\mathrm{tr}(\mathbf{T}^c) &= \frac{1}{2} e_{ijk} e_{\alpha mn} T_{mj} T_{nk} \mathbf{e}_\alpha \cdot \mathbf{e}_i = I_1(\mathbf{T}^c) \\
&= \frac{1}{2} e_{ijk} e_{\alpha mn} T_{mj} T_{nk} \delta_{i\alpha} = \frac{1}{2} e_{ijk} e_{imn} T_{mj} T_{nk} \\
&= \frac{1}{2}\left(\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj}\right) T_{mj} T_{nk} = \frac{1}{2}\left(T_{jj}T_{kk} - T_{jk}T_{kj}\right) \\
&= I_2(\mathbf{T})
\end{aligned}$$

Determinants

For the tensors $\mathbf{A}$ and $\mathbf{B}$, use direct methods to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$

Select linearly independent tensors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent. Now,

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

$$= \det \mathbf{A} \times \det \mathbf{B}$$

$$\det \alpha \boldsymbol{C} = \epsilon^{ijk}(\alpha C_i^1)(\alpha C_j^2)(\alpha C_k^3) = \alpha^3 \det \boldsymbol{C}$$

For any invertible tensor we show that $\det(\boldsymbol{S}^{\mathrm{C}}) = (\det \boldsymbol{S})^2$

The inverse of tensor $\boldsymbol{S}$,

$$\mathbf{S}^{-1} = (\det \mathbf{S})^{-1}(\mathbf{S}^{\mathrm{cT}})$$

let the scalar $\alpha = \det \mathbf{S}$. We can see clearly that,

$$\mathbf{S}^{\mathrm{c}} = \alpha \mathbf{S}^{-T}$$

Taking the determinant of this equation, we have,

$$\det(\mathbf{S}^{\mathrm{C}}) = \alpha^3 \det(\mathbf{S}^{-T}) = \alpha^3 \det(\mathbf{S}^{-1})$$

as the transpose operation has no effect on the value of a determinant. Noting that the determinant of an inverse is the inverse of the determinant, we have,

$$\det(\boldsymbol{S}^{\mathrm{C}}) = \alpha^3 \det(\boldsymbol{S}^{-1}) = \frac{\alpha^3}{\alpha} = (\det \boldsymbol{S})^2$$

Cofactor

Show that $(\alpha \boldsymbol{S})^{\mathrm{C}} = \alpha^2 \boldsymbol{S}^{\mathrm{C}}$

Ans

$$(\alpha \boldsymbol{S})^{\mathrm{C}} = (\det(\alpha \boldsymbol{S}))(\alpha \boldsymbol{S})^{-\mathrm{T}} = (\alpha^3 \det(\boldsymbol{S}))\alpha^{-1}\boldsymbol{S}^{-\mathrm{T}}$$
$$= (\alpha^2 \det(\boldsymbol{S}))\boldsymbol{S}^{-\mathrm{T}} = \alpha^2 \boldsymbol{S}^{\mathrm{C}}$$

Show that $(\boldsymbol{S}^{-1})^{\mathrm{C}} = (\det \boldsymbol{S})^{-1} \boldsymbol{S}^{\mathrm{T}}$

Ans.

$$(\boldsymbol{S}^{-1})^{\mathrm{C}} = \det(\boldsymbol{S}^{-1})\,(\boldsymbol{S}^{-1})^{-T} = (\det \boldsymbol{S})^{-1} \boldsymbol{S}^{\mathrm{T}}$$

(d) Show that $\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1} = (\det \boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$

Ans.

$$\boldsymbol{S}^{\mathrm{C}} = \det(\boldsymbol{S})\,\boldsymbol{S}^{-T}$$

Consequently,

$$\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1} = (\det \boldsymbol{S})^{-1}(\boldsymbol{S}^{-T})^{-1} = (\det \boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$$

(e) Show that $\left(\boldsymbol{S}^{\mathrm{C}}\right)^{\mathrm{C}} = (\det \boldsymbol{S})\boldsymbol{S}$

Ans.

$$\boldsymbol{S}^{\mathrm{C}} = \det(\boldsymbol{S})\,\boldsymbol{S}^{-T}$$

So that,

$$\left(\boldsymbol{S}^{\mathrm{C}}\right)^{\mathrm{C}} = (\det \boldsymbol{S}^{\mathrm{C}})\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-T} = (\det \boldsymbol{S})^{2}\left[\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1}\right]^{T}$$

$$= (\det \boldsymbol{S})^{2}\left[(\det \boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\right]^{T} = (\det \boldsymbol{S})^{2}(\det \boldsymbol{S})^{-1}\boldsymbol{S} = (\det \boldsymbol{S})\boldsymbol{S}$$

as required.

* 4. Let $\mathbf{\Omega}$ be skew with axial vector $\boldsymbol{\omega}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\mathbf{\Omega u} \times \mathbf{\Omega v} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$ and, hence conclude that $\mathbf{\Omega}^{\mathrm{C}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.
*

$$\begin{aligned}\mathbf{\Omega u} \times \mathbf{\Omega v} &= (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) = (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) \\ &= [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\omega}]\mathbf{v} = [\boldsymbol{\omega} \cdot (\mathbf{u} \times \mathbf{v})]\boldsymbol{\omega} \\ &= (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

But by definition, the cofactor must satisfy,

$$\mathbf{\Omega u} \times \mathbf{\Omega v} = \mathbf{\Omega}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

which compared with the previous equation yields the desired result that

$$\mathbf{\Omega}^{\mathrm{C}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}).$$

Orthogonal Tensors

Given a Euclidean Vector Space $\mathcal{E}$, a tensor $\boldsymbol{Q}$ is said to be orthogonal if, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{E}$,

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}$$

Specifically, we can allow $\boldsymbol{a} = \boldsymbol{b}$, so that

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{a}$$

Or

$$||\boldsymbol{Q}\boldsymbol{a}|| = ||\boldsymbol{a}||$$

In which case the mapping leaves the magnitude unaltered.

Orthogonal Tensors

Let $\boldsymbol{q} = \boldsymbol{Q}\boldsymbol{a}$

$$(\boldsymbol{Q}\boldsymbol{a}) \cdot (\boldsymbol{Q}\boldsymbol{b}) = \boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

By definition of the transpose, we have that,

$$\boldsymbol{q} \cdot \boldsymbol{Q}\boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{q} = \boldsymbol{b} \cdot \boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{a} = \boldsymbol{b} \cdot \boldsymbol{a}$$

Clearly, $\boldsymbol{Q}^T\boldsymbol{Q} = \mathbf{1}$

A condition necessary and sufficient for a tensor $\boldsymbol{Q}$ *to be orthogonal is that* $\boldsymbol{Q}$ *be invertible and its inverse equal to its transpose.*

Orthogonal

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{Q}\right) = \left(\det\boldsymbol{Q}^{\boldsymbol{T}}\right)(\det\boldsymbol{Q}) = (\det\boldsymbol{Q})^2 = 1$$

Which clearly shows that

$$(\det\boldsymbol{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "*proper orthogonal*".

Rotation & Reflection

A rotation is a proper orthogonal tensor while a reflection is not.

Rotation

* **Let $\boldsymbol{Q}$ be a rotation. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$**

This question is the same as showing that the cofactor of $\boldsymbol{Q}$ is $\boldsymbol{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\boldsymbol{T}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

where

$$\mathbf{T} = \text{cof}(\boldsymbol{Q}) = \det(\boldsymbol{Q})\, \boldsymbol{Q}^{-\mathrm{T}}$$

Now that $\boldsymbol{Q}$ is a rotation, $\det(\boldsymbol{Q}) = 1$, and

$$\boldsymbol{Q}^{-\mathrm{T}} = (\boldsymbol{Q}^{-1})^{T} = (\boldsymbol{Q}^{\mathrm{T}})^{T} = \boldsymbol{Q}$$

This implies that $\boldsymbol{T} = \boldsymbol{Q}$ and consequently,

$$\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

For a proper orthogonal tensor Q, show that the eigenvalue equation always yields an eigenvalue of +1. This means that there is always a solution for the equation,

$$\boldsymbol{Qu} = \boldsymbol{u}$$

For any invertible tensor,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

For a proper orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = 1$. It therefore follows that,

$$\mathbf{Q}^{\mathrm{C}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

It is easily shown that $\mathrm{tr}\mathbf{Q}^{\mathrm{C}} = I_2(\mathbf{Q})$ (HW Show this Romano 26)

Characteristic equation for $\boldsymbol{Q}$ is,

$$\det(\mathbf{Q} - \lambda\mathbf{1}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

Or,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

Which is obviously satisfied by $\lambda = 1$.

The Eigenvalue Problem

Recall that a tensor $\boldsymbol{T}$ is a linear transformation for $\boldsymbol{u} \in \mathscr{V}$

$$\boldsymbol{T}: \mathscr{V} \to \mathscr{V}$$

states that $\exists\ \boldsymbol{w} \in \mathscr{V}$ such that,

$$\boldsymbol{T}\boldsymbol{u} \equiv \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{w}$$

Generally, $\boldsymbol{u}$ and its image, $\boldsymbol{w}$ are independent vectors for an arbitrary tensor $\boldsymbol{T}$. The eigenvalue problem considers the special case when there is a linear dependence between $\boldsymbol{u}$ and $\boldsymbol{w}$.

Eigenvalue Problem

Here the image $\boldsymbol{w} = \lambda\boldsymbol{u}$ where $\lambda \in \mathscr{R}$

$$\boldsymbol{Tu} = \lambda\boldsymbol{u}$$

The vector $\boldsymbol{u}$, if it can be found, that satisfies the above equation, is called an eigenvector while the scalar λ is its corresponding eigenvalue.

The eigenvalue problem examines the existence of the eigenvalue and the corresponding eigenvector as well as their consequences.

In order to obtain such solutions, it is useful to write out this equation in its component form:

$$T^i_j u^j \mathbf{g}_i = \lambda u^i \mathbf{g}_i$$

so that,

$$\left(T^i_j - \lambda \delta^i_j\right) u^j \mathbf{g}_i = \mathbf{o}$$

the zero vector. Each component must vanish identically so that we can write

$$\left(T^i_j - \lambda \delta^i_j\right) u^j = 0$$

From the fundamental law of algebra, the above equations can only be possible for arbitrary values of u^j if the determinant,

$$\left|T_j^i - \lambda\delta_j^i\right|$$

Vanishes identically. Which, when written out in full, yields,

$$\begin{vmatrix} T_1^1 - \lambda & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 - \lambda & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 - \lambda \end{vmatrix} = 0$$

Expanding, we have,

$$-T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 + T_1^1T_2^2T_3^3 + T_2^1T_1^2\lambda - T_1^1T_2^2\lambda + T_3^1T_1^3\lambda + T_3^2T_2^3\lambda - T_1^1T_3^3\lambda - T_2^2T_3^3\lambda + T_1^1\lambda^2 + T_2^2\lambda^2 + T_3^3\lambda^2 - \lambda^3 = 0$$

$$= -T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 + T_1^1T_2^2T_3^3 + \left(T_2^1T_1^2 - T_1^1T_2^2 + T_3^1T_1^3 + T_3^2T_2^3 - T_1^1T_3^3 - T_2^2T_3^3\right)\lambda + \left(T_1^1 + T_2^2 + T_3^3\right)\lambda^2 - \lambda^3 = 0$$

or

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Principal Invariants Again

- This is the characteristic equation for the tensor $\boldsymbol{T}$. From here we are able, in the best cases, to find the three eigenvalues. Each of these can be used in to obtain the corresponding eigenvector.
- The above coefficients are the same invariants we have seen earlier!

Positive Definite Tensors

A tensor $\boldsymbol{T}$ is Positive Definite if for all $\boldsymbol{u} \in \mathscr{V}$,

$$\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{u} > 0$$

It is easy to show that the eigenvalues of a symmetric, positive definite tensor are all greater than zero. (HW: Show this, and its converse that if the eigenvalues are greater than zero, the tensor is symmetric and positive definite. Hint, use the spectral decomposition.)

Cayley- Hamilton Theorem

* We now state without proof (See Dill for proof) the important ***Caley-Hamilton*** theorem: Every tensor satisfies its own characteristic equation. That is, the characteristic equation not only applies to the eigenvalues but must be satisfied by the tensor $\mathbf{T}$ itself. This means,

$$\mathbf{T}^3 - I_1\mathbf{T}^2 + I_2\mathbf{T} - I_3\mathbf{1} = \boldsymbol{O}$$

is also valid.

* This fact is used in continuum mechanics to obtain the ***spectral decomposition*** of important material and spatial tensors.