1. The trilinear mapping, $[\,,\,,\,]: \mathscr{V} \times \mathscr{V} \times \mathscr{V} \to \mathscr{R}$ from the product set $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to real space is defined by: $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

In component form,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = e_{ijk} a_i b_j c_k$$

Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield,

$$\begin{aligned} e_{jki} b_j c_k a_i &= [\mathbf{b}, \mathbf{c}, \mathbf{a}] = e_{ijk} b_i c_j a_k \\ &= e_{kij} c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = e_{ijk} c_i a_j b_k \end{aligned}$$

The other results follow from antisymmetric arrangements and the nature of e_{ijk}.

2. Given that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are linearly dependent.

Suppose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that,

$$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= e_{ijk} a_i b_j c_k = e_{ijk}(\alpha b_i + \beta c_i) b_j c_k \\ &= \alpha e_{ijk} b_i b_j c_k + \beta e_{ijk} c_i b_j c_k \\ &= 0 \end{aligned}$$

Note that $b_i b_j c_k$ is symmetric in i and j, $c_i b_j c_k$ is symmetric in i and k and e_{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.

3. Show that the product of a symmetric and an antisymmetric object vanishes.

Consider the product sum, $e_{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and e_{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here.

$$e_{ijk} b_i b_j c_k = -e_{jik} b_i b_j c_k = -e_{jik} b_j b_i c_k = -e_{ijk} b_i b_j c_k = 0$$

The first equality on account of the antisymmetry of e_{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself.

This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.

4. Show that the product

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathbf{B}$$

Can be written in indicial notation as, $a_{ij}a_{jk} = b_{ik}$.

To show this, apply summation convention and see that,

for $i = 1,\ k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1,\ k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1,\ k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2,\ k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2,\ k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2,\ k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3,\ k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3,\ k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3,\ k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

It is necessary to go through this manual process to gain the experience of seeing this transparently in future. Similary, $\mathbf{AA}^{\mathrm{T}} = \mathbf{B}$ can be written in indicial notation as, $a_{ij}a_{kj} = b_{ik}$ which again becomes clear after a manual expansion after invoking the summation convention.

5. If $\forall\, \mathbf{v} \in \mathscr{V}$, $\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\mathbf{a} = \mathbf{b}$; If $\forall\, \mathbf{v} \in \mathscr{V}$, $\mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$, Show that $\mathbf{a} = \mathbf{b}$

$\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} \Rightarrow (\mathbf{a} - \mathbf{b})\mathbf{v} = \mathbf{o}$. Since $\mathbf{v}$ is arbitrary, let $\mathbf{v} = \mathbf{a} - \mathbf{b}$. We therefore have

$$|\mathbf{a}-\mathbf{b}|^2 = \mathbf{0}$$

$$\Rightarrow\ \mathbf{a}-\mathbf{b} = \mathbf{o}$$

It is only the zero vector that has a magnitude of zero. Therefore $\mathbf{a}=\mathbf{b}$.

Secondly, we are given that $\forall\ \mathbf{v} \in \mathscr{V}, \boldsymbol{a}\times\mathbf{v} = \mathbf{b}\times\mathbf{v}$,

Now take a dot product with $\boldsymbol{a}$, we have that,

$$\boldsymbol{a}\cdot\mathbf{b}\times\mathbf{v} = (\boldsymbol{a}\times\mathbf{b})\cdot\mathbf{v} = 0 = \mathbf{o}\cdot\mathbf{v}$$

for all $\mathbf{v}$ proving from the first part, that $\boldsymbol{a}\times\mathbf{b}=\mathbf{o}$. This shows that $\boldsymbol{a}\times$ $\mathbf{b}$ are collinear. We can therefore write that $\mathbf{b}=\alpha\boldsymbol{a}$

Hence, $\boldsymbol{a}\times\mathbf{v} = \mathbf{b}\times\mathbf{v} = \alpha\boldsymbol{a}\times\mathbf{v}$ where α is a scalar. So that

$$(\boldsymbol{a}\times\mathbf{v})(1-\alpha) = \mathbf{o} \Rightarrow 1 = \alpha$$

showing that $\boldsymbol{a}=\mathbf{b}$ as was required.

6. Given that,

$$e_{rst}e_{ijk} = \begin{vmatrix}\delta_{ri} & \delta_{rj} & \delta_{rk}\\ \delta_{si} & \delta_{sj} & \delta_{sk}\\ \delta_{ti} & \delta_{tj} & \delta_{tk}\end{vmatrix}$$ **Show that** $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj}-\delta_{si}\delta_{rj}$

Expanding the equation, we have:

$$\begin{aligned} e_{rsk}e_{ijk} &= \delta_{ki}\begin{vmatrix}\delta_{rj} & \delta_{rk}\\ \delta_{sj} & \delta_{sk}\end{vmatrix} - \delta_{kj}\begin{vmatrix}\delta_{ri} & \delta_{rk}\\ \delta_{si} & \delta_{sk}\end{vmatrix} + 3\begin{vmatrix}\delta_{ri} & \delta_{rj}\\ \delta_{si} & \delta_{sj}\end{vmatrix}\\ &= \delta_{ki}(\delta_{rj}\delta_{sk}-\delta_{sj}\delta_{rk}) - \delta_{kj}(\delta_{ri}\delta_{sk}-\delta_{si}\delta_{rk})\\ &\quad + 3(\delta_{ri}\delta_{sj}-\delta_{si}\delta_{rj})\\ &= \delta_{rj}\delta_{si}-\delta_{sj}\delta_{ri}-\delta_{ri}\delta_{sj}+\delta_{si}\delta_{rj}+3(\delta_{ri}\delta_{sj}-\delta_{si}\delta_{rj})\\ &= \delta_{ri}\delta_{sj}-\delta_{si}\delta_{rj}\end{aligned}$$

7. Given that that $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj}-\delta_{si}\delta_{rj}$, show that $e_{rjk}e_{ijk} = 2\delta_{ri}$

Contracting one more index, we have:

$$\begin{aligned} e_{rjk}e_{ijk} &= \delta_{ri}\delta_{jj}-\delta_{ji}\delta_{rj}\\ &= 3\delta_{ri}-\delta_{ri}\\ &= 2\delta_{ri}\end{aligned}$$

These results are useful in several situations.

8. Show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.

Let $\mathbf{z} = \mathbf{v}\times\mathbf{w} = e_{ijk}v_i w_j \mathbf{e}_k$

$$\begin{aligned}\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\times\mathbf{z} &= e_{\alpha\beta\gamma}u_\alpha z_\beta \mathbf{e}_\gamma = e_{\alpha\beta\gamma}u_\alpha z_\beta \mathbf{e}_\gamma \\ &= \epsilon_{\alpha\beta\gamma}u^\alpha \epsilon^{ij\beta} v_i w_j \mathbf{g}^\gamma = e_{ij\beta}e_{\gamma\alpha\beta}u_\alpha v_i w_j \mathbf{e}_\gamma \\ &= \left(\delta_{i\gamma}\delta_{j\alpha} - \delta_{i\alpha}\delta_{j\gamma}\right)u_\alpha v_i w_j \mathbf{e}_\gamma \\ &= u_j v_\gamma w_j \mathbf{e}_\gamma - u_i v_i w_\gamma \mathbf{e}_\gamma \\ &= (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\end{aligned}$$

9. . In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i\mathbf{e}_i$. Show that (a) $x_{,j} = \frac{x_j}{x}$, (b) $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$, (c) $x_{,ii} = \frac{2}{x}$, (d) If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ $U_{,ii} = 0$ and $\operatorname{div}\left(\frac{\mathbf{r}}{x}\right) = \frac{2}{x}$.

$(a)\ \ x = \sqrt{x_i x_i}$

$$x_{,j} = \frac{\partial\sqrt{x_i x_i}}{\partial x_j} = \frac{\partial\sqrt{x_i x_i}}{\partial(x_i x_i)}\times\frac{\partial(x_i x_i)}{\partial x_j} = \frac{1}{2\sqrt{x_i x_i}}\left[x_i\delta_{ij} + x_i\delta_{ij}\right] = \frac{x_j}{x}.$$

$$\begin{aligned}(b)\ x_{,ij} &= \frac{\partial}{\partial x_j}\left(\frac{\partial\sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right) = \frac{x\frac{\partial x_i}{\partial x_j} - x_i\frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2} \\ &= \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}\end{aligned}$$

$$(c)\ \ x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3} = \frac{3}{x} - \frac{(x)^2}{(x)^3} = \frac{2}{x}.$$

$(d)\ U = \frac{1}{x}$ so that

$$U_{,j} = \frac{\partial\frac{1}{x}}{\partial x_j} = \frac{\partial\frac{1}{x}}{\partial x}\times\frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$

Consequently,

$$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right) = \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i\frac{\partial}{\partial x_j}(x^3)}{x^6}$$

$$= \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial(x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6} = \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6}$$

$$= \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

$$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5} = \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$

$$\text{div}\left(\frac{\mathbf{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x}x_{j,j} + \left(\frac{1}{x}\right)_{,j} = \frac{3}{x} + x_j\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\frac{dx}{dx_j}\right)$$

$$= \frac{3}{x} + x_j\left[-\left(\frac{1}{x^2}\right)\frac{x_j}{x}\right] = \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$

10. Show that $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0$

Using the index: $\mathbf{u}\times\mathbf{v} = e_{ijk}u_i v_j \mathbf{e}_k$

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = u_\alpha \mathbf{e}_\alpha \cdot (e_{ijk}u_i v_j \mathbf{e}_k)$$
$$= u_\alpha(e_{ijk}u_i v_j)\, \mathbf{e}_\alpha \cdot \mathbf{e}_k$$

But the product $\mathbf{e}_\alpha \cdot \mathbf{e}_k = \delta_{\alpha k}$. Therefore

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = u_\alpha(e_{ijk}u_i v_j)\delta_{\alpha k}$$

Simplifying: $u_\alpha \delta_{\alpha k} = u_k$

$$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = e_{ijk}u_i v_j u_k = 0$$

The expression becomes zero because indexes i, k are symmetrical in one term and anti symmetrical in the other.

11. Show that $e_{ijk}u_i v_j u_k = 0$

In the expression, $e_{ijk}u_i v_j u_k$, swapping i and k in the alternating symbol, we have,

$$e_{ijk}u_i v_j u_k = -e_{kji}u_i v_j u_k$$
$$= -e_{kji}u_k v_j u_i$$

Since a swapping of u_i with u_k does not alter the equation. But we observe that i and k are dummy indices. Consequently, using one for the other throughout the equation leaves things unchanged. Therefore,

$$\begin{aligned} e_{ijk}u_i v_j u_k &= -e_{kji}u_i v_j u_k \\ &= -e_{kji}u_k v_j u_i \\ &= -e_{ijk}u_i v_j u_k \end{aligned}$$

Meaning that the expression is the negative of itself. Therefore it must be zero. $e_{ijk}u_i v_j u_k = 0$.

This is a general principle that if you have the product of a symmetric object and an antisymmetric object in the same expression, the expression vanishes.

12. Show that the Kronecker delta expression $\delta_{ii} = 3$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$$

13. Show that $e_{ijk}e_{ijk} = 6$;

$$e_{ijk}e_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji}$$

But we have shown that: $\delta_{ii} = \delta_{jj} = 3$

Using the substitution property of the Kronecker delta, $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$

$$e_{ijk}e_{ijk} = (3 \times 3) - 3 = 9 - 3 = 6$$

14. Explain why $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$

In Cartesian coordinates, by the definition of the cross product, we know that,

$$\begin{aligned} \mathbf{e}_1 \times \mathbf{e}_2 &= \mathbf{e}_3 \\ \mathbf{e}_2 \times \mathbf{e}_3 &= \mathbf{e}_1 \\ \mathbf{e}_3 \times \mathbf{e}_1 &= \mathbf{e}_2 \end{aligned}$$

Furthermore, because a change in the order of a dot product negates the sign,

$$\begin{aligned} \mathbf{e}_2 \times \mathbf{e}_1 &= -\mathbf{e}_3 \\ \mathbf{e}_3 \times \mathbf{e}_2 &= -\mathbf{e}_1 \end{aligned}$$

$$\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$$

Lastly,

$$\mathbf{e}_1 \times \mathbf{e}_1 = 0$$
$$\mathbf{e}_2 \times \mathbf{e}_2 = 0$$
$$\mathbf{e}_3 \times \mathbf{e}_3 = 0$$

The summary of all these results is simply, $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$.

15. Given that $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$, show that $\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = e_{ijk}$

Since we are free to change a dummy index, we have that $\mathbf{e}_i \times \mathbf{e}_j = e_{ij\alpha}\mathbf{e}_\alpha$, take a dot product of both sides with $\mathbf{e}_k$

$$\begin{aligned}\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k &= e_{ij\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_k \\ &= e_{ij\alpha}\delta_{\alpha k} \\ &= e_{ijk}\end{aligned}$$

16. Show that the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in Cartesian coordinates is $e_{ijk}a_i b_j \mathbf{e}_k$. where $\boldsymbol{a} = a_i\mathbf{e}_i$ and $\boldsymbol{b} = b_i\mathbf{e}_i$

Express vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as $\boldsymbol{a} = a_i\mathbf{e}_i$ and $\boldsymbol{b} = b_i\mathbf{e}_i$. Using the above result, we can write that,

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j e_{ijk}\mathbf{e}_k.\end{aligned}$$

17. Simplify the following by employing the substitution properties of the Kronecker delta

$(a) e_{ijk}\delta_{kn}, (b) e_{ijk}\delta_{is}\delta_{jm}\ (c) e_{ijk}\delta_{is}\delta_{jm}\ (d)\ a_{ij}\delta_{in}\ (e)\delta_{ij}\delta_{jn}\ (f)\delta_{ij}\delta_{jn}\delta_{ni}$

$(a)\ e_{ijn}\ (b) e_{sjk}\ (c) e_{smn}\ (d) a_{nj} (e)\delta_{nj}\ (f)\delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{33} = 3$

18. Given that, $I_{ij} = \iiint_V (x_m x_m \delta_{ij} - x_i x_j)\, \rho(x_1, x_2, x_3) dx_1 dx_2 dx_3)$ is the moment of inertia along the axis $i - j$ where x $= x_1, y = x_2, z = x_3$ and $\rho(x_1, x_2, x_3)$ is scalar density of the material find all the components of the tensor.

$$I_{11} = \iiint_V (y^2 + z^2)\, \rho(x, y, z) dxdydz, \quad I_{21} = I_{12} = \iiint_V xy\, \rho(x, y, z) dxdydz,$$

$$I_{22} = \iiint_V (z^2 + x^2)\, \rho(x, y, z) dxdydz, \quad I_{32} = I_{23} = \iiint_V yz\, \rho(x, y, z) dxdydz,$$

$$I_{31} = I_{13} = \iiint_V xy\, \rho(x, y, z) dxdydz, \quad I_{33} = \iiint_V (x^2 + y^2)\, \rho(x, y, z) dxdydz$$

19. Show that the Cylindrical Polar basis vectors,

$$\boldsymbol{e}_r(r, \phi, z) = \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{js}$$

$$\boldsymbol{e}_\phi(r, \phi, z) = -\sin\phi\, \boldsymbol{i} + \cos\phi\, \boldsymbol{j}$$

$$\boldsymbol{e}_z(r, \phi, z) = \boldsymbol{k}$$

constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

$$\|\boldsymbol{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$

$$\|\boldsymbol{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$

$$\|\boldsymbol{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1.

Now lets take them in pairs:

$$\boldsymbol{e}_r \cdot \boldsymbol{e}_\phi = -\cos\phi \sin\phi + \cos\phi \sin\phi = 0$$

$$\boldsymbol{e}_\phi \cdot \boldsymbol{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$

$$\boldsymbol{e}_z \cdot \boldsymbol{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$

So that they are pairwise orthogonal

20. Show that the Spherical Polar basis vectors

$$\boldsymbol{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$$
$$\boldsymbol{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}$$
$$\boldsymbol{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}.$$

Constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

Follow the same procedure as in the above question and obtain the similar result for the spherical polar case.

21. Show that the contraction of a symmetric object with an antisymmetric object equals zero. For example, given that $a_{mn}, m, n = 1,2,3$ is antisymmetric, Show that $a_{mn}x_m x_n = 0$.

(a) It is easily seen that $x_m x_n = x_n x_m$ hence symmetric. If a_{mn} is anti-symmetric, then the contraction $a_{mn}x_m x_n$ must necessarily vanish.

(b) Given that $a_{mn}x_m x_n = 0$ for arbitrary values of $x_n, n = 1,2,3$ then we can write,

$$a_{mn}x_m x_n = -a_{mn}x_m x_n$$

because zero is also a negative of itself. Swapping the roles of x_m and x_n on the RHS of the above, we can write,

$$\begin{aligned} a_{mn}x_m x_n &= -a_{mn}x_m x_n \\ &= -a_{mn}x_n x_m \\ &= -a_{nm}x_n x_m \end{aligned}$$

after swapping the roles of the two dummy indices. We therefore consolidate on the LHS by writing,

$$a_{mn}x_m x_n + a_{nm}x_n x_m = 0$$
$$(a_{mn} + a_{nm})x_m x_n = 0$$

Notice that the quantity in the parenthesis is always symmetric. And also note the contraction of two symmetric tensors can only vanish if one or both tensors vanish. Here, $x_m x_n$ is a product of arbitrary tensors. We are left with the fact that

$$a_{mn} + a_{nm} = 0$$

or,

$$a_{mn} = -a_{nm}$$

which is the definition of anti-symmetry.

22. Noting that $e_{ijk}\sigma_{jk} = 0$ observe that e_{ijk} is perfectly antisymmetric. What does this tell about σ_{jk}?

It tells that σ_{ij} is symmetric. The contraction of a symmetric tensor with an anti-symmetric is zero. Since we know that e_{ijk} is anti-symmetric, the given result of the contraction shows that σ_{ij} is symmetric.

23. Given that A_{mn} and B_{mn} are symmetric, Let $A_{mn}\, x_m x_n = B_{mn} x_m x_n$ for arbitrary values of $x^i, i = 1,2,3$, show that $A_{mn} = B_{mn}$ for all values of $\boldsymbol{m}, \boldsymbol{n}$

We can place the RHS of the equation with the LHS to obtain,

$$(A_{mn} - B_{mn})x_m x_n = 0.$$

By the arguments in Number Next, it is clear than in this case,

$$(A_{mn} - B_{mn}) = 0$$

This proves the desired result that

$$A_{mn} = B_{mn}.$$

24. Given that $a_{ij} = B_i B_j$, where B_1, B_2 and $B_3$are constants Calculate the determinant $\left|a_{ij}\right|$

The determinant

$$|A| = e_{ij} a_{1i} a_{2j} = e_{ij} B_1 B_i B_2 B_j = B_1 B_2 \left(e_{ij} B_i B_j\right) = 0$$

The last equality results again from the fact that the contraction of a symmetric object with an anti-symmetric object results in zero.

25. If A_{ij} is symmetric and B_{ij} is antisymmetric, find the value of $C = A_{ij} B_{ij}$

We are given that,

$C = A_{ij} B_{ij}$

$= A_{ji}B_{ij}$ since A_{ij} is symmetric $= -A_{ji}\, B_{ji}$ since B_{ij} is anti symmetric
$= -A_{ij}B_{ij}$ Interchanging the roles of i and j
$= 0$

Because zero is the only scalar that can be negative of itself. An interchange of dummy indices is a valid step. Note we could not do that for a free index! This result is an important one. The contraction of a symmetric quantity with an anti-symmetric one results in zero.

26. Show that the second-order system T_{ij} can be expressed as the sum of a symmetric system and an anti-symmetric system. Find an expression for these.

We desire to find symmetric and anti-symmetric tensors A_{ij}, B_{ij} respectively such that

$$T_{ij} = A_{ij} + B_{ij}.$$

Let us in fact assume that this is so and see if we can find A_{ij}, B_{ij} with these properties satisfying the equation above. We begin by transposing the equation:

$$T_{ji} = A_{ji} + B_{ji}$$

We now add the two equations to obtain,

$$T_{ij} + T_{ji} = A_{ij} + A_{ji} + B_{ij} + B_{ji} = 2A_{ij}$$

as the last two terms cancel themselves out on account of anti-symmetry while the first two add on account of symmetry. We can therefore see that we have a unique value for A_{ij} that is,

$$A_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)$$

In the same way, a subtraction instead of addition would have led to,

$$T_{ij} - T_{ji} = A_{ij} - A_{ji} + B_{ij} - B_{ji} = 2B_{ij}$$

$$B_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)$$

This is a general rule that any second order indexed quantity can be made the sum of two parts: One symmetric the other, anti-symmetric.

27. The angle $0 \leq \theta \leq \pi$ between two skew lines in space is defined as the angle between their direction vectors when these vectors are placed at the origin. Show that for two lines with direction numbers a_i and $b_i, i = 1; 2; 3$ the cosine of the angle between these lines satisfies

$$\cos\theta = \frac{a_i b_i}{\sqrt{(a_i a_i)}\sqrt{(b_i b_i)}}$$

First note that the dot product of the two line vectors is,

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i = |a||b|\cos\theta = \surd(a_i a_i)\surd(b_i b_i)\cos\theta$$

From where it is obvious that,

$$\cos\theta = \frac{a_i b_i}{\sqrt{(a_i a_i)}\sqrt{(b_i b_i)}}$$

28. Let $\lambda = A_{ij}x_i x_j$ where $A_{ij} = A_{ji}$. **Calculate** $(a)\,\frac{\partial\lambda}{\partial x_m}, (b)\,\frac{\partial^2\lambda}{\partial x_m \partial x_k}$

$$\lambda = A_{ij}x_i x_j = A_{mk}x_m x_k$$

$$\frac{\partial\lambda}{\partial x_m} = A_{mk}x_k + A_{lk}x_l\frac{\partial x_k}{\partial x_m}$$

$$= A_{mk}x_k + A_{lk}x_l\delta_{km} = A_{mk}x_k + A_{lk}x_l\delta_{km}$$

$$= A_{mk}x_k + A_{lm}x_l = 2A_{mk}x_k, \text{and furthermore,}$$

$$\frac{\partial^2\lambda}{\partial x_m \partial x_k} = 2A_{mk}.$$

Remember that in the above we made use of the liberty to alter the dummy indices to conform to the requirements of the derivative. The substitutionary attribute of the Kronecker delta has been used to advantage here.

29. Show that the product of a symmetric and an antisymmetric object vanishes.

Consider the product sum, $\epsilon_{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and ϵ^{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here.

$$\epsilon_{ijk} b_i b_j c_k = -\epsilon_{jik} b_i b_j c_k = -\epsilon_{jik} b_j b_i c_k = -\epsilon_{ijk} b_i b_j c_k = 0$$

The first equality on account of the antisymmetry of ϵ_{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself.

This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.

30. Write the following equations in indicial form: $A_1^2 + A_2^2 + A_3^2 = 0$, and $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$

(i) $A_i A_i = 0$, (ii) $\frac{\partial^2 \phi}{\partial x_i \partial x_i} = 0$