1. **Divergence of a product: Given that $\boldsymbol{\varphi}$ is a scalar field and v a vector field, show that $\mathbf{div}(\boldsymbol{\varphi}\mathrm{v}) = (\mathbf{grad}\boldsymbol{\varphi})\cdot \mathrm{v} + \boldsymbol{\varphi}\,\mathbf{div}\,\mathrm{v}$**

$$\begin{aligned}\mathrm{grad}(\varphi\mathbf{v}) &= (\varphi v^i)_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \varphi_{,j}\, v^i \mathbf{g}_i \otimes \mathbf{g}^j + \varphi v^i{}_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \mathbf{v} \otimes (\mathrm{grad}\,\varphi) + \varphi\,\mathrm{grad}\,\mathbf{v}\end{aligned}$$

Now, $\mathrm{div}(\varphi\mathbf{v}) = \mathrm{tr}\big(\mathrm{grad}(\varphi\mathbf{v})\big)$. Taking the trace of the above, we have:

$$\mathrm{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\mathrm{grad}\,\varphi) + \varphi\,\mathrm{div}\,\mathbf{v}$$

2. **Show that $\mathbf{grad}(\mathrm{u}\cdot\mathrm{v}) = (\mathbf{grad}\,\mathrm{u})^{\mathbf{T}}\mathrm{v} + (\mathbf{grad}\,\mathrm{v})^{\mathbf{T}}\mathrm{u}$**

$\mathbf{u}\cdot\mathbf{v} = u^i v_i$ is a scalar sum of components.

$$\begin{aligned}\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) &= (u^i v_i)_{,j}\,\mathbf{g}^j \\ &= u^i{}_{,j}\, v_i \mathbf{g}^j + u^i v_{i,j}\,\mathbf{g}^j\end{aligned}$$

Now $\mathrm{grad}\,\mathbf{u} = u^i{}_{,j}\,\mathbf{g}_i \otimes \mathbf{g}^j$ swapping the bases, we have that,

$$(\mathrm{grad}\,\mathbf{u})^{\mathrm{T}} = u^i{}_{,j}\,(\mathbf{g}^j \otimes \mathbf{g}_i).$$

Writing $\mathbf{v} = v_k \mathbf{g}^k$, we have that, $(\mathrm{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} = u^i{}_{,j}\, v_k (\mathbf{g}^j \otimes \mathbf{g}_i)\mathbf{g}^k = u^i{}_{,j}\, v_k \mathbf{g}^j \delta_i^k = u^i{}_{,j}\, v_i \mathbf{g}^j$

It is easy to similarly show that $u^i v_{i,j}\,\mathbf{g}^j = (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$. Clearly,

$$\text{grad}(\mathbf{u}\cdot\mathbf{v}) = \left(u^i v_i\right)_{,j}\,\mathbf{g}^j = u^i{}_{,j}\, v_i \mathbf{g}^j + u^i v_{i,j}\,\mathbf{g}^j$$
$$= (\text{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\text{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$$

As required.

3. Show that $\textbf{grad}(\mathrm{u}\times\mathrm{v}) = (\mathrm{u}\times)\textbf{grad}\,\mathrm{v} - (\mathrm{v}\times)\textbf{grad}\,\mathrm{u}$

$$\mathbf{u}\times\mathbf{v} = \epsilon^{ijk}u_j v_k \mathbf{g}_i$$

Recall that the gradient of this vector is the tensor,

$$\text{grad}(\mathbf{u}\times\mathbf{v}) = \left(\epsilon^{ijk}u_j v_k\right)_{,l}\,\mathbf{g}_i\otimes\mathbf{g}^l$$
$$= \epsilon^{ijk}u_{j,l}\, v_k \mathbf{g}_i\otimes\mathbf{g}^l + \epsilon^{ijk}u_j v_{k,l}\,\mathbf{g}_i\otimes\mathbf{g}^l$$
$$= -\epsilon^{ikj}u_{j,l}\, v_k \mathbf{g}_i\otimes\mathbf{g}^l + \epsilon^{ijk}u_j v_{k,l}\,\mathbf{g}_i\otimes\mathbf{g}^l$$
$$= -(\mathbf{v}\times)\text{grad}\,\mathbf{u} + (\mathbf{u}\times)\text{grad}\,\mathbf{v}$$

4. Show that $\textbf{div}\,(\mathrm{u}\times\mathrm{v}) = \mathrm{v}\cdot\textbf{curl}\,\mathrm{u} - \mathrm{u}\cdot\textbf{curl}\,\mathrm{v}$

We already have the expression for $\text{grad}(\mathbf{u}\times\mathbf{v})$ above; remember that

$$\text{div}\,(\mathbf{u}\times\mathbf{v}) = \text{tr}[\text{grad}(\mathbf{u}\times\mathbf{v})]$$
$$= -\epsilon^{ikj}u_{j,l}\, v_k \mathbf{g}_i\cdot\mathbf{g}^l + \epsilon^{ijk}u_j v_{k,l}\,\mathbf{g}_i\cdot\mathbf{g}^l$$
$$= -\epsilon^{ikj}u_{j,l}\, v_k \delta_i^l + \epsilon^{ijk}u_j v_{k,l}\,\delta_i^l$$
$$= -\epsilon^{ikj}u_{j,i}\, v_k + \epsilon^{ijk}u_j v_{k,i} = \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}$$

5. **Given a scalar point function $\boldsymbol{\phi}$ and a vector field** v**, show that curl** $(\boldsymbol{\phi}\mathrm{v}) = \boldsymbol{\phi}$ **curl** $\mathrm{v} + ($**grad** $\boldsymbol{\phi}) \times \mathrm{v}$.

$$\begin{aligned}\text{curl }(\phi\mathbf{v}) &= \epsilon^{ijk}(\phi v_k)_{,j}\,\mathbf{g}_i \\ &= \epsilon^{ijk}\left(\phi_{,j}\, v_k + \phi v_{k,j}\right)\mathbf{g}_i \\ &= \epsilon^{ijk}\phi_{,j}\, v_k\mathbf{g}_i + \epsilon^{ijk}\phi v_{k,j}\,\mathbf{g}_i \\ &= (\text{grad }\phi)\times\mathbf{v} + \phi\text{ curl }\mathbf{v}\end{aligned}$$

6. **Show that div** $(\mathrm{u}\otimes\mathrm{v}) = ($**div** $\mathrm{v})\mathrm{u} + ($**grad** $\mathrm{u})\mathrm{v}$

$\mathbf{u}\otimes\mathbf{v}$ is the tensor, $u^i v^j\mathbf{g}_i\otimes\mathbf{g}_j$. The gradient of this is the third order tensor,

$$\text{grad }(\mathbf{u}\otimes\mathbf{v}) = \left(u^i v^j\right)_{,k}\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mathbf{g}^k$$

And by divergence, we mean the contraction of the last basis vector:

$$\begin{aligned}\text{div }(\mathbf{u}\otimes\mathbf{v}) &= \left(u^i v^j\right)_{,k}\left(\mathbf{g}_i\otimes\mathbf{g}_j\right)\mathbf{g}^k \\ &= \left(u^i v^j\right)_{,k}\mathbf{g}_i\delta_j^k = \left(u^i v^j\right)_{,j}\mathbf{g}_i \\ &= u^i{}_{,j}\, v^j\mathbf{g}_i + u^i v^j{}_{,j}\,\mathbf{g}_i \\ &= (\text{grad }\mathbf{u})\mathbf{v} + (\text{div }\mathbf{v})\mathbf{u}\end{aligned}$$

7. For a scalar field ϕ and a tensor field T show that grad $(\phi\mathrm{T}) = \phi$grad $\mathrm{T} + \mathrm{T} \otimes$ gradϕ. Also show that div $(\phi\mathrm{T}) = \phi$ div $\mathrm{T} + \mathrm{T}$gradϕ

$$\begin{aligned}
\mathrm{grad}(\phi\mathbf{T}) &= \left(\phi T^{ij}\right)_{,k} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\
&= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\
&= \mathbf{T} \otimes \mathrm{grad}\phi + \phi\mathrm{grad}\,\mathbf{T}
\end{aligned}$$

Furthermore, we can contract the last two bases and obtain,

$$\begin{aligned}
\mathrm{div}(\phi\mathbf{T}) &= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^k \\
&= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \delta_j^k \\
&= T^{ik}\phi_{,k}\mathbf{g}_i + \phi T^{ik}{}_{,k}\mathbf{g}_i \\
&= \mathbf{T}\mathrm{grad}\phi + \phi\,\mathrm{div}\,\mathbf{T}
\end{aligned}$$

8. For two arbitrary tensors S and T, show that grad$(\mathrm{ST}) = ($grad $\mathrm{S}^{\mathrm{T}})^{\mathrm{T}}\mathrm{T} + \mathrm{S}$ grad T

$$\begin{aligned}
\mathrm{grad}(\mathbf{ST}) &= \left(S_{ij}T^{jk}\right)_{,\alpha} \mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\
&= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\
&= \left(T^{kj}S_{ji,\alpha} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha \\
&= (\mathrm{grad}\,\mathbf{S}^{\mathrm{T}})^{\mathrm{T}}\mathbf{T} + \mathbf{S}\,\mathrm{grad}\,\mathbf{T}
\end{aligned}$$

9. For two arbitrary tensors S and T, show that div$(\mathrm{ST}) = ($grad $\mathrm{S}) : \mathrm{T} + \mathrm{T}$ div S

$$\mathrm{grad}(\mathbf{ST}) = \left(S_{ij}T^{jk}\right)_{,\alpha} \mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha$$

$$= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \otimes \mathbf{g}_k \otimes \mathbf{g}^\alpha$$

$$\text{div}(\mathbf{ST}) = \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i(\mathbf{g}_k \cdot \mathbf{g}^\alpha)$$

$$= \left(S_{ij,\alpha}T^{jk} + S_{ij}T^{jk}{}_{,\alpha}\right)\mathbf{g}^i \delta_k^\alpha$$

$$= \left(S_{ij,k}T^{jk} + S_{ij}T^{jk}{}_{,k}\right)\mathbf{g}^i$$

$$= (\text{grad}\,\mathbf{S}) : \mathbf{T} + \mathbf{S}\,\text{div}\,\mathbf{T}$$

10. **For two arbitrary vectors,** u **and** v**, show that** $\mathbf{grad}(\mathrm{u} \times \mathrm{v}) = (\mathrm{u}\,\times)\mathbf{grad}\mathrm{v} - (\mathrm{v}\,\times)\mathbf{grad}\mathrm{u}$

$$\text{grad}(\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk}u_j v_k\right)_{,l}\,\mathbf{g}_i \otimes \mathbf{g}^l$$

$$= \left(\epsilon^{ijk}u_{j,l}\,v_k + \epsilon^{ijk}u_j v_{k,l}\right)\mathbf{g}_i \otimes \mathbf{g}^l$$

$$= \left(u_{j,l}\,\epsilon^{ijk}v_k + v_{k,l}\,\epsilon^{ijk}u_j\right)\mathbf{g}_i \otimes \mathbf{g}^l$$

$$= -(\mathbf{v}\,\times)\text{grad}\mathbf{u} + (\mathbf{u}\,\times)\text{grad}\mathbf{v}$$

11. **For a vector field** u**, show that** $\mathbf{grad}(\mathrm{u}\,\times)$ **is a third ranked tensor. Hence or otherwise show that** $\mathbf{div}(\mathrm{u}\,\times) = -\mathbf{curl}\,\mathrm{u}$**.**

The second–order tensor $(\mathbf{u}\,\times)$ is defined as $\epsilon^{ijk}u_j\mathbf{g}_i \otimes \mathbf{g}_k$. Taking the covariant derivative with an independent base, we have

$$\text{grad}(\mathbf{u}\,\times) = \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}^l$$

This gives a third order tensor as we have seen. Contracting on the last two bases,

$$\begin{aligned}\text{div}(\mathbf{u}\times) &= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \otimes \mathbf{g}_k \cdot \mathbf{g}^l \\ &= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \delta_k^l \\ &= \epsilon^{ijk} u_{j,k}\, \mathbf{g}_i \\ &= -\text{curl}\, \mathbf{u}\end{aligned}$$

12. Show that div $(\boldsymbol{\phi}\text{I}) = \textbf{grad}\ \boldsymbol{\phi}$

Note that $\phi\mathbf{I} = \left(\phi g_{\alpha\beta}\right)\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$. Also note that

$$\text{grad}\ \phi\mathbf{I} = \left(\phi g_{\alpha\beta}\right)_{,i} \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \otimes \mathbf{g}^i$$

The divergence of this third order tensor is the contraction of the last two bases:

$$\begin{aligned}\text{div}\ (\phi\mathbf{I}) = \text{tr}(\text{grad}\ \phi\mathbf{I}) &= \left(\phi g_{\alpha\beta}\right)_{,i} \left(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\right)\mathbf{g}^i = \left(\phi g_{\alpha\beta}\right)_{,i} \mathbf{g}^\alpha g^{\beta i} \\ &= \phi_{,i}\, g_{\alpha\beta} g^{\beta i} \mathbf{g}^\alpha \\ &= \ \phi_{,i}\, \delta_\alpha^i \mathbf{g}^\alpha = \ \phi_{,i}\, \mathbf{g}^i = \text{grad}\ \phi\end{aligned}$$

13. Show that curl $(\boldsymbol{\phi}\text{I}) = (\ \textbf{grad}\ \boldsymbol{\phi}) \times$

Note that $\phi\mathbf{I} = \left(\phi g_{\alpha\beta}\right)\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$, and that curl $\mathbf{T} = \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$ so that,

$$\begin{aligned}\text{curl}\ (\phi\mathbf{I}) &= \epsilon^{ijk} (\phi g_{\alpha k})_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha \\ &= \epsilon^{ijk} \left(\phi_{,j}\, g_{\alpha k}\right)\mathbf{g}_i \otimes \mathbf{g}^\alpha = \epsilon^{ijk} \phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k \\ &= (\ \text{grad}\ \phi) \times\end{aligned}$$

14. **Show that the dyad** $\mathrm{u} \otimes \mathrm{v}$ **is NOT, in general symmetric**: $\mathrm{u} \otimes \mathrm{v} = \mathrm{v} \otimes \mathrm{u} - (\mathrm{u} \times \mathrm{v}) \times$

$$\mathbf{u} \times \mathbf{v} = \epsilon^{ijk} u_j v_k \, \mathbf{g}_i$$

$$\begin{aligned}
\big((\mathbf{u} \times \mathbf{v}) \times\big) &= \epsilon_{\alpha i \beta} \epsilon^{ijk} u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= -\left(\delta_\alpha^j \delta_\beta^k - \delta_\alpha^k \delta_\beta^j\right) u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= \left(-u_\alpha v_\beta + u_\beta v_\alpha\right) \mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}
\end{aligned}$$

15. **Show that curl** $(\mathrm{v} \times) = (\mathbf{div}\ \mathrm{v})\mathrm{I} - \mathbf{grad}\ \mathrm{v}$

$$(\mathbf{v} \times) = \epsilon^{\alpha\beta k} v_\beta \, \mathbf{g}_\alpha \otimes \mathbf{g}_k$$

$$\text{curl}\ \mathbf{T} = \epsilon^{ijk} T_{\alpha k}{}_{,j} \, \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

so that

$$\begin{aligned}
\text{curl}\ (\mathbf{v} \times) &= \epsilon^{ijk} \epsilon^{\alpha\beta k} v_\beta{}_{,j} \, \mathbf{g}_i \otimes \mathbf{g}_\alpha \\
&= \left(g^{i\alpha} g^{j\beta} - g^{i\beta} g^{j\alpha}\right) v_\beta{}_{,j} \, \mathbf{g}_i \otimes \mathbf{g}_\alpha \\
&= v^j{}_{,j} \, \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha - v^i{}_{,j} \, \mathbf{g}_i \otimes \mathbf{g}^j \\
&= (\text{div}\ \mathbf{v})\mathbf{I} - \text{grad}\ \mathbf{v}
\end{aligned}$$

16. Show that div $(\mathrm{u} \times \mathrm{v}) = \mathrm{v} \cdot \mathbf{curl}\, \mathrm{u} - \mathrm{u} \cdot \mathbf{curl}\, \mathrm{v}$

$$\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk} u_j v_k\right)_{,i}$$

Noting that the tensor ϵ^{ijk} behaves as a constant under a covariant differentiation, we can write,

$$\begin{aligned}\operatorname{div}(\mathbf{u} \times \mathbf{v}) &= \left(\epsilon^{ijk} u_j v_k\right)_{,i} \\ &= \epsilon^{ijk} u_{j,i}\, v_k + \epsilon^{ijk} u_j v_{k,i} \\ &= \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}\end{aligned}$$

17. Given a scalar point function $\boldsymbol{\phi}$ **and a vector field** v**, show that curl** $(\boldsymbol{\phi}\mathrm{v}) = \boldsymbol{\phi}$ **curl** $\mathrm{v} + (\mathbf{grad}\, \boldsymbol{\phi}) \times \mathrm{v}$.

$$\begin{aligned}\operatorname{curl}(\phi \mathbf{v}) &= \epsilon^{ijk} (\phi v_k)_{,j}\, \mathbf{g}_i \\ &= \epsilon^{ijk} \left(\phi_{,j}\, v_k + \phi v_{k,j}\right) \mathbf{g}_i \\ &= \epsilon^{ijk} \phi_{,j}\, v_k \mathbf{g}_i + \epsilon^{ijk} \phi v_{k,j}\, \mathbf{g}_i \\ &= (\operatorname{grad} \phi) \times \mathbf{v} + \phi \operatorname{curl} \mathbf{v}\end{aligned}$$

18. Show that curl $(\mathbf{grad}\, \boldsymbol{\phi}) = 0$

For any tensor $\mathbf{v} = v_\alpha \mathbf{g}^\alpha$

$$\operatorname{curl} \mathbf{v} = \epsilon^{ijk} v_{k,j}\ \mathbf{g}_i$$

Let $\mathbf{v} =$ grad ϕ. Clearly, in this case, $v_k = \phi_{,k}$ so that $v_{k,j} = \phi_{,kj}$. It therefore follows that,

$$\text{curl}\ (\text{grad}\ \phi) = \epsilon^{ijk}\phi_{,kj}\,\mathbf{g}_i = \mathbf{o}.$$

The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the scalar field is immaterial. This will be true only if the scalar field is continuous – a proposition we have assumed in the above.

19. **Show that curl** $(\mathbf{grad}\ \text{v}) = 0$

For any tensor $\mathbf{T} = \text{T}_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}$

$$\text{curl}\ \mathbf{T} = \epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^{\alpha}$$

Let $\mathbf{T} =$ grad $\mathbf{v}$. Clearly, in this case, $\text{T}_{\alpha\beta} = v_{\alpha,\beta}$ so that $T_{\alpha k,j} = v_{\alpha,kj}$. It therefore follows that,

$$\text{curl}\ (\text{grad}\ \mathbf{v}) = \epsilon^{ijk}v_{\alpha,kj}\,\mathbf{g}_i \otimes \mathbf{g}^{\alpha} = \mathbf{0}.$$

The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the vector field is immaterial. This will be true only if the vector field is continuous – a proposition we have assumed in the above.

20. Show that curl $(\mathbf{grad}\ \mathrm{v})^{\mathbf{T}} = \mathbf{grad}(\mathbf{curl}\ \mathrm{v})$

From previous derivation, we can see that, curl $\mathbf{T} = \epsilon^{ijk} T_{\alpha k},_j\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$. Clearly,

$$\text{curl } \mathbf{T}^{\mathrm{T}} = \epsilon^{ijk} T_{k\alpha},_j\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

so that curl (grad $\mathbf{v})^{\mathrm{T}} = \epsilon^{ijk} v_{k},_{\alpha j}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha$. But curl $\mathbf{v} = \epsilon^{ijk} v_k,_j\ \mathbf{g}_i$. The gradient of this is,

$$\text{grad}(\text{curl } \mathbf{v}) = \left(\epsilon^{ijk} v_k,_j\right),_\alpha \mathbf{g}_i \otimes \mathbf{g}^\alpha = \epsilon^{ijk} v_k,_{j\alpha}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha = \text{curl (grad } \mathbf{v})^{\mathrm{T}}$$

21. Show that div $(\mathbf{grad}\ \boldsymbol{\phi} \times \mathbf{grad}\ \boldsymbol{\vartheta}) = \mathbf{0}$

$$\text{grad } \phi \times \text{grad } \theta = \epsilon^{ijk} \phi,_j\, \theta,_k\, \mathbf{g}_i$$

The gradient of this vector is the tensor,

$$\begin{aligned}\text{grad}(\text{grad } \phi \times \text{grad } \theta) &= \left(\epsilon^{ijk} \phi,_j\, \theta,_k\right),_l\, \mathbf{g}_i \otimes \mathbf{g}^l \\ &= \epsilon^{ijk} \phi,_{jl}\, \theta,_k\, \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} \phi,_j\, \theta,_{kl}\, \mathbf{g}_i \otimes \mathbf{g}^l\end{aligned}$$

The trace of the above result is the divergence we are seeking:

$$\begin{aligned}\text{div (grad } \phi \times \text{grad } \theta) &= \text{tr}[\text{grad}(\text{grad } \phi \times \text{grad } \theta)] \\ &= \epsilon^{ijk} \phi,_{jl}\, \theta,_k\, \mathbf{g}_i \cdot \mathbf{g}^l + \epsilon^{ijk} \phi,_j\, \theta,_{kl}\, \mathbf{g}_i \cdot \mathbf{g}^l \\ &= \epsilon^{ijk} \phi,_{jl}\, \theta,_k\, \delta_i^l + \epsilon^{ijk} \phi,_j\, \theta,_{kl}\, \delta_i^l \\ &= \epsilon^{ijk} \phi,_{ji}\, \theta,_k + \epsilon^{ijk} \phi,_j\, \theta,_{ki} = 0\end{aligned}$$

Each term vanishing on account of the contraction of a symmetric tensor with an antisymmetric.

22. Show that curl curl v = grad(div v) − grad^{2}v

Let $\mathbf{w} = \text{curl}\ \mathbf{v} \equiv \epsilon^{ijk} v_{k},_{j}\ \mathbf{g}_i$. But curl $\mathbf{w} \equiv \epsilon^{\alpha\beta\gamma} w_{\gamma},_{\beta}\ \mathbf{g}_{\alpha}$. Upon inspection, we find that $w_\gamma = g_{\gamma i}\epsilon^{ijk} v_{k},_{j}$ so that

$$\text{curl}\ \mathbf{w} \equiv \epsilon^{\alpha\beta\gamma}\left(g_{\gamma i}\epsilon^{ijk} v_{k},_{j}\right),_{\beta}\ \mathbf{g}_{\alpha} = g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} v_{k},_{j\beta}\ \mathbf{g}_{\alpha}$$

Now, it can be shown that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$ so that,

$$\begin{aligned}\text{curl}\ \mathbf{w} &= \left(g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}\right) v_{k},_{j\beta}\ \mathbf{g}_{\alpha} \\ &= v^{\beta},_{j\beta}\ \mathbf{g}^{j} - g^{\beta j} v^{\alpha},_{j\beta}\ \mathbf{g}_{\alpha} \\ &= \text{grad}(\text{div}\ \mathbf{v}) - \text{grad}^2\mathbf{v}\end{aligned}$$

Also recall that the Laplacian (grad^2) of a scalar field ϕ is, $\text{grad}^2\phi = g^{ij}\phi,_{ij}$. In Cartesian coordinates, this becomes,

$$\text{grad}^2\phi = g^{ij}\phi,_{ij} = \delta_{ij}\ \phi,_{ij} = \ \phi,_{ii}$$

as the unit (metric) tensor now degenerates to the Kronecker delta in this special case. For a vector field, $\text{grad}^2\mathbf{v} = g^{\beta j} v^{\alpha},_{j\beta}\ \mathbf{g}_{\alpha}$.

Also note that while grad is a vector operator, the Laplacian (grad^2) is a scalar operator.

23. Show that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$

Note that

$$g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g_{\gamma i}\begin{vmatrix} g^{i\alpha} & g^{i\beta} & g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix} = \begin{vmatrix} g_{\gamma i}g^{i\alpha} & g_{\gamma i}g^{i\beta} & g_{\gamma i}g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_\gamma^\alpha & \delta_\gamma^\beta & \delta_\gamma^\gamma \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \delta_\gamma^\alpha \begin{vmatrix} g^{j\beta} & g^{j\gamma} \\ g^{k\beta} & g^{k\gamma} \end{vmatrix} - \delta_\gamma^\beta \begin{vmatrix} g^{j\alpha} & g^{j\gamma} \\ g^{k\alpha} & g^{k\gamma} \end{vmatrix} + \delta_\gamma^\gamma \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= \begin{vmatrix} g^{j\beta} & g^{j\alpha} \\ g^{k\beta} & g^{k\alpha} \end{vmatrix} - \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} + 3\begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} = \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$$

24. Given that $\boldsymbol{\varphi}(\boldsymbol{t}) = |\mathrm{A}(\boldsymbol{t})|$, Show that $\dot{\boldsymbol{\varphi}}(\boldsymbol{t}) = \frac{\mathrm{A}}{|\mathrm{A}(\boldsymbol{t})|} : \dot{\mathrm{A}}$

$$\varphi^2 \equiv \mathbf{A}: \mathbf{A}$$

Now,

$$\frac{d}{dt}(\varphi^2) = 2\varphi\frac{d\varphi}{dt} = \frac{d\mathbf{A}}{dt}: \mathbf{A} + \mathbf{A}: \frac{d\mathbf{A}}{dt} = 2\mathbf{A}: \frac{d\mathbf{A}}{dt}$$

as inner product is commutative. We can therefore write that

$$\frac{d\varphi}{dt} = \frac{\mathbf{A}}{\varphi} : \frac{d\mathbf{A}}{dt} = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$$

as required.

25. Given a tensor field T**, obtain the vector** $\mathrm{w} \equiv \mathrm{T}^{\mathbf{T}}\mathrm{v}$ **and show that its divergence is** $\mathrm{T}: (\boldsymbol{\nabla}\mathrm{v}) + \mathrm{v} \cdot \textbf{div}\ \mathrm{T}$

The gradient of $\mathbf{w}$ is the tensor, $\left(T_{ji}v^j\right)_{,k}\mathbf{g}^i \otimes \mathbf{g}^k$. Therefore, divergence of $\boldsymbol{w}$ (the trace of the gradient) is the scalar sum, $T_{ji}v^j{}_{,k}\, g^{ik} + T_{ji,k}\, v^j g^{ik}$. Expanding, we obtain,

$$\begin{aligned}
\operatorname{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) &= T_{ji}v^j{}_{,k}\, g^{ik} + T_{ji,k}\, v^j g^{ik} \\
&= T_j^k{}_{,k}\, v^j + T_j^k v^j{}_{,k} \\
&= (\operatorname{div}\mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^{\mathrm{T}}\operatorname{grad}\mathbf{v}) \\
&= (\operatorname{div}\mathbf{T}) \cdot \mathbf{v} + \mathbf{T}: (\operatorname{grad}\mathbf{v})
\end{aligned}$$

Recall that scalar product of two vectors is commutative so that

$$\operatorname{div}\left(\mathbf{T}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{T}: (\operatorname{grad}\mathbf{v}) + \mathbf{v} \cdot \operatorname{div}\mathbf{T}$$

26. **For a second-order tensor** T **define curl** $\mathrm{T} \equiv \boldsymbol{\epsilon^{ijk} T_{\alpha k,j}}\ \mathrm{g}_{\boldsymbol{i}} \otimes \mathrm{g}^{\boldsymbol{\alpha}}$ **show that for any constant vector** a**,** $(\mathbf{curl}\ \mathrm{T})\ a = \mathbf{curl}\ (\mathrm{T}^{\mathbf{T}} a)$

Express vector $\boldsymbol{a}$ in the invariant form with covariant components as $\boldsymbol{a} = a^{\beta}\mathbf{g}_{\beta}$. It follows that

$$
\begin{aligned}
(\text{curl } \mathbf{T})\ \boldsymbol{a} &= \epsilon^{ijk} T_{\alpha k,j}\ (\mathbf{g}_i \otimes \mathbf{g}^{\alpha})\boldsymbol{a} \\
&= \epsilon^{ijk} T_{\alpha k,j}\ a^{\beta}(\mathbf{g}_i \otimes \mathbf{g}^{\alpha})\mathbf{g}_{\beta} \\
&= \epsilon^{ijk} T_{\alpha k,j}\ a^{\beta}\mathbf{g}_i \delta^{\alpha}_{\beta} \\
&= \epsilon^{ijk} (T_{\alpha k})_{,j}\ \mathbf{g}_i a^{\alpha} \\
&= \epsilon^{ijk} (T_{\alpha k} a^{\alpha})_{,j}\ \mathbf{g}_i
\end{aligned}
$$

The last equality resulting from the fact that vector $\boldsymbol{a}$ is a constant vector. Clearly,

$$(\text{curl } \mathbf{T})\ \boldsymbol{a} = \text{curl } (\mathbf{T}^{\mathrm{T}}\boldsymbol{a})$$

27. **For any two vectors** u **and** v**, show that** $\mathbf{curl}\ (\mathrm{u} \otimes \mathrm{v}) = [(\mathbf{grad}\ \mathrm{u})\mathrm{v}\, \times]^{\boldsymbol{T}} + (\mathbf{curl}\ \mathrm{v}) \otimes u$ **where** $\mathrm{v}\, \times$ **is the skew tensor** $\boldsymbol{\epsilon^{ikj} v_k}\ \mathrm{g}_{\boldsymbol{i}} \otimes \mathrm{g}_{\boldsymbol{j}}$**.**

Recall that the curl of a tensor $\boldsymbol{T}$ is defined by curl $\boldsymbol{T} \equiv \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^{\alpha}$. Clearly therefore,

$$
\begin{aligned}
\text{curl } (\mathbf{u} \otimes \mathbf{v}) &= \epsilon^{ijk} (u_{\alpha} v_k)_{,j}\ \mathbf{g}_i \otimes \mathbf{g}^{\alpha} \\
&= \epsilon^{ijk} \left(u_{\alpha,j}\, v_k + u_{\alpha} v_{k,j}\right) \mathbf{g}_i \otimes \mathbf{g}^{\alpha}
\end{aligned}
$$

$$= \epsilon^{ijk} u_{\alpha},_j\, v_k\, \mathbf{g}_i \otimes \mathbf{g}^\alpha + \epsilon^{ijk} u_\alpha v_k,_j\ \ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$
$$= \left(\epsilon^{ijk} v_k\, \mathbf{g}_i\right) \otimes \left(u_{\alpha},_j\, \mathbf{g}^\alpha\right) + \left(\epsilon^{ijk} v_k,_j\ \ \mathbf{g}_i\right) \otimes (u_\alpha \mathbf{g}^\alpha)$$
$$= \left(\epsilon^{ijk} v_k\, \mathbf{g}_i \otimes \mathbf{g}_j\right)\left(u_{\alpha},_\beta\, \mathbf{g}^\beta \otimes \mathbf{g}^\alpha\right) + \left(\epsilon^{ijk} v_k,_j\ \ \mathbf{g}_i\right) \otimes (u_\alpha \mathbf{g}^\alpha)$$
$$= -(\mathbf{v}\,\times)(\text{grad}\,\mathbf{u})^{\boldsymbol{T}} + \ (\text{curl}\,\mathbf{v}) \otimes \mathbf{u}$$
$$= [(\text{grad}\,\mathbf{u})\mathbf{v}\,\times]^{\boldsymbol{T}} + \ (\text{curl}\,\mathbf{v}) \otimes \mathbf{u}$$

upon noting that the vector cross is a skew tensor.

28. Show that curl (u × v) = div(u ⊗ v − v ⊗ u)

The vector $\mathbf{w} \equiv \mathbf{u} \times \mathbf{v} = w_{\boldsymbol{k}} \mathbf{g}^{\boldsymbol{k}} = \epsilon_{k\alpha\beta} u^\alpha v^\beta \mathbf{g}^{\boldsymbol{k}}$ and curl $\mathbf{w} = \boldsymbol{\epsilon}^{\boldsymbol{ijk}} w_k,_j\ \mathbf{g}_i$. Therefore,

$$\begin{aligned}
\text{curl}\,(\mathbf{u} \times \mathbf{v}) &= \epsilon^{ijk} w_k,_j\ \mathbf{g}_i \\
&= \epsilon^{ijk} \epsilon_{k\alpha\beta} \left(u^\alpha v^\beta\right),_j \mathbf{g}_i \\
&= \left(\delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha\right)\left(u^\alpha v^\beta\right),_j \mathbf{g}_i \\
&= \left(\delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha\right)\left(u^\alpha,_j\, v^\beta + u^\alpha v^\beta,_j\,\right)\mathbf{g}_i \\
&= \left[u^i,_j\, v^j + u^i v^j,_j - \left(u^j,_j\, v^i + u^j v^i,_j\,\right)\right]\mathbf{g}_i \\
&= \left[\left(u^i v^j\right),_j - \left(u^j v^i\right),_j\,\right]\mathbf{g}_i \\
&= \text{div}(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u})
\end{aligned}$$

since $\text{div}(\mathbf{u} \otimes \mathbf{v}) = \left(u^i v^j\right),_\alpha \mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^\alpha = \left(u^i v^j\right),_j \mathbf{g}_i$.

29. **Given a scalar point function $\boldsymbol{\phi}$ and a second-order tensor field T, show that $\mathbf{curl}\ (\boldsymbol{\phi}\mathrm{T}) = \boldsymbol{\phi}\ \mathbf{curl}\ \mathrm{T} + \big((\mathbf{grad}\ \boldsymbol{\phi})\ \times\big)\mathrm{T}^{\mathbf{T}}$ where $[(\mathbf{grad}\ \boldsymbol{\phi})\ \times]$ is the skew tensor $\boldsymbol{\epsilon^{ijk}\phi_{,j}}\ \mathrm{g}_{\boldsymbol{i}} \otimes \mathrm{g}_{\boldsymbol{k}}$**

$$
\begin{aligned}
\text{curl}\ (\phi\mathbf{T}) &\equiv \epsilon^{ijk}(\phi T_{\alpha k})_{,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\big(\phi_{,j} T_{\alpha k} + \phi T_{\alpha k,j}\big)\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\phi_{,j} T_{\alpha k}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha + \phi\epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \big(\epsilon^{ijk}\phi_{,j}\ \mathbf{g}_i \otimes \mathbf{g}_k\big)\big(T_{\alpha\beta}\mathbf{g}^\beta \otimes \mathbf{g}^\alpha\big) + \phi\epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \phi\ \text{curl}\ \mathbf{T} + \big((\text{grad}\ \phi)\ \times\big)\mathbf{T}^{\mathrm{T}}
\end{aligned}
$$

30. **For a second-order tensor field T, show that $\mathbf{div}(\mathbf{curl}\ \mathrm{T}) = \mathbf{curl}(\mathbf{div}\ \mathrm{T}^{\mathbf{T}})$**

Define the second order tensor S as

$$
\text{curl}\ \mathbf{T} \equiv \epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha = S^i_{.\alpha}\mathbf{g}_i \otimes \mathbf{g}^\alpha
$$

The gradient of $\boldsymbol{S}$ is $S^i_{.\alpha,\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta = \epsilon^{ijk}T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

Clearly,

$$
\begin{aligned}
\text{div}(\text{curl}\ \mathbf{T}) &= \epsilon^{ijk}T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = \epsilon^{ijk}T_{\alpha k,j\beta}\ \mathbf{g}_i\ g^{\alpha\beta} \\
&= \epsilon^{ijk}T^{\beta}{}_{k,j\beta}\ \mathbf{g}_i = \text{curl}(\text{div}\ \mathbf{T}^{\mathrm{T}})
\end{aligned}
$$

31. Show that if φ defined in the space spanned by orthogonal coordinates x^i, then $\nabla^2(x^i\varphi) = 2\frac{\partial\varphi}{\partial x^i} + x^i\nabla^2\varphi$.

By definition, $\nabla^2(x^i\varphi) = g^{jk}(x^i\varphi)_{,jk}$. Expanding, we have

$$g^{jk}(x^i\varphi)_{,jk} = g^{jk}(x^i{}_{,j}\varphi + x^i\varphi_{,j})_{,k} = g^{jk}(\delta^i_j\varphi + x^i\varphi_{,j})_{,k}$$
$$= g^{jk}(\delta^i_j\varphi_{,k} + x^i{}_{,k}\varphi_{,j} + x^i\varphi_{,jk})$$
$$= g^{jk}(\delta^i_j\varphi_{,k} + \delta^i_k\varphi_{,j} + x^i\varphi_{,jk})$$
$$= g^{ik}\varphi_{,k} + g^{ij}\varphi_{,j} + x^i g^{jk}\varphi_{,jk}$$

When the coordinates are orthogonal, this becomes,

$$\frac{2}{(h_i)^2}\frac{\partial\Phi}{\partial x^i} + x^i\nabla^2\Phi$$

where we have suspended the summation rule and h_i is the square root of the appropriate metric tensor component.

32. In Cartesian coordinates, If the volume V is enclosed by the surface S, the position vector $r = x^i\mathrm{g}_i$ and n is the external unit normal to each surface element, show that $\frac{1}{6}\int_S \nabla(r\cdot r)\cdot n dS$ equals the volume contained in V.

$$\boldsymbol{r}\cdot\boldsymbol{r} = x^i x^j \mathbf{g}_i\cdot\mathbf{g}_j = x^i x^j g_{ij}$$

By the Divergence Theorem,

$$\int_S \nabla(\boldsymbol{r}\cdot\boldsymbol{r})\cdot\boldsymbol{n}dS = \int_V \nabla\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{r})]dV = \int_V \partial_l\left[\partial_k\left(x^i x^j g_{ij}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\,dV$$
$$= \int_V \partial_l\left[g_{ij}\left(x^i{}_{,k}\,x^j + x^i x^j{}_{,k}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\,dV = \int_V g_{ij}g^{lk}\left(\delta^i_k x^j + x^i\delta^j_k\right)_{,l}\ dV$$
$$= \int_V 2g_{ik}g^{lk}x^i{}_{,l}\ dV = \int_V 2\delta^l_i\delta^i_l\ dV = 6\int_V dV$$

33. For any Euclidean coordinate system, show that div u × v = v curl u − u curl v

Given the contravariant vector u^i and v^i with their associated vectors u_i and v_i, the contravariant component of the above cross product is $\epsilon^{ijk}u_jv_k$ .The required divergence is simply the contraction of the covariant x^i derivative of this quantity:

$$\left(\epsilon^{ijk}u_jv_k\right)_{,i} = \epsilon^{ijk}u_{j,i}v_k + \epsilon^{ijk}u_jv_{k,i}$$

where we have treated the tensor ε^{ijk} as a constant under the covariant derivative. Cyclically rearranging the RHS we obtain,

$$\left(\epsilon^{ijk}u_jv_k\right)_{,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jki}v_{k,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jik}v_{k,i}$$

where we have used the anti-symmetric property of the tensor ϵ^{ijk}. The last expression shows clearly that

div $\mathbf{u}\times\mathbf{v} = \mathbf{v}$ curl $\mathbf{u} - \mathbf{u}$ curl $\mathbf{v}$

as required.

34. **For a general tensor field** T **show that,** $\mathbf{curl}(\mathbf{curl}\ \mathrm{T}) = [\mathbf{grad}^2(\mathbf{tr}\ \mathrm{T}) - \mathbf{div}(\mathbf{div}\ \mathrm{T})]\mathrm{I} + \mathbf{grad}(\mathbf{div}\ \mathrm{T}) + \left(\mathbf{grad}(\mathbf{div}\ \mathrm{T})\right)^{\mathbf{T}} - \mathbf{grad}\left(\mathbf{grad}\ (\mathbf{tr}\ T)\right) - \mathbf{grad}^2\mathrm{T}^{\mathbf{T}}$

$$\begin{aligned}\mathrm{curl}\ \boldsymbol{T} &= \boldsymbol{\epsilon}^{\alpha st} T_{\beta t,s}\,\mathbf{g}_\alpha \otimes \mathbf{g}^\beta \\ &= S^{\alpha}_{.\beta}\mathbf{g}_\alpha \otimes \mathbf{g}^\beta\end{aligned}$$

$$\mathrm{curl}\ \boldsymbol{S} = \epsilon^{ijk} S^{\alpha}_{.k,j}\,\mathbf{g}_i \otimes \mathbf{g}_\alpha$$

so that

$$\begin{aligned}\mathrm{curl}\ \boldsymbol{S} = \mathrm{curl}(\mathrm{curl}\ \boldsymbol{T}) &= \epsilon^{ijk}\epsilon^{\alpha st} T_{kt,sj}\,\mathbf{g}_i \otimes \mathbf{g}_\alpha \\ &= \begin{vmatrix} g^{i\alpha} & g^{is} & g^{it} \\ g^{j\alpha} & g^{js} & g^{jt} \\ g^{k\alpha} & g^{ks} & g^{kt} \end{vmatrix} T_{kt,sj}\,\mathbf{g}_i \otimes \mathbf{g}_\alpha \\ &= \begin{bmatrix} g^{i\alpha}\left(g^{js}g^{kt} - g^{jt}g^{ks}\right) + g^{is}\left(g^{jt}g^{k\alpha} - g^{j\alpha}g^{kt}\right) \\ +g^{it}\left(g^{j\alpha}g^{ks} - g^{js}g^{k\alpha}\right) \end{bmatrix} T_{kt,sj}\,\mathbf{g}_i \otimes \mathbf{g}_\alpha \\ &= \left[g^{js}T^{t}_{.t,sj} - T^{sj}_{..},_{sj}\right](\mathbf{g}^\alpha \otimes \mathbf{g}_\alpha) + \left[T^{\alpha j}_{..},_{sj} - g^{j\alpha}T^{t}_{.t,sj}\right](\mathbf{g}^s \otimes \mathbf{g}_\alpha) \\ &\quad + \left[g^{j\alpha}T^{s.}_{.t,sj} - g^{js}T^{\alpha.}_{.t,sj}\right](\mathbf{g}^t \otimes \mathbf{g}_\alpha) \\ &= [\mathrm{grad}^2(\mathrm{tr}\ \mathbf{T}) - \mathrm{div}(\mathrm{div}\ \mathbf{T})]\mathbf{I} + \left(\mathrm{grad}(\mathrm{div}\ \mathbf{T})\right)^{\mathrm{T}} - \mathrm{grad}\left(\mathrm{grad}\ (\mathrm{tr}\mathbf{T})\right) \\ &\quad + \left(\mathrm{grad}(\mathrm{div}\ \mathbf{T})\right) - \mathrm{grad}^2\mathbf{T}^{\mathrm{T}}\end{aligned}$$

35. When T is symmetric, show that tr(curl T) vanishes.

$$\text{curl}\,\mathbf{T} = \boldsymbol{\epsilon}^{ijk} T_{\beta k,j}\, \mathbf{g}_i \otimes \mathbf{g}^\beta$$

$$\text{tr}(\text{curl}\,\mathbf{T}) = \epsilon^{ijk} T_{\beta k,j}\, \mathbf{g}_i \cdot \mathbf{g}^\beta$$

$$= \epsilon^{ijk} T_{\beta k,j}\, \delta_i^\beta = \epsilon^{ijk} T_{ik,j}$$

which obviously vanishes on account of the symmetry and antisymmetry in i and k. In this case,

$$\text{curl}(\text{curl}\,\mathbf{T}) = [\nabla^2(\text{tr}\,\mathbf{T}) - \text{div}(\text{div}\,\mathbf{T})]\mathbf{1} - \text{grad}\big(\text{grad}\,(\text{tr}\boldsymbol{T})\big) + 2\big(\text{grad}(\text{div}\,\boldsymbol{T})\big) - \nabla^2\mathbf{T}$$

as $\big(\text{grad}(\text{div}\,\mathbf{T})\big)^{\mathrm{T}} = \text{grad}(\text{div}\,\mathbf{T})$ if the order of differentiation is immaterial and $\mathbf{T}$ is symmetric.

36. For a scalar function ϕ and a vector v show that the divergence of the vector vϕ is equal to, v $\cdot$ grad ϕ + ϕ div v

$$\text{grad}\,(\mathbf{v}\phi) = \big(v^i\phi\big)_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^j = \big(\phi v^i{}_{,j} + v^i\phi_{,j}\big)\mathbf{g}_i \otimes \mathbf{g}^j$$

Taking the trace of this equation,

$$\text{div}\,\mathbf{v}\phi = \text{tr}\big(\text{grad}\,(\mathbf{v}\phi)\big) = \big(\phi v^i{}_{,j} + v^i\phi_{,j}\big)\mathbf{g}_i \cdot \mathbf{g}^j$$

$$= \big(\phi v^i{}_{,j} + v^i\phi_{,j}\big)\delta_j^i$$

$$= \phi v^i{}_{,i} + v^i \phi_{,i}$$
$$= \phi \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad}\phi$$

Hence the result.

37. Show that curl u × v = (v · **grad** u) + (u · **div** v) − (v · **div** u) − (u · **grad** v)

Taking the associated (covariant) vector of the expression for the cross product in the last example, it is straightforward to see that the LHS in indicial notation is,

$$\epsilon^{lmi}\left(\epsilon_{ijk} u^j v^k\right)_{,m}$$

Expanding in the usual way, noting the relation between the alternating tensors and the Kronecker deltas,

$$\begin{aligned}
\epsilon^{lmi}\left(\epsilon_{ijk} u^j v^k\right)_{,m} &= \delta^{lmi}_{jki}\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right) \\
&= \delta^{lm}_{jk}\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right) \\
&= \begin{vmatrix} \delta^l_j & \delta^m_j \\ \delta^l_k & \delta^m_k \end{vmatrix}\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right) \\
&= \left(\delta^l_j \delta^m_k - \delta^l_k \delta^m_j\right)\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right) \\
&= \delta^l_j \delta^m_k u^j{}_{,m} v^k - \delta^l_j \delta^m_k u^j v^k{}_{,m} + \delta^l_k \delta^m_j u^j{}_{,m} v^k - \delta^l_k \delta^m_j u^j v^k{}_{,m} \\
&= u^l{}_{,m} v^m - u^m{}_{,m} v^l + u^l v^m{}_{,m} - u^m v^l{}_{,m}
\end{aligned}$$

Which is the result we seek in indicial notation.

38. **. In Cartesian coordinates let x denote the magnitude of the position vector $\mathrm{r} = x_i \mathrm{e}_i$. Show that (a) $x_{,j} = \frac{x_j}{x}$, (b) $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$, (c) $x_{,ii} = \frac{2}{x}$, (d) If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ $U_{,ii} = 0$ and div $\left(\frac{\mathrm{r}}{x}\right) = \frac{2}{x}$.**

$$(a)\ \ x = \sqrt{x_i x_i}$$

$$x_{,j} = \frac{\partial \sqrt{x_i x_i}}{\partial x_j} = \frac{\partial \sqrt{x_i x_i}}{\partial (x_i x_i)} \times \frac{\partial (x_i x_i)}{\partial x_j} = \frac{1}{2\sqrt{x_i x_i}}\left[x_i \delta_{ij} + x_i \delta_{ij}\right] = \frac{x_j}{x}.$$

$$(b)\ x_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial \sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right) = \frac{x\frac{\partial x_i}{\partial x_j} - x_i \frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$$

$$(c)\ \ x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3} = \frac{3}{x} - \frac{(x)^2}{(x)^3} = \frac{2}{x}.$$

(d) $U = \frac{1}{x}$ so that

$$U_{,j} = \frac{\partial \frac{1}{x}}{\partial x_j} = \frac{\partial \frac{1}{x}}{\partial x} \times \frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$

Consequently,

$$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right) = \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i \frac{\partial}{\partial x_j}(x^3)}{x^6}$$

$$= \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial(x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6} = \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

$$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5} = \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$

$$\text{div}\left(\frac{\mathbf{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x}x_{j,j} + \left(\frac{1}{x}\right)_{,j} = \frac{3}{x} + x_j\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\frac{dx}{dx_j}\right)$$

$$= \frac{3}{x} + x_j\left[-\left(\frac{1}{x^2}\right)\frac{x_j}{x}\right] = \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$

39. Define the dyadic and squared times products of tensors as, $(\mathrm{A} \otimes \mathrm{B})\mathrm{C} = (\mathrm{B}:\mathrm{C})\mathrm{A}$ **and** $(\mathrm{A} \boxtimes \mathrm{B})\mathrm{C} = \mathrm{ACB}^{\mathbf{T}}$ **Show that** $(\mathrm{A} \boxtimes \mathrm{B})\,(\mathrm{C} \otimes \mathrm{D}) = \mathrm{ACB}^{\mathbf{T}} \otimes \mathrm{D}$

$$\begin{aligned}(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D})\mathbf{E} &= (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{D}:\mathbf{E})\mathbf{C} \\ &= (\mathbf{ACB}^{\mathrm{T}})\mathbf{D}:\mathbf{E} \\ &= (\mathbf{ACB}^{\mathrm{T}} \otimes \mathbf{D})\mathbf{E}\end{aligned}$$

so that

$$(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = \mathbf{ACB}^{\mathrm{T}} \otimes \mathbf{D}$$

40. Define the dyadic and squared times products of tensors as, $(\mathrm{A} \otimes \mathrm{B})\mathrm{C} = (\mathrm{B}:\mathrm{C})\mathrm{A}$ and $(\mathrm{A} \boxtimes \mathrm{B})\mathrm{C} = \mathrm{ACB}^{\mathbf{T}}$ For vectors a, b and tensors A, B show that $(\mathrm{A} \boxtimes \mathrm{B})\,(\mathrm{a} \otimes \mathrm{b}) = \mathrm{Aa} \otimes \mathrm{Bb}$.

$$(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{a} \otimes \mathbf{b}) = \mathbf{A}(\mathbf{a} \otimes \mathbf{b})\mathbf{B}^{\mathrm{T}} = \mathbf{Aa} \otimes \mathbf{Bb}$$

41. Define the dyadic and squared times products of tensors as, $(\mathrm{A} \otimes \mathrm{B})\mathrm{C} = (\mathrm{B}:\mathrm{C})\mathrm{A}$ and $(\mathrm{A} \boxtimes \mathrm{B})\mathrm{C} = \mathrm{ACB}^{\mathbf{T}}$ For vectors $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d show that $(\mathrm{a} \otimes \mathrm{b}) \boxtimes (\mathrm{c} \otimes \mathrm{d}) = (\mathrm{a} \otimes \mathrm{c}) \otimes (\mathrm{b} \otimes \mathrm{d})$

For a tensor $\mathbf{E}$,

$$\begin{aligned}
\big((\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d})\big)\mathbf{E} &= (\mathbf{a} \otimes \mathbf{b})\mathbf{E}(\mathbf{d} \otimes \mathbf{c}) \\
&= (\mathbf{a} \otimes \mathbf{c})[(\mathbf{E}^{\mathrm{T}}\mathbf{b}) \cdot \mathbf{d}] \\
&= (\mathbf{a} \otimes \mathbf{c})\,\mathrm{tr}\big((\mathbf{d} \otimes \mathbf{b})\mathbf{E}\big) \\
&= (\mathbf{a} \otimes \mathbf{c})[(\mathbf{b} \otimes \mathbf{d}):\mathbf{E}] \\
&= \big((\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d})\big)\mathbf{E}
\end{aligned}$$

so that $(\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d})$.

42. Define the tensor basis $\mathrm{G}^{ij} \equiv \mathrm{g}^i \otimes \mathrm{g}^j$, observe that unlike the scalar component g_{ij}, the tensor G^{ij} is not symmetrical in its indices; furthermore, show that $\mathbb{I} \equiv g_{ij}\mathrm{G}^{ij} \boxtimes g_{\alpha\beta}\mathrm{G}^{\alpha\beta}$ is the fourth order unit tensor.

By the definition of $\mathbf{G}^{ij} \equiv \mathbf{g}^i \otimes \mathbf{g}^j$, It is immediately clear that $\mathbf{G}^{ij} \equiv \left[\mathbf{G}^{ji}\right]^{\mathrm{T}}$. It is therefore not symmetric in its components. We further observe that $g_{ij}\mathbf{G}^{ij}$ is the component representation of the second-order unit tensor.

Lastly, $\mathbb{I}$ is the fourth-order unit tensor. This is evident because, given any second-order tensor $\mathbf{T}$, $\mathbb{I}\mathbf{T} = \mathbf{T}$. To show this to be true, take any component representation of $\mathbf{T}$ and expand $\mathbb{I}\mathbf{T}$:

$$
\begin{aligned}
\mathbb{I}\mathbf{T} &= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)\mathbf{T} \\
&= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)T_{kl}\mathbf{g}^i \otimes \mathbf{g}^j \\
&= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)T_{kl}\mathbf{G}^{kl} \\
&= g_{ij}g_{\alpha\beta}T_{kl}\left(\mathbf{G}^{ij} \boxtimes \mathbf{G}^{\alpha\beta}\right)\mathbf{G}^{kl} \\
&= g_{ij}g_{\alpha\beta}T_{kl}\mathbf{G}^{ij}\mathbf{G}^{kl}\mathbf{G}^{\beta\alpha} \\
&= g_{ij}g_{\alpha\beta}T_{kl}g^{jk}g^{l\beta}\mathbf{G}^{i\alpha} \\
&= \delta_i^k\delta_\beta^l T_{kl}\mathbf{G}^{i\alpha} = T_{i\alpha}\mathbf{G}^{i\alpha} \\
&= \mathbf{T}
\end{aligned}
$$

Showing that, $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$

43. Given that $\mathbb{I} = \mathrm{I} \boxtimes \mathrm{I}$, show that, $\mathbb{I} = g_{ij} g_{kl} \mathrm{G}^{ij} \boxtimes \mathrm{G}^{kl} = g_{ik} g_{jl} \mathrm{G}^{ij} \otimes \mathrm{G}^{kl}$

The first expression is recognizable as $\mathbf{I} \boxtimes \mathbf{I}$ since

$$\begin{aligned} \mathbb{I} = \mathbf{I} \boxtimes \mathbf{I} &= g_{ij} \mathbf{G}^{ij} \boxtimes g_{\alpha\beta} \mathbf{G}^{\alpha\beta} \\ &= g_{ij} g_{\alpha\beta} \mathbf{G}^{ij} \boxtimes \mathbf{G}^{\alpha\beta} \end{aligned}$$

Let us see how the second expression operates on a second-order tensor:

$$\begin{aligned} g_{ik} g_{jl} \left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \mathbf{T} &= g_{ik} g_{jl} \left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) T_{\alpha\beta} \mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta} \\ &= g_{ik} g_{jl} T_{\alpha\beta} \left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta} \\ &= g_{ik} g_{jl} T_{\alpha\beta} \mathbf{G}^{ij} \left(\mathbf{G}^{kl} : \left(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}\right)\right) \\ &= g_{ik} g_{jl} T_{\alpha\beta} \mathbf{G}^{ij} g^{k\alpha} g^{l\beta} = \delta_i^{\alpha} \delta_j^{\beta} T_{\alpha\beta} \mathbf{G}^{ij} \\ &= T_{ij} \mathbf{G}^{ij} = \mathbf{T} \end{aligned}$$

confirming that $g_{ik} g_{jl} \mathbf{G}^{ij} \otimes \mathbf{G}^{kl} = \mathbb{I} = g_{ik} g_{jl} \left(\mathbf{g}^{i} \otimes \mathbf{g}^{j}\right) \otimes \left(\mathbf{g}^{k} \otimes \mathbf{g}^{l}\right)$.

44. For a second-order tensor A show that $\mathrm{A}\mathbb{I} = \mathbb{I}\mathrm{A} = \mathrm{A}$ where $\mathbb{I}$ is the fourth-order unit tensor.

Note that $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$. Therefore, $\mathbf{A}\mathbb{I} = \mathbf{A}(\mathbf{I} \boxtimes \mathbf{I}) = \mathbf{I}^{\mathrm{T}} \mathbf{A} \mathbf{I} = \mathbf{A}$. Similarly, $\mathbb{I}\mathbf{A} = (\mathbf{I} \boxtimes \mathbf{I})\mathbf{A} = \mathbf{I}\mathbf{A}\mathbf{I}^{\mathrm{T}} = \mathbf{A}$ since the identity tensor is symmetric and hence selftranspose.

45. The transposer tensor $\mathbb{T}$ turns a second-order tensor into its transpose: $\mathbb{T}\mathrm{S} = \mathrm{S}^{\mathbf{T}} = \mathrm{S}\mathbb{T}$; show that $\mathbb{T} = \boldsymbol{g_{il}g_{jk}}\mathrm{G}^{\boldsymbol{ij}} \otimes \mathrm{G}^{\boldsymbol{kl}}$

$$
\begin{aligned}
\mathbb{T}\mathbf{S} &= g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\mathbf{S} \\
&= g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\left(S^{\alpha\beta}\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\left(\mathbf{G}^{kl}:\left(\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}\right)\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\left(\mathbf{g}^{k} \cdot \mathbf{g}_{\beta}\right)\left(\mathbf{g}^{l} \cdot \mathbf{g}_{\alpha}\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\delta_{\beta}^{k}\delta_{\alpha}^{l} = S_{ji}\mathbf{G}^{ij} \\
&= \mathbf{S}^{\mathrm{T}} \\
\mathbf{S}\mathbb{T} &= \mathbf{S}g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \\
&= \left(S^{\alpha\beta}\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}\right)g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\left(\left(\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}\right):\mathbf{G}^{ij}\right)\mathbf{G}^{kl} \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{kl}\left(\mathbf{g}^{i} \cdot \mathbf{g}_{\alpha}\right)\left(\mathbf{g}^{j} \cdot \mathbf{g}_{\beta}\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{kl}\delta_{\alpha}^{i}\delta_{\beta}^{j} = S^{ij}\mathbf{G}_{ji} \\
&= \mathbf{S}^{\mathrm{T}}
\end{aligned}
$$

46. Define the symmetrizer, $\mathbb{S}$ and anti symmetrizer, $\mathbb{W}$ tensors as fourth order tensors that return the symmetric and antisymmetric parts of a second-order tensor; show that $\mathbb{S} = \frac{1}{2}(\mathbb{I} + \mathbb{T})$ and $\mathbb{W} = \frac{1}{2}(\mathbb{I} - \mathbb{T})$.

Consider a tensor **A**.

$$\mathbb{S}\mathbf{A} = \frac{1}{2}(\mathbb{I} + \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \operatorname{sym}\mathbf{A}$$

Similarly,

$$\mathbb{W}\mathbf{A} = \frac{1}{2}(\mathbb{I} - \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \operatorname{skw}\mathbf{A}$$

47. For any second-order tensor A Show that $\mathbb{S}\mathrm{A} = \mathrm{A}\mathbb{S}$, and that $\mathbb{W}\mathrm{A} = \mathrm{A}\mathbb{W}$ where $\mathbb{S}$ is the fourth-order symmetrizer tensor. [Hint: $\mathrm{A}\mathbb{I} = \mathbb{I}\mathrm{A}, \mathbb{T}\mathrm{S} = \mathrm{S}\mathbb{T}$]

Consider a tensor **A**.

$$\mathbb{S}\mathbf{A} = \frac{1}{2}(\mathbb{I} + \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \operatorname{sym}\mathbf{A}$$

$$\mathbf{A}\mathbb{S} = \mathbf{A}\left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right) = \frac{1}{2}(\mathbf{A}\mathbb{I} + \mathbf{A}\mathbb{T}) = \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \operatorname{sym}\mathbf{A}$$

so that $\mathbb{S}\mathbf{A} = \mathbf{A}\mathbb{S} = \operatorname{sym}\mathbf{A}$. Similarly,

$$\mathbb{W}\mathbf{A} = \frac{1}{2}(\mathbb{I} - \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \operatorname{skw}\mathbf{A}$$

$$\mathbf{A}\mathbb{W} = \mathbf{A}\left(\frac{1}{2}(\mathbb{I} - \mathbb{T})\right) = \frac{1}{2}(\mathbf{A}\mathbb{I} - \mathbf{A}\mathbb{T}) = \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \operatorname{skw}\mathbf{A}$$

showing that $\mathbb{W}\mathbf{A} = \mathbf{A}\mathbb{W} = \operatorname{skw}\mathbf{A}$

48. For the fourth order tensors $\mathbb{S}, \mathbb{T}$, and $\mathbb{W}$ show that (a) $\mathbb{T}\mathbb{T} = \mathbb{I}$, (b) $\mathbb{T}\mathbb{S} = \mathbb{S}\mathbb{T}$, (c) $\mathbb{S}\mathbb{S} = \mathbb{S}$ (d) $\mathbb{W}\mathbb{W} = \mathbb{W}$ and (e) $\mathbb{S}\mathbb{W} = \mathbb{W}\mathbb{S} = \mathbb{O}$.

(a) An indicial proof $\mathbb{T}\mathbb{T} = \mathbb{I}$ is straightforward. A direct proof is however more illuminating: Consider the double transpose:

$$\mathbb{T}\mathbb{T}\mathbf{A} = \mathbb{T}\mathbf{A}^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A} = \mathbb{I}\mathbf{A}$$

showing clearly that $\mathbb{T}\mathbb{T} = \mathbb{I}$.

(b)

$$\mathbb{T}\mathbb{S} = \mathbb{T}\left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right) = \frac{1}{2}(\mathbb{T}\mathbb{I} + \mathbb{T}\mathbb{T}) = \frac{1}{2}(\mathbb{T} + \mathbb{I}) = \mathbb{S}$$

$$\mathbb{S}\mathbb{T} = \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\mathbb{T} = \frac{1}{2}(\mathbb{I}\mathbb{T} + \mathbb{T}\mathbb{T}) = \frac{1}{2}(\mathbb{T} + \mathbb{I}) = \mathbb{S}$$

so that $\mathbb{T}\mathbb{S} = \mathbb{S}\mathbb{T} = \mathbb{S}$

(c) For a second-order tensor $\mathbf{A}$

$$
\begin{aligned}
\mathbb{S}\mathbb{S}\mathbf{A} &= \mathbb{S}(\operatorname{sym}\mathbf{A}) \\
&= \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\operatorname{sym}\mathbf{A} \\
&= \frac{1}{2}\operatorname{sym}\mathbf{A} + \frac{1}{2}\operatorname{sym}\mathbf{A} \\
&= \operatorname{sym}\mathbf{A} = \mathbb{S}\mathbf{A}
\end{aligned}
$$

so that $\mathbb{S}\mathbb{S} = \mathbb{S}$.

(d) For a second-order tensor $\mathbf{A}$

$$
\begin{aligned}
\mathbb{W}\mathbb{W}\mathbf{A} &= \mathbb{W}(\operatorname{skw}\mathbf{A}) \\
&= \left(\frac{1}{2}(\mathbb{I} - \mathbb{T})\right)\operatorname{skw}\mathbf{A} = \frac{1}{2}\operatorname{skw}\mathbf{A} + \frac{1}{2}\operatorname{skw}\mathbf{A} \\
&= \operatorname{skw}\mathbf{A} = \mathbb{W}\mathbf{A}
\end{aligned}
$$

(e) For a second-order tensor $\mathbf{A}$

$$
\begin{aligned}
\mathbb{S}\mathbb{W}\mathbf{A} &= \mathbb{S}(\operatorname{skw}\mathbf{A}) \\
&= \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\operatorname{skw}\mathbf{A} \\
&= \frac{1}{2}\operatorname{skw}\mathbf{A} - \frac{1}{2}\operatorname{skw}\mathbf{A} \\
&= \mathbb{O}\mathbf{A} = \mathbf{O}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{W}\mathbb{S}\mathbf{A} &= \mathbb{W}\,\mathrm{sym}\,\mathbf{A} \\
&= \left(\frac{1}{2}(\mathbb{I}-\mathbb{T})\right)\mathrm{sym}\,\mathbf{A} \\
&= \frac{1}{2}\mathrm{sym}\,\mathbf{A} - \frac{1}{2}\mathrm{sym}\,\mathbf{A} = \mathbf{O}
\end{aligned}
$$

showing that $\mathbb{S}\mathbb{W} = \mathbb{W}\mathbb{S} = \mathbb{O}$ the fourth-order zero tensor.

49. Given that, in indicial notation, the transposer $\mathbb{T} = \boldsymbol{g}_{il}\boldsymbol{g}_{jk}G^{\boldsymbol{ij}} \otimes G^{\boldsymbol{kl}}$, show that $\mathbb{T}\mathbb{T} = \mathbb{I}$.

$$
\begin{aligned}
\mathbb{T}\mathbb{T} &= \left(g_{il}g_{jk}\boldsymbol{G}^{ij} \otimes \boldsymbol{G}^{kl}\right)\left(g^{\alpha\gamma}g^{\beta\delta}\boldsymbol{G}_{\alpha\beta} \otimes \boldsymbol{G}_{\delta\gamma}\right) \\
&= g_{il}g_{jk}g^{\alpha\gamma}g^{\beta\delta}\boldsymbol{G}^{ij} \otimes \boldsymbol{G}_{\delta\gamma}\left(\boldsymbol{G}^{kl} : \boldsymbol{G}_{\alpha\beta}\right) \\
&= g_{il}g_{jk}g^{\alpha\gamma}g^{\beta\delta}\boldsymbol{G}^{ij} \otimes \boldsymbol{G}_{\delta\gamma}\left(\delta_\alpha^k\delta_\beta^l\right) \\
&= g_{il}g_{jk}g^{k\gamma}g^{l\delta}\boldsymbol{G}^{ij} \otimes \boldsymbol{G}_{\delta\gamma} \\
&= g_{il}g_{jk}g^{k\gamma}g^{l\delta}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}_\delta \otimes \mathbf{g}_\gamma\right) \\
&= g_{il}g_{jk}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}^l \otimes \mathbf{g}^k\right) \\
&= g_{ik}g_{jl}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}^k \otimes \mathbf{g}^l\right) \\
&= \mathbb{I}
\end{aligned}
$$

50. **The position vector in the above example** $\mathrm{r} = \boldsymbol{x_i}\mathrm{e}_{\boldsymbol{i}}$. **Show that (a) div** $\mathrm{r} = 3$, (b) **div** $(\mathrm{r} \otimes \mathrm{r}) = 4\mathrm{r}$, (c) **div** $\mathrm{r} = \mathbf{3}$, **and (d) grad** $\mathrm{r} = 1$ **and (e) curl** $(\mathrm{r} \otimes \mathrm{r}) = -\mathrm{r} \times$

$$\begin{aligned}
\operatorname{grad} \boldsymbol{r} &= x_{i,\boldsymbol{j}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \\
&= \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} = \mathbf{1} \\
\operatorname{div} \boldsymbol{r} &= x_{i,\boldsymbol{j}}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_{\boldsymbol{j}} \\
&= \delta_{ij}\delta_{ij} = \delta_{jj} = 3. \boldsymbol{r} \otimes \boldsymbol{r} = x_i\boldsymbol{e}_i \otimes x_j\boldsymbol{e}_j = x_i x_j \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \operatorname{grad}(\boldsymbol{r} \otimes \boldsymbol{r}) \\
&= \left(x_i x_j\right)_{,k} \boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \otimes \boldsymbol{e}_{\boldsymbol{k}} = \left(x_{i,k}\, x_j + x_i x_{j,k}\right)\boldsymbol{e}_i \otimes \boldsymbol{e}_{\boldsymbol{j}} \cdot \boldsymbol{e}_{\boldsymbol{k}} \\
&= \left(\delta_{ik}\boldsymbol{x}_{\boldsymbol{j}} + \boldsymbol{x}_{\boldsymbol{i}}\delta_{jk}\right)\delta_{jk}\boldsymbol{e}_i = \left(\delta_{ik}\boldsymbol{x}_{\boldsymbol{k}} + \boldsymbol{x}_{\boldsymbol{i}}\delta_{jj}\right)\boldsymbol{e}_i \\
&= 4x_i\boldsymbol{e}_i = 4\boldsymbol{r} \\
\operatorname{curl}(\boldsymbol{r} \otimes \boldsymbol{r}) &= \epsilon_{\alpha\beta\gamma}\left(x_i x_\gamma\right)_{,\beta}\, \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha\beta\gamma}\left(x_{i,\beta}\, x_\gamma + x_i x_{\gamma,\beta}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha\beta\gamma}\left(\delta_{i\beta}x_\gamma + x_i\delta_{\gamma\beta}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} \\
&= \epsilon_{\alpha i\gamma}x_\gamma\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} + \epsilon_{\alpha\beta\beta}x_i\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} = -\epsilon_{\alpha\gamma i}x_\gamma\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_{\boldsymbol{i}} = -\boldsymbol{r} \times
\end{aligned}$$

51. **Define the magnitude of tensor** A **as,** $|\mathrm{A}| = \sqrt{\mathbf{tr}(\mathrm{A}\mathrm{A}^{\mathbf{T}})}$ **Show that** $\frac{\boldsymbol{\partial}|\mathrm{A}|}{\boldsymbol{\partial}\mathrm{A}} = \frac{\mathrm{A}}{|\mathrm{A}|}$

By definition, given a scalar α, the derivative of a scalar function of a tensor $f(\mathbf{A})$ is

$$\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} f(\mathbf{A} + \alpha\mathbf{B})$$

for any arbitrary tensor $\mathbf{B}$.

In the case of $f(\mathbf{A}) = |\mathbf{A}|$,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}}:\mathbf{B} = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}|\mathbf{A}+\alpha\mathbf{B}|$$

$$|\mathbf{A}+\alpha\mathbf{B}| = \sqrt{\operatorname{tr}(\mathbf{A}+\alpha\mathbf{B})(\mathbf{A}+\alpha\mathbf{B})^{\mathrm{T}}} = \sqrt{\operatorname{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{B}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{A}\mathbf{B}^{\mathrm{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}$$

Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write,

$$\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}|\mathbf{A}+\alpha\mathbf{B}| = \lim_{\alpha\to 0}\frac{\operatorname{tr}(\mathbf{B}\mathbf{A}^{\mathrm{T}})+\operatorname{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})+2\alpha\operatorname{tr}(\mathbf{B}\mathbf{B}^{\mathrm{T}})}{2\sqrt{\operatorname{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{B}\mathbf{A}^{\mathrm{T}}+\alpha\mathbf{A}\mathbf{B}^{\mathrm{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}} = \frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}} = \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

So that,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}}:\mathbf{B} = \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

or,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

as required since $\mathbf{B}$ is arbitrary.

52. **Show that** $\frac{\partial I_3(\mathrm{S})}{\partial \mathrm{S}} = \frac{\partial \det(\mathrm{S})}{\partial \mathrm{S}} = \mathrm{S}^{\mathrm{c}}$ **the cofactor of** S.

Clearly $\mathbf{S}^{\mathrm{c}} = \det(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}} = \boldsymbol{I}_3(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}}$. Details of this for the contravariant components of a tensor is presented below. Let

$$\det(\mathbf{S}) \equiv |\mathbf{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha\delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha\delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha\delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\boldsymbol{S}$

53. For a scalar variable $\boldsymbol{\alpha}$, if the tensor $\mathrm{T} = \mathrm{T}(\boldsymbol{\alpha})$ and $\dot{\mathrm{T}} \equiv \frac{d\mathrm{T}}{d\alpha}$, Show that $\frac{d}{d\alpha}\mathbf{det}(\mathrm{T}) = \mathbf{det}(\mathrm{T})\,\mathbf{tr}(\dot{\mathrm{T}}\mathrm{T}^{-1})$

Let $\boldsymbol{A} \equiv \dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$ so that, $\dot{\boldsymbol{T}} = \boldsymbol{A}\boldsymbol{T}$. In component form, we have $\dot{T}_j^i = A_m^i T_j^m$. Therefore,

$$\frac{d}{d\alpha}\det(\boldsymbol{T}) = \frac{d}{d\alpha}\left(e^{ijk}T_i^1T_j^2T_k^3\right) = e^{ijk}\left(\dot{T}_i^1T_j^2T_k^3 + T_i^1\dot{T}_j^2T_k^3 + T_i^1T_j^2\dot{T}_k^3\right)$$

$$= e^{ijk}\left(A_l^1 T_i^l T_j^2 T_k^3 + T_i^1 A_m^2 T_j^m T_k^3 + T_i^1 T_j^2 A_n^3 T_k^n\right)$$

$$= e^{ijk}\left[\left(A_1^1 T_i^1 + \boxed{A_2^1 T_i^2} + \boxed{A_3^1 T_i^3}\right) T_j^2 T_k^3 + T_i^1\left(\boxed{A_1^2 T_j^1} + A_2^2 T_j^2 + \boxed{A_3^2 T_j^3}\right) T_k^3 + T_i^1 T_j^2\left(\boxed{A_1^3 T_k^1} + \boxed{A_2^3 T_k^2} + A_3^3 T_k^3\right)\right]$$

All the boxed terms in the above equation vanish on account of the contraction of a symmetric tensor with an antisymmetric one.

(For example, the first boxed term yields, $e^{ijk} A_2^1 T_i^2 T_j^2 T_k^3$

Which is symmetric as well as antisymmetric in i and j. It therefore vanishes. The same is true for all other such terms.)

$$\frac{\partial}{\partial \alpha}\det(\mathbf{T}) = e^{ijk}\left[\left(A_1^1 T_i^1\right) T_j^2 T_k^3 + T_i^1\left(A_2^2 T_j^2\right) T_k^3 + T_i^1 T_j^2\left(A_3^3 T_k^3\right)\right]$$

$$= A_m^m e^{ijk} T_i^1 T_j^2 T_k^3 = \mathrm{tr}\left(\dot{\mathbf{T}}\mathbf{T}^{-1}\right)\det(\mathbf{T})$$

as required.

54. Prove Liouville's Theorem that for a scalar variable α, if the tensor $\mathrm{T} = \mathrm{T}(\alpha)$ and $\dot{\mathrm{T}} \equiv \frac{d\mathrm{T}}{d\alpha}$, $\frac{d}{d\alpha}\det(\mathrm{T}) = \det(\mathrm{T})\,\mathrm{tr}(\dot{\mathrm{T}}\mathrm{T}^{-1})$ by direct methods.

We choose three constant, linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ so that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\det(\mathbf{T}) = [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]$$

Differentiating both sides, noting that the RHS is a product,

$$[\mathbf{a},\mathbf{b},\mathbf{c}]\frac{d}{d\alpha}\det(\mathbf{T})$$
$$=\frac{d}{d\alpha}[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]$$
$$=\left[\frac{d\mathbf{T}}{d\alpha}\mathbf{a},\mathbf{Tb},\mathbf{Tc}\right]+\left[\mathbf{Ta},\frac{d\mathbf{T}}{d\alpha}\mathbf{b},\mathbf{Tc}\right]+\left[\mathbf{Ta},\mathbf{Tb},\frac{d\mathbf{T}}{d\alpha}\mathbf{c}\right]$$
$$=\left[\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}\right]+\left[\mathbf{Ta},\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Tb},\mathbf{Tc}\right]+\left[\mathbf{Ta},\mathbf{Tb},\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\mathbf{Tc}\right]$$
$$=\operatorname{tr}\left(\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\right)[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]$$

Clearly, $\frac{d}{d\alpha}\det(\mathbf{T})=\det(\mathbf{T})\operatorname{tr}\left(\frac{d\mathbf{T}}{d\alpha}\mathbf{T}^{-1}\right)$

55. Without breaking down into components, use Liouville's theorem, $\frac{\partial}{\partial\alpha}\mathbf{det}(\mathrm{T})=\mathbf{det}(\mathrm{T})\,\mathbf{tr}\left(\dot{\mathrm{T}}\mathrm{T}^{-1}\right)$, to establish the fact that $\frac{\partial\mathbf{det}(\mathrm{T})}{\partial\mathrm{T}}=\mathrm{T}^{c}$

Start from Liouville's Theorem, given a scalar parameter such that $\mathbf{T}=\mathbf{T}(\alpha)$,

$$\frac{\partial}{\partial\alpha}\big(\det(\mathbf{T})\big)=\det(\mathbf{T})\operatorname{tr}\left[\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)\mathbf{T}^{-1}\right]=[\det(\mathbf{T})\,\mathbf{T}^{-\mathbf{T}}]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$

By the simple rules of multiple derivative,

$$\frac{\partial}{\partial\alpha}\big(\det(\mathbf{T})\big)=\left[\frac{\partial}{\partial\mathbf{T}}\big(\det(\mathbf{T})\big)\right]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$

It therefore follows that,

$$\left[\frac{\partial}{\partial \mathbf{T}}\big(\det(\mathbf{T})\big) - [\det(\mathbf{T})\,\mathbf{T}^{-\mathrm{T}}]\right] : \left(\frac{\partial \mathbf{T}}{\partial \alpha}\right) = 0$$

Hence

$$\frac{\partial}{\partial \mathbf{T}}\big(\det(\mathbf{T})\big) = [\det(\mathbf{T})\,\mathbf{T}^{-\mathrm{T}}] = \mathbf{T}^{\mathbf{c}}$$

56. If T is invertible, show that $\frac{\partial}{\partial \mathrm{T}}(\log \det(\mathrm{T})) = \mathrm{T}^{-\mathrm{T}}$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T})) &= \frac{\partial(\log\det(\mathbf{T}))}{\partial \det(\mathbf{T})}\,\frac{\partial \det(\mathbf{T})}{\partial \mathbf{T}} \\ &= \frac{1}{\det(\mathbf{T})}\mathbf{T}^{\mathrm{c}} = \frac{1}{\det(\mathbf{T})}\det(\mathbf{T})\,\mathbf{T}^{-\mathrm{T}} \\ &= \mathbf{T}^{-\mathrm{T}}\end{aligned}$$

57. If T is invertible, show that $\frac{\partial}{\partial \mathrm{T}}\big(\log \det(\mathrm{T}^{-1})\big) = -\mathrm{T}^{-\mathrm{T}}$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T}^{-1})) &= \frac{\partial(\log\det(\mathbf{T}^{-1}))}{\partial \det(\mathbf{T}^{-1})}\,\frac{\partial \det(\mathbf{T}^{-1})}{\partial \mathbf{T}^{-1}}\,\frac{\partial \mathbf{T}^{-1}}{\partial \mathbf{T}} \\ &= \frac{1}{\det(\mathbf{T}^{-1})}\mathbf{T}^{-\mathbf{c}}(-\mathbf{T}^{-1} \boxtimes \mathbf{T}^{-\mathrm{T}})\end{aligned}$$

$$= -\frac{1}{\det(\mathbf{T}^{-1})}\det(\mathbf{T}^{-1})\,\mathbf{T}^{\mathrm{T}}(-\mathbf{T}^{-1} \boxtimes \mathbf{T}^{-\mathrm{T}})$$
$$= -\mathbf{T}^{-\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{T}^{-\mathrm{T}}$$
$$= -\mathbf{T}^{-\mathrm{T}}$$

58. Given that A is a constant tensor, Show that $\frac{\partial}{\partial \mathrm{S}}\mathbf{tr}(\mathrm{AS}) = \mathrm{A}^{\mathbf{T}}$

In invariant components terms, let $\mathbf{A} = \mathrm{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$ and let $\mathbf{S} = \mathrm{S}_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}$.

$$\mathbf{AS} = \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}\left(\mathbf{g}_i \otimes \mathbf{g}_j\right)\left(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}\right)$$
$$= \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}\left(\mathbf{g}_i \otimes \mathbf{g}^{\beta}\right)\delta_j^{\alpha}$$
$$= \mathrm{A}^{ij}\mathrm{S}_{j\beta}\left(\mathbf{g}_i \otimes \mathbf{g}^{\beta}\right)$$
$$\mathrm{tr}(\mathbf{AS}) = \mathrm{A}^{ij}\mathrm{S}_{j\beta}\left(\mathbf{g}_i \cdot \mathbf{g}^{\beta}\right)$$
$$= \mathrm{A}^{ij}\mathrm{S}_{j\beta}\delta_i^{\beta} = \mathrm{A}^{ij}\mathrm{S}_{ji}$$
$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \frac{\partial}{\partial \mathrm{S}_{\alpha\beta}}\mathrm{tr}(\mathbf{AS})\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}$$
$$= \frac{\partial \mathrm{A}^{ij}\mathrm{S}_{ji}}{\partial \mathrm{S}_{\alpha\beta}}\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta}$$
$$= \mathrm{A}^{ij}\delta_j^{\alpha}\delta_i^{\beta}\mathbf{g}_{\alpha} \otimes \mathbf{g}_{\beta} = \mathrm{A}^{ij}\mathbf{g}_j \otimes \mathbf{g}_i = \mathbf{A}^{\mathrm{T}} = \frac{\partial}{\partial \mathbf{S}}(\mathbf{A}^{\mathrm{T}} : \mathbf{S})$$

as required.

59. Given that A **and** B **are constant tensors, show that** $\frac{\partial}{\partial \mathrm{S}}\mathbf{tr}(\mathrm{ASB}^{\mathbf{T}}) = \mathrm{A}^{\mathbf{T}}\mathrm{B}$

First observe that $\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS})$. If we write, $\mathbf{C} \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}$, it is obvious from the above that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{CS}) = \mathbf{C}^{\mathrm{T}}$. Therefore,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathbf{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$$

60. Given that A **and** B **are constant tensors, show that** $\frac{\partial}{\partial \mathrm{S}}\mathbf{tr}(\mathrm{AS}^{\mathbf{T}}\mathrm{B}^{\mathbf{T}}) = \mathrm{B}^{\mathbf{T}}\mathrm{A}$

Observe that $\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS}^{\mathrm{T}}) = \mathrm{tr}[\mathbf{S}(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}] = \mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}]$

[The transposition does not alter trace; neither does a cyclic permutation. Ensure you understand why each equality here is true.] Consequently,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \frac{\partial}{\partial \mathbf{S}}\mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}] = [(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}]^{\mathbf{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$$

61. Let S be a symmetric and positive definite tensor and let $\boldsymbol{I_1}(\mathrm{S}), \boldsymbol{I_2}(\mathrm{S})$ and $\boldsymbol{I_3}(\mathrm{S})$ be the three principal invariants of S show that *(a)* $\frac{\partial I_1(\mathrm{S})}{\partial \mathrm{S}} = \mathrm{I}$ the identity tensor, *(b)* $\frac{\partial I_2(\mathrm{S})}{\partial \mathrm{S}} = \boldsymbol{I_1}(\mathrm{S})\mathrm{I} - \mathrm{S}$ and (c) $\frac{\partial I_3(\mathrm{S})}{\partial \mathrm{S}} = \boldsymbol{I_3}(\mathrm{S})\, \mathrm{S}^{-1}$

$\frac{\partial \boldsymbol{I}_1(\boldsymbol{S})}{\partial \boldsymbol{S}}$ can be written in the invariant component form as,

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_i^j} \mathbf{g}_i \otimes \mathbf{g}^j$$

Recall that $I_1(\mathbf{S}) = \mathrm{tr}(\mathbf{S}) = S_\alpha^\alpha$ hence

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_i^j} \mathbf{g}_i \otimes \mathbf{g}^j = \frac{\partial S_\alpha^\alpha}{\partial S_i^j} \mathbf{g}_i \otimes \mathbf{g}^j$$
$$= \delta_\alpha^i \delta_j^\alpha \mathbf{g}_i \otimes \mathbf{g}^j = \delta_j^i \mathbf{g}_i \otimes \mathbf{g}^j$$
$$= \mathbf{I}$$

which is the identity tensor as expected.

$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}}$ in a similar way can be written in the invariant component form as,

$$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2} \frac{\partial I_1(\mathbf{S})}{\partial S_i^j} \left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta \right] \mathbf{g}_i \otimes \mathbf{g}^j$$

where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)]$. Consequently,

$$\begin{aligned}\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} &= \frac{1}{2}\frac{\partial}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \frac{1}{2}\left[\delta_\alpha^i \delta_j^\alpha S_\beta^\beta + \delta_\beta^i \delta_j^\beta S_\alpha^\alpha - \delta_\beta^i \delta_j^\alpha S_\alpha^\beta - \delta_\alpha^i \delta_j^\beta S_\beta^\alpha\right]\mathbf{g}_i \otimes \mathbf{g}^j \\ &= \frac{1}{2}\left[\delta_j^i S_\beta^\beta + \delta_j^i S_\alpha^\alpha - S_i^j - S_i^j\right]\mathbf{g}_i \otimes \mathbf{g}^j = \left(\delta_j^i S_\alpha^\alpha - S_i^j\right)\mathbf{g}_i \otimes \mathbf{g}^j \\ &= I_1(\mathbf{S})\mathbf{1} - \mathbf{S}\end{aligned}$$

$$\det(\boldsymbol{S}) \equiv |\boldsymbol{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\begin{aligned}\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta &= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \\ &= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha \delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha \delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha \delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \\ &= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \\ &= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\end{aligned}$$

Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\boldsymbol{S}$

62. For a tensor field Ξ, The volume integral in the region $\boldsymbol{\Omega} \subset \mathcal{E}$, $\int_{\boldsymbol{\Omega}} (\mathbf{grad}\ \Xi)\ \boldsymbol{dv} = \int_{\boldsymbol{\partial\Omega}} \Xi \otimes n\ \boldsymbol{ds}$ where n is the outward drawn normal to $\boldsymbol{\partial\Omega}$ – the boundary of $\boldsymbol{\Omega}$. Show that for a vector field f

$$\int_{\boldsymbol{\Omega}} (\mathbf{div}\ f)\ \boldsymbol{dv} = \int_{\boldsymbol{\partial\Omega}} f \cdot n\ \boldsymbol{ds}$$

Replace $\boldsymbol{\Xi}$ by the vector field $\boldsymbol{f}$ we have,

$$\int_{\Omega} (\text{grad}\ \boldsymbol{f})\ dv = \int_{\partial\Omega} \boldsymbol{f} \otimes \boldsymbol{n}\ ds$$

Taking the trace of both sides and noting that both trace and the integral are linear operations, therefore we have,

$$\begin{aligned}
\int_{\Omega} (\text{div}\ \boldsymbol{f})\ dv &= \int_{\Omega} \text{tr}(\text{grad}\ \boldsymbol{f})\ dv \\
&= \int_{\partial\Omega} \text{tr}(\boldsymbol{f} \otimes \boldsymbol{n})\ ds \\
&= \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\ ds
\end{aligned}$$

63. Show that for a scalar function Hence the divergence theorem becomes, $\int_{\Omega}(\mathbf{grad}\ \boldsymbol{\phi})\ \boldsymbol{dv} = \int_{\partial\Omega} \boldsymbol{\phi} n\ \boldsymbol{ds}$

Recall that for a vector field, that for a vector field $\boldsymbol{f}$

$$\int_{\Omega}(\text{div}\ \boldsymbol{f})\ dv = \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}\ ds$$

if we write, $\mathbf{f} = \phi\boldsymbol{a}$ where $\boldsymbol{a}$ is an arbitrary constant vector, we have,

$$\int_{\Omega}(\text{div}[\phi\boldsymbol{a}])\ dv = \int_{\partial\Omega} \phi\boldsymbol{a}\cdot\mathbf{n}\ ds = \boldsymbol{a}\cdot\int_{\partial\Omega}\phi\mathbf{n}\ ds$$

For the LHS, note that, $\text{div}[\phi\boldsymbol{a}] = \text{tr}(\text{grad}[\phi\boldsymbol{a}])$

$$\text{grad}[\phi\boldsymbol{a}] = \left(\phi a^i\right)_{,j}\mathbf{g}_i\otimes\mathbf{g}^j = a^i\phi_{,j}\,\mathbf{g}_i\otimes\mathbf{g}^j$$

The trace of which is,

$$a^i\phi_{,j}\,\mathbf{g}_i\cdot\mathbf{g}^j = a^i\phi_{,j}\,\delta_i^j = a^i\phi_{,i} = \boldsymbol{a}\cdot\text{grad}\ \phi$$

For the arbitrary constant vector $\boldsymbol{a}$, we therefore have that,

$$\int_{\Omega}(\text{div}[\phi\boldsymbol{a}])\ dv = \boldsymbol{a}\cdot\int_{\Omega}\text{grad}\ \phi\ \ dv = \boldsymbol{a}\cdot\int_{\partial\Omega}\phi\mathbf{n}\ ds$$

$$\int_{\Omega}\text{grad}\ \phi\ \ dv = \int_{\partial\Omega}\phi\mathbf{n}\ ds$$

64. Given tensors A, B **and** C **we define the tensor square product with operator,** $\boxtimes$**, in the equation,** $(\mathrm{A} \boxtimes \mathrm{B})\mathrm{C} = \mathrm{ACB}^{\mathbf{T}}$**, show that the square product of two tensors** A **and** B **has the component form,** $\mathrm{A} \boxtimes \mathrm{B} = \boldsymbol{A_{ik}B_{jl}}\mathrm{g}^{\boldsymbol{i}} \otimes \mathrm{g}^{\boldsymbol{j}} \otimes \mathrm{g}^{\boldsymbol{k}} \otimes \mathrm{g}^{\boldsymbol{l}}$

Let $\mathbf{A} = A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{B} = B_{kl}\mathbf{g}^k \otimes \mathbf{g}^l$, $\mathbf{C} = C^{\alpha\beta}\mathbf{g}_\alpha \otimes \boldsymbol{g}_\beta$. If we assume that $\mathbf{A} \boxtimes \mathbf{B} = A_{ik}B_{jl}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l$, Then the product,

$$
\begin{aligned}
(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} &= A_{ik}B_{jl}\left(\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l\right)\left(C^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right) \\
&= A_{ik}B_{jl}C^{\alpha\beta}\delta^k_\alpha\delta^l_\beta\mathbf{g}^i \otimes \mathbf{g}^j \\
&= A_{ik}B_{jl}C^{kl}\mathbf{g}^i \otimes \mathbf{g}^j
\end{aligned}
$$

To complete the proof, we only need to show that the above result equals $\mathbf{ACB}^{\mathrm{T}}$. To do so, we change the basis order in $\mathbf{B}$ so that,

$$
\begin{aligned}
\mathbf{ACB}^{\mathrm{T}} &= \left(A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right)\left(C^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\left(B_{kl}\mathbf{g}^l \otimes \mathbf{g}^k\right) \\
&= A_{ij}C^{\alpha\beta}B_{kl}\mathbf{g}^i \otimes \mathbf{g}^k\delta^j_\alpha\delta^l_\beta = A_{ij}C^{jl}B_{kl}\mathbf{g}^i \otimes \mathbf{g}^k \\
&= A_{ik}B_{jl}C^{kl}\mathbf{g}^i \otimes \mathbf{g}^j \\
&= (\mathbf{A} \boxtimes \mathbf{B})\mathbf{C}
\end{aligned}
$$

We can therefore conclude that $\mathbf{A} \boxtimes \mathbf{B} = A_{ik}B_{jl}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l$.

.

65. Given tensors A, B **and** C **we define the tensor square product with operator,** ⊠, **in the equation,** $(\mathrm{A} \boxtimes \mathrm{B})\mathrm{C} = \mathrm{ACB}^{\mathbf{T}}$**, and that** $\mathrm{A} \boxtimes \mathrm{B} = \boldsymbol{A_{ik}B_{jl}}\mathrm{g}^{\boldsymbol{i}} \otimes \mathrm{g}^{\boldsymbol{j}} \otimes \mathrm{g}^{\boldsymbol{k}} \otimes \mathrm{g}^{\boldsymbol{l}}$**and show** $\mathrm{A}(\mathrm{B} \boxtimes \mathrm{C}) = (\mathrm{B}^{\mathbf{T}} \boxtimes \mathrm{C}^{\mathbf{T}})\mathrm{A}$.

By the definition of the tensor square product, the last expression,

$$(\mathbf{B}^{\mathrm{T}} \boxtimes \mathbf{C}^{\mathrm{T}})\mathbf{A} = \mathbf{B}^{\mathrm{T}}\mathbf{AC}$$

We therefore need to show that $\mathbf{A}(\mathbf{B} \boxtimes \mathbf{C}) = \mathbf{B}^{\mathrm{T}}\mathbf{AC}$. Let, $\mathbf{A} = A^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$, $\mathbf{B} = B_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{C} = C_{kl}\mathbf{g}^k \otimes \mathbf{g}^l$

$$\begin{aligned}\mathbf{A}(\mathbf{B} \boxtimes \mathbf{C}) &= \left(A^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\left(B_{ik}C_{jl}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l\right)\\ &= A^{\alpha\beta}B_{ik}C_{jl}\mathbf{g}^k \otimes \mathbf{g}^l\delta^i_\alpha\delta^j_\beta\\ &= A^{ij}B_{ik}C_{jl}\mathbf{g}^k \otimes \mathbf{g}^l\end{aligned}$$

$$\begin{aligned}\mathbf{B}^{\mathrm{T}}\mathbf{AC} &= B_{ij}A^{\alpha\beta}C_{kl}\left(\mathbf{g}^j \otimes \mathbf{g}^i\right)\left(\mathbf{g}_\alpha \otimes \boldsymbol{g}_\beta\right)\left(\mathbf{g}^k \otimes \mathbf{g}^l\right)\\ &= B_{ij}A^{\alpha\beta}C_{kl}\mathbf{g}^j \otimes \mathbf{g}^l\delta^i_\alpha\delta^k_\beta\\ &= B_{ij}A^{ik}C_{kl}\mathbf{g}^j \otimes \mathbf{g}^l\\ &= A^{ij}B_{ik}C_{jl}\mathbf{g}^k \otimes \mathbf{g}^l = \mathbf{A}(\mathbf{B} \boxtimes \mathbf{C})\end{aligned}$$

We therefore conclude that $\mathbf{A}(\mathbf{B} \boxtimes \mathbf{C}) = \mathbf{B}^{\mathrm{T}}\mathbf{AC} = (\mathbf{B}^{\mathrm{T}} \boxtimes \mathbf{C}^{\mathrm{T}})\mathbf{A}$.

66. Use the expression $(\mathrm{S}+\mathrm{T})^{\mathbf{c}} = \mathrm{S}^{\mathbf{c}} + \mathrm{T}^{\mathbf{c}} + \mathrm{T}^{\mathbf{T}}\mathrm{S}^{\mathbf{T}} + \mathrm{S}^{\mathbf{T}}\mathrm{T}^{\mathbf{T}} - \mathbf{tr}(\mathrm{T})\mathrm{S}^{\mathbf{T}} - \mathbf{tr}(\mathrm{S})\mathrm{T}^{\mathbf{T}} + [\mathbf{tr}(\mathrm{S})\mathbf{tr}(\mathrm{T}) - \mathbf{tr}(\mathrm{ST})]\mathrm{I}$ **to obtain an expression for the fourth order tensor,** $\frac{\partial \mathrm{S}^{\mathbf{c}}}{\partial \mathrm{S}} \equiv \frac{\partial\, \mathbf{cof}\, \mathrm{S}}{\partial \mathrm{S}}$.

We observe that the Gateaux differential,

$$
\begin{aligned}
\operatorname{cof}(\mathbf{S}+\alpha d\mathbf{S}) &= \left(\frac{\partial}{\partial \mathbf{S}} \operatorname{cof}\mathbf{S}\right) d\mathbf{S} \\
&= \left[\frac{\partial}{\partial \alpha} \operatorname{cof}(\mathbf{S}+\alpha d\mathbf{S})\right]_{\alpha=0} \\
&= \left[\frac{\partial}{\partial \alpha}\Big(\mathbf{S}^{\mathrm{c}} + (\alpha d\mathbf{S})^{\mathrm{c}} + (\alpha d\mathbf{S})^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}(\alpha d\mathbf{S})^{\mathrm{T}} - \operatorname{tr}\big((\alpha d\mathbf{S})\big)\mathbf{S}^{\mathrm{T}}\right. \\
&\quad \left. - \operatorname{tr}(\mathbf{S})(\alpha d\mathbf{S})^{\mathrm{T}} + \Big[\operatorname{tr}(\mathbf{S})\operatorname{tr}\big((\alpha d\mathbf{S})\big) - \operatorname{tr}\big(\mathbf{S}(\alpha d\mathbf{S})\big)\Big]\mathbf{I}\Big)\right]_{\alpha=0}
\end{aligned}
$$

This can be further simplified as,

$$
\begin{aligned}
&= \left[\frac{\partial}{\partial \alpha}\Big(\mathbf{S}^{\mathrm{c}} + \alpha^2 (d\mathbf{S})^{\mathrm{c}} + \alpha (d\mathbf{S})^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \alpha \mathbf{S}^{\mathrm{T}}(d\mathbf{S})^{\mathrm{T}} - \alpha \operatorname{tr}(d\mathbf{S})\mathbf{S}^{\mathrm{T}} - \alpha \operatorname{tr}(\mathbf{S})(d\mathbf{S})^{\mathrm{T}}\right. \\
&\quad \left. + [\alpha \operatorname{tr}(\mathbf{S})\operatorname{tr}(d\mathbf{S}) - \alpha \operatorname{tr}(\mathbf{S}d\mathbf{S})]\mathbf{I}\Big)\right]_{\alpha=0} \\
&= \Big[2\alpha (d\mathbf{S})^{\mathrm{c}} + (d\mathbf{S})^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}(d\mathbf{S})^{\mathrm{T}} - \operatorname{tr}(d\mathbf{S})\mathbf{S}^{\mathrm{T}} - \operatorname{tr}(\mathbf{S})(d\mathbf{S})^{\mathrm{T}} \\
&\quad + \big(\operatorname{tr}(\mathbf{S})\operatorname{tr}(d\mathbf{S}) - \operatorname{tr}(\mathbf{S}d\mathbf{S})\big)\mathbf{I}\Big]_{\alpha=0}
\end{aligned}
$$

$$= (d\mathbf{S})^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}(d\mathbf{S})^{\mathrm{T}} - \mathrm{tr}(d\mathbf{S})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})(d\mathbf{S})^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(d\mathbf{S}) - \mathrm{tr}(\mathbf{S}d\mathbf{S})]\mathbf{I}$$
$$= [(\mathbf{I} \boxtimes \mathbf{S})\mathbb{T} + (\mathbf{S}^{\mathrm{T}} \boxtimes \mathbf{I})\mathbb{T} - \mathbf{S}^{\mathrm{T}} \otimes \mathbf{I} - \mathrm{tr}(\mathbf{S})\mathbb{T} + \mathrm{tr}(\mathbf{S})(\mathbf{I} \otimes \mathbf{I}) - \mathbf{I} \otimes \mathbf{S}^{\mathrm{T}}]d\mathbf{S}$$

from which we can conclude that,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{cof}\,\mathbf{S} = \Big((\mathbf{I} \boxtimes \mathbf{S}) + (\mathbf{S}^{\mathrm{T}} \boxtimes \mathbf{I})\Big)\mathbb{T} - (\mathbf{S}^{\mathrm{T}} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S})\big((\mathbf{I} \otimes \mathbf{I}) - \mathbb{T}\big)$$

67. Use indicial notation to obtain an expression for the fourth order tensor, $\frac{\partial \mathrm{S}^{\mathrm{c}}}{\partial \mathrm{S}} \equiv \frac{\partial\,\mathrm{cof}\,\mathrm{S}}{\partial \mathrm{S}}$. Hint: $\mathrm{S}^{\mathrm{c}} = \frac{1}{2}\boldsymbol{\delta}_{ijk}^{lmn}\boldsymbol{S}_m^j\boldsymbol{S}_n^k\mathrm{g}_l \otimes \mathrm{g}^i$

$$\frac{\partial \mathbf{S}^{\mathrm{c}}}{\partial \mathbf{S}} \equiv \frac{\partial\,\mathrm{cof}\,\mathbf{S}}{\partial \mathbf{S}} = \frac{1}{2}\delta_{ijk}^{lmn}\frac{\partial}{\partial S_\beta^\alpha}\left(S_m^j S_n^k\right)\mathbf{g}_l \otimes \mathbf{g}^i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}_\beta$$
$$= \frac{1}{2}\delta_{ijk}^{lmn}\left(\delta_\alpha^j\delta_m^\beta S_n^k + \delta_\alpha^k\delta_n^\beta S_m^j\right)\mathbf{g}_l \otimes \mathbf{g}^i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}_\beta$$
$$= \frac{1}{2}\delta_{ijk}^{lmn}S_n^k\mathbf{g}_l \otimes \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_m + \frac{1}{2}\delta_{ijk}^{lmn}S_m^j\mathbf{g}_l \otimes \mathbf{g}^i \otimes \mathbf{g}^k \otimes \mathbf{g}_n$$
$$= \delta_{ijk}^{lmn}S_n^k\mathbf{g}_l \otimes \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_m$$

68. Use the fact that for any two tensors A and B, $\mathbb{S}(\mathrm{A} \boxtimes \mathrm{B} + \mathrm{B} \boxtimes \mathrm{A})\mathbb{S} = 2\mathbb{S}(\mathrm{A} \boxtimes \mathrm{B})\mathbb{S}$ and the derivative of the cofactor to show that for the right Cauchy-Green tensor,

$$\frac{\partial}{\partial \mathbf{C}} \operatorname{cof} \mathbf{C} = \big((\mathbf{I} \boxtimes \mathbf{C}) + (\mathbf{C} \boxtimes \mathbf{I})\big)\mathbb{T} - (\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}) + \operatorname{tr}(\mathbf{C})\big((\mathbf{I} \otimes \mathbf{I}) - \mathbb{T}\big)$$

so that

$$\mathbb{S}\left(\frac{\partial}{\partial \mathbf{C}} \operatorname{cof} \mathbf{C}\right)\mathbb{S} = \mathbb{S}\big((\mathbf{I} \boxtimes \mathbf{C}) + (\mathbf{C} \boxtimes \mathbf{I})\big)\mathbb{S} - (\mathbf{C} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}) + \operatorname{tr}(\mathbf{C})\big((\mathbf{I} \otimes \mathbf{I}) - \mathbb{S}\big)$$

69. Given any scalar $\boldsymbol{k} > 0$, for the scalar-valued tensor function, $\boldsymbol{f}(\mathrm{S}) = \mathbf{tr}(\mathrm{S}^{\boldsymbol{k}})$, show that, $\frac{\boldsymbol{d}}{\boldsymbol{d}\mathrm{S}} \boldsymbol{f}(\mathrm{S}) = \boldsymbol{k}(\mathrm{S}^{\boldsymbol{k}-\mathbf{1}})^{\boldsymbol{T}}$.

When $k = 1$,

$$\begin{aligned} Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\bigg|_{\alpha=0} \\ &= \frac{d}{d\alpha} \operatorname{tr}(\boldsymbol{S} + \alpha d\boldsymbol{S})\bigg|_{\alpha=0} = \operatorname{tr}(\mathbf{1}\, d\boldsymbol{S}) \\ &= \mathbf{I}: d\boldsymbol{S} \end{aligned}$$

So that,

$$\frac{d}{d\boldsymbol{S}}\ \mathrm{tr}(\boldsymbol{S}) = \ \mathbf{I}.$$

When $k = 2$, The Gateaux differential in this case,

$$\begin{aligned}
Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \frac{d}{d\alpha}\mathrm{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})^2\}\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha}\mathrm{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\}\Big|_{\alpha=0} = \mathrm{tr}\left[\frac{d}{d\alpha}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\right]\Big|_{\alpha=0} \\
&= \mathrm{tr}[d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S}) + (\boldsymbol{S} + \alpha d\boldsymbol{S})d\boldsymbol{S}]|_{\alpha=0} \\
&= \mathrm{tr}[d\boldsymbol{S}\,\boldsymbol{S} + \boldsymbol{S}\,d\boldsymbol{S}] = 2(\boldsymbol{S})^{\mathrm{T}}\colon d\boldsymbol{S}
\end{aligned}$$

When $k = 3$,

$$\begin{aligned}
Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \frac{d}{d\alpha}\mathrm{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})^3\}\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha}\mathrm{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\}\Big|_{\alpha=0} \\
&= \mathrm{tr}\left[\frac{d}{d\alpha}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\right]\Big|_{\alpha=0} \\
&= \mathrm{tr}[d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S}) + (\boldsymbol{S} + \alpha d\boldsymbol{S})d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S}) \\
&\quad + (\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})d\boldsymbol{S}]|_{\alpha=0} \\
&= \mathrm{tr}[d\boldsymbol{S}\,\boldsymbol{S}\,\boldsymbol{S} + \boldsymbol{S}\,d\boldsymbol{S}\,\boldsymbol{S} + \boldsymbol{S}\,\boldsymbol{S}\,\boldsymbol{dS}] = 3(\boldsymbol{S}^2)^{\mathrm{T}}\colon d\boldsymbol{S}
\end{aligned}$$

It easily follows by induction that, $\frac{d}{d\mathbf{S}}\ f(\mathbf{S}) = k(\mathbf{S}^{k-1})^T$.

70. Find the derivative of the second principal invariant of the tensor S

$$\frac{d}{d\boldsymbol{S}} I_2(\boldsymbol{S}) = \frac{1}{2}\frac{d}{d\boldsymbol{S}}[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)]$$
$$= \frac{1}{2}[2\mathrm{tr}(\boldsymbol{S})\mathbf{1} - 2\,\boldsymbol{S}^{\mathrm{T}}]$$
$$= \mathrm{tr}(\boldsymbol{S})\mathbf{1} - \boldsymbol{S}^{\mathrm{T}}$$

71. Find the derivative of the third principal invariant of the tensor S

By Cayley-Hamilton, in terms of traces only,

$$I_3(\boldsymbol{S}) = \frac{1}{6}[\mathrm{tr}^3(\boldsymbol{S}) - 3\mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + 2\mathrm{tr}(\boldsymbol{S}^3)]$$

$$\frac{d}{d\boldsymbol{S}} I_3(\boldsymbol{S}) = \frac{1}{6}\frac{d}{d\boldsymbol{S}}[\mathrm{tr}^3(\boldsymbol{S}) - 3\mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + 2\mathrm{tr}(\boldsymbol{S}^3)]$$
$$= \frac{1}{6}[3\mathrm{tr}^2(\boldsymbol{S})\mathbf{I} - 3\mathrm{tr}(\boldsymbol{S}^2)\mathbf{I} - 3\mathrm{tr}(\boldsymbol{S})2\boldsymbol{S}^T + 2\times 3(\boldsymbol{S}^2)^T]$$
$$= I_2\mathbf{I} - I_1(\boldsymbol{S})\boldsymbol{S}^T + \boldsymbol{S}^{2T}$$

72. By the representation theorem, every real isotropic tensor function is a function of its principal invariants. Show that for every isotropic function $\boldsymbol{f}(\mathrm{S})$ the derivative $\frac{\partial \boldsymbol{f}(\mathrm{S})}{\partial \mathrm{S}}$ can be represented as a quadratic function of $\mathrm{S}^{\mathbf{T}}$.

By the representation theorem,

$$f(\mathbf{S}) = \phi\big(I_1(\mathbf{S}), I_2(\mathbf{S}), I_3(\mathbf{S})\big)$$

consequently,

$$\begin{aligned}
\frac{\partial f(\mathbf{S})}{\partial \mathbf{S}} &= \frac{\partial f(\mathbf{S})}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{S}} \\
&= \frac{\partial f(\mathbf{S})}{\partial I_1}\mathbf{I} + \frac{\partial f(\mathbf{S})}{\partial I_2}(\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}) + \frac{\partial f(\mathbf{S})}{\partial I_3}(I_2\mathbf{I} - I_1(\mathbf{S})\mathbf{S}^T + \mathbf{S}^{2T}) \\
&= \left(\frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2}I_1(\mathbf{S}) + \frac{\partial f(\mathbf{S})}{\partial I_3}I_2(\mathbf{S})\right)\mathbf{I} - \left(\frac{\partial f(\mathbf{S})}{\partial I_2} + \frac{\partial f(\mathbf{S})}{\partial I_3}I_1(\mathbf{S})\right)\mathbf{S}^{\mathrm{T}} \\
&\quad + \frac{\partial f(\mathbf{S})}{\partial I_3}\mathbf{S}^{2T} \\
&= \alpha_0\mathbf{I} + \alpha_1\mathbf{S}^{\mathrm{T}} + \alpha_2\mathbf{S}^{2T}
\end{aligned}$$

where $\alpha_0 = \frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2}I_1(\mathbf{S}) + \frac{\partial f(\mathbf{S})}{\partial I_3}I_2(\mathbf{S})$ $\alpha_1 = -\frac{\partial f(\mathbf{S})}{\partial I_2} - \frac{\partial f(\mathbf{S})}{\partial I_3}I_1(\mathbf{S})$ and $\alpha_2 = \frac{\partial f(\mathbf{S})}{\partial I_3}$ is the desired quadratic representation of the derivative.

73. Show that the derivative of S^2 is $(S \boxtimes I + I \boxtimes S^T)$

The Gateaux differential of $\mathbf{F}(\mathbf{S}) = \mathbf{S}^2$ is,

$$
\begin{aligned}
\mathbf{F}(\mathbf{S}, d\mathbf{S}) &= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \mathbf{F}(\mathbf{S} + \alpha d\mathbf{S}) \\
&= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} (\mathbf{S}^2 + \alpha \mathbf{S} d\mathbf{S} + \alpha d\mathbf{S}(\mathbf{S}) + \alpha^2 (d\mathbf{S})^2) \\
&= \mathbf{S} d\mathbf{S} + d\mathbf{S}(\mathbf{S}) \\
&= (\mathbf{S} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{S}^\mathrm{T}) d\mathbf{S} \\
&= \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}} \right) d\mathbf{S}
\end{aligned}
$$

so that, $\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}} = \mathbf{S} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{S}^\mathrm{T}$.

74. For three tensors A, B, and C, show that $A(B \boxtimes C) = (C \boxtimes B)A$

Let $\mathbf{A} = A^{\alpha\beta} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$, $\mathbf{B} = B_{ij} \mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{C} = C_{kl} \mathbf{g}^k \otimes \mathbf{g}^l$ so that, $\mathbf{B} \boxtimes \mathbf{C} = B_{ik} C_{jl} \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l$

$$
\begin{aligned}
\mathbf{A}(\mathbf{B} \boxtimes \mathbf{C}) &= \left(A^{\alpha\beta} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta \right) \left(B_{ik} C_{jl} \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l \right) \\
&= A^{\alpha\beta} B_{ik} C_{jl} \delta^i_\alpha \delta^j_\beta \mathbf{g}^k \otimes \mathbf{g}^l = A^{ij} B_{ik} C_{jl} \mathbf{g}^k \otimes \mathbf{g}^l
\end{aligned}
$$

$$= B_{ik}C_{jl}\mathbf{g}^k \otimes \mathbf{g}^l \delta^i_\alpha \delta^j_\beta A^{\alpha\beta}$$
$$= B_{ik}C_{jl}\left(\mathbf{g}^k \otimes \mathbf{g}^l \otimes \left(\mathbf{g}^i \otimes \mathbf{g}^j\right)\right)\left(A^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)$$

75. For any positive integer n Given that the derivative $\frac{\partial \mathrm{S}^n}{\partial \mathrm{S}} = \sum_{r=1}^{n} \mathrm{S}^{n-r} \boxtimes (\mathrm{S}^{\mathrm{T}})^{r-1} = \mathrm{S}^n{}_{,\mathrm{S}}$, and that $\mathrm{A}(\mathrm{B} \boxtimes \mathrm{C}) = \mathrm{B}^{\mathrm{T}}\mathrm{AC} = (\mathrm{B}^{\mathrm{T}} \boxtimes \mathrm{C}^{\mathrm{T}})\mathrm{A}$ show that the derivative, $\frac{\partial \operatorname{tr} \mathrm{S}^n}{\partial \mathrm{S}} = n\left(\mathrm{S}^{n-1}\right)^{\mathrm{T}}$

Applying the rules of differentiation,

$$\frac{\partial \operatorname{tr} \mathbf{S}^n}{\partial \mathbf{S}} = \left(\frac{\partial \operatorname{tr} \mathbf{S}^n}{\partial \mathbf{S}^n}\right)\left(\frac{\partial \mathbf{S}^n}{\partial \mathbf{S}}\right)$$
$$= \mathbf{I}\left(\sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1}\right) = \sum_{r=1}^{n} \mathbf{I}(\mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1})$$
$$= \sum_{r=1}^{n} (\mathbf{S}^{n-r})^{\mathrm{T}}\mathbf{I}(\mathbf{S}^{\mathrm{T}})^{r-1} = \sum_{r=1}^{n} (\mathbf{S}^{n-r+r-1})^{\mathrm{T}} = \sum_{r=1}^{n} (\mathbf{S}^{n-1})^{\mathrm{T}}$$
$$= n(\mathbf{S}^{n-1})^{\mathrm{T}}$$

76. Show that if $\mathrm{F}(\mathrm{S}) = \mathrm{S}^{-1}$, then $\frac{\partial \mathrm{F}(\mathrm{S})}{\partial \mathrm{S}} = -\, \mathrm{S}^{-1} \boxtimes \mathrm{S}^{-\mathrm{T}} = -(\mathrm{S} \boxtimes \mathrm{S}^{\mathrm{T}})^{-1}$.

Observe that the differential of the unit tensor is the zero tensor;

$$D(\mathbf{SS}^{-1}) = D(\mathbf{I}) = \mathbf{O}$$

Now, the Gateaux derivative follows the same product rules as the regular derivative, hence,

$$D(\mathbf{SS}^{-1}) = D(\mathbf{S})\mathbf{S}^{-1} + \mathbf{S}\, D(\mathbf{S}^{-1})$$

so that

$$\begin{aligned} D(\mathbf{S}^{-1}) &= -\,\mathbf{S}^{-1} D(\mathbf{S})\mathbf{S}^{-1} \\ &= -(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}})\, d\mathbf{S} \\ &= \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right) d\mathbf{S} \end{aligned}$$

so that,

$$\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}} = -\,\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}} = -(\mathbf{S} \boxtimes \mathbf{S}^{\mathrm{T}})^{-1}$$

77. Given that $\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{S}} = -\,\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathbf{T}} = -(\mathbf{S} \boxtimes \mathbf{S}^{\mathbf{T}})^{-1}$ **Show that** $\frac{\partial \mathbf{S}^{-\mathbf{T}}}{\partial \mathbf{S}} = -\,\mathbb{T}(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathbf{T}}) = -(\mathbf{S}^{-\mathbf{T}} \boxtimes \mathbf{S}^{-1})\mathbb{T} = -\mathbb{T}(\mathbf{S} \boxtimes \mathbf{S}^{\mathbf{T}})^{-1}$

The first and the last results are immediately obvious by invoking the rule that the transpose of a derivative is the derivative of the transpose. Hence we operate the transposer on the given result and obtain,

$$\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{S}} = -\,\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}} = -(\mathbf{S} \boxtimes \mathbf{S}^{\mathrm{T}})^{-1}$$

implies that

$$\frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}} = -\,\mathbb{T}(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}}) = -\mathbb{T}(\mathbf{S} \boxtimes \mathbf{S}^{\mathrm{T}})^{-1}$$

For the middle result, consider that,

$$\frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}} = \frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}^{\mathrm{T}}} \frac{\partial \mathbf{S}^{\mathrm{T}}}{\partial \mathbf{S}}$$

Replacing $\mathbf{S}$ by $\mathbf{S}^{\mathrm{T}}$ in the result, $\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{S}} = -\,\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}}$, we have $\frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}^{\mathrm{T}}} = -\,\mathbf{S}^{-\mathrm{T}} \boxtimes \mathbf{S}^{-1}. \frac{\partial \mathbf{S}^{\mathrm{T}}}{\partial \mathbf{S}} = \mathbb{T}$ because if $\mathbf{F}(\mathbf{S}) = \mathbf{S}^{\mathrm{T}}$

$$\begin{aligned} D\mathbf{F}(\mathbf{S}, d\mathbf{S}) &= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} (\mathbf{S} + \alpha d\mathbf{S})^{\mathrm{T}} \\ &= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} (\mathbf{S}^{\mathrm{T}} + \alpha d\mathbf{S}^{\mathrm{T}}) \\ &= d\mathbf{S}^{\mathrm{T}} = \mathbb{T} d\mathbf{S} \\ &= \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}} \right) d\mathbf{S} \end{aligned}$$

Consequently,

$$\frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}} = \frac{\partial \mathbf{S}^{-\mathrm{T}}}{\partial \mathbf{S}^{\mathrm{T}}}\frac{\partial \mathbf{S}^{\mathrm{T}}}{\partial \mathbf{S}} = -(\mathbf{S}^{-\mathrm{T}} \boxtimes \mathbf{S}^{-1})\mathbb{T}$$

78. **Show that** $\frac{\partial \mathrm{S}^{\mathbf{T}}\mathrm{S}}{\partial \mathrm{S}} = (\mathrm{I} \boxtimes \mathrm{S}^{\mathbf{T}})\mathbb{T} + (\mathrm{S}^{\mathbf{T}} \boxtimes \mathrm{I})$

The differential $D\mathbf{F}(\mathbf{S}, d\mathbf{S})$ when $\mathbf{F}(\mathbf{S}) = \mathbf{S}^{\mathrm{T}}\mathbf{S}$ is,

$$\begin{aligned} D(\mathbf{S}^{\mathrm{T}}\mathbf{S}) &= d\mathbf{S}^{\mathrm{T}}\,\mathbf{S} + \mathbf{S}^{\mathrm{T}}\,d\mathbf{S} \\ &= \left((\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}})\mathbb{T} + \mathbf{S}^{\mathrm{T}} \boxtimes \mathbf{I}\right) d\mathbf{S} \\ &= \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right) d\mathbf{S} \end{aligned}$$

so that

$$\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}})\mathbb{T} + (\mathbf{S}^{\mathrm{T}} \boxtimes \mathbf{I})$$

79. **Show that** $\frac{\partial \mathrm{S}\mathrm{S}^{\mathbf{T}}}{\partial \mathrm{S}} = (\mathrm{I} \boxtimes \mathrm{S}) + (\mathrm{S} \boxtimes \mathrm{I})\mathbb{T}$

The differential $D\mathbf{F}(\mathbf{S}, d\mathbf{S})$ when $\mathbf{F}(\mathbf{S}) = \mathbf{S}\mathbf{S}^{\mathrm{T}}$ is,

$$\begin{aligned} D(\mathbf{S}^{\mathrm{T}}\mathbf{S}) &= d\mathbf{S}\,\mathbf{S}^{\mathrm{T}} + \mathbf{S}d\mathbf{S}^{\mathrm{T}} \\ &= \left((\mathbf{I} \boxtimes \mathbf{S}) + (\mathbf{S} \boxtimes \mathbf{I})\mathbb{T}\right) d\mathbf{S} \\ &= \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right) d\mathbf{S} \end{aligned}$$

so that

$$\frac{\partial \mathbf{S}\mathbf{S}^{\mathrm{T}}}{\partial \mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{S}) + (\mathbf{S} \boxtimes \mathbf{I})\mathbb{T}.$$

80. **For the integers $\boldsymbol{n} > \mathbf{0}$, if $\mathrm{F(S)} = \mathrm{S}^{-\boldsymbol{n}}$ for any differentiable tensor, show by mathematical induction that the derivative**

$$\frac{\boldsymbol{\partial}\mathrm{F(S)}}{\boldsymbol{\partial}\mathrm{S}} = \sum_{\boldsymbol{r}=-\boldsymbol{n}+\mathbf{1}}^{\mathbf{0}} -\mathrm{S}^{-\boldsymbol{n}-\boldsymbol{r}} \boxtimes (\mathrm{S}^{\mathrm{T}})^{\boldsymbol{r}-\mathbf{1}} = \mathrm{S}^{-\boldsymbol{n}}{}_{,\mathrm{S}}$$

Case $n = 1$

$$\mathbf{S}^{-1}{}_{,\mathbf{S}} = -\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}}$$

as expected.

Case $n = 2$

$$\mathbf{S}^{-2}{}_{,\mathbf{S}} = -\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-2\mathrm{T}} - \mathbf{S}^{-2} \boxtimes \mathbf{S}^{-\mathrm{T}}$$

Assume it is true for arbitrary n we use the fact that $\frac{\partial \mathbf{UV}}{\partial \mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{V}^{\mathrm{T}})\frac{\partial \mathbf{U}}{\partial \mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I})\frac{\partial \mathbf{V}}{\partial \mathbf{S}}$ to obtain

$$\mathbf{S}^{-n-1}{}_{,\mathbf{S}} = (\mathbf{S}^{-n}\mathbf{S}^{-1})_{,\mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{S}^{-\mathrm{T}})\frac{\partial \mathbf{S}^{-n}}{\partial \mathbf{S}} + (\mathbf{S}^{-n} \boxtimes \mathbf{I})\frac{\partial \mathbf{S}^{-\mathbf{1}}}{\partial \mathbf{S}}$$

$$= (\mathbf{I} \boxtimes \mathbf{S}^{-\mathrm{T}}) \sum_{r=1}^{n} \mathbf{S}^{-n-r-1} \boxtimes (\mathbf{S}^{\mathrm{T}})^{-r} + (\mathbf{S}^{-n} \boxtimes \mathbf{I})(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}})$$

$$= \sum_{r=1}^{n} \mathbf{S}^{-n-r-1} \boxtimes (\mathbf{S}^{\mathrm{T}})^{-r-1} + (\mathbf{S}^{-n-1} \boxtimes \mathbf{S}^{-\mathrm{T}})$$

$$= \sum_{r=0}^{n+1} \mathbf{S}^{-r-n+1} \boxtimes (\mathbf{S}^{\mathrm{T}})^{-r}$$

81. For the positive integer n, if $\mathrm{F}(\mathrm{S}) = \mathrm{S}^{n}$ for any differentiable tensor, show by mathematical induction that the derivative

$$\frac{\partial \mathrm{F}(\mathrm{S})}{\partial \mathrm{S}} = \mathrm{S}^{n-1} \boxtimes \mathrm{I} + \mathrm{S}^{n-2} \boxtimes \mathrm{S}^{\mathrm{T}} + \mathrm{S}^{n-3} \boxtimes \mathrm{S}^{2\mathrm{T}} + \cdots + \mathrm{S} \boxtimes \mathrm{S}^{(n-2)\mathrm{T}} + \mathrm{I} \boxtimes \mathrm{S}^{(n-1)\mathrm{T}}$$

$$= \sum_{r=1}^{n} \mathrm{S}^{n-r} \boxtimes (\mathrm{S}^{\mathrm{T}})^{r-1} = \mathrm{S}^{n}{}_{,\mathrm{S}}$$

Case $n = 1$

$$\mathbf{S}_{,\mathbf{S}} = \mathbf{S}^{0} \boxtimes (\mathbf{S}^{\mathrm{T}})^{0} = \mathbf{I} \boxtimes \mathbf{I} = \mathbb{I}$$

as expected.

Case $n = 2$

$$\mathbf{S}^2,_{\mathbf{S}} = \mathbf{S} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}$$

Assume it is true for arbitrary n we use the fact that $\frac{\partial \mathbf{UV}}{\partial \mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{V}^{\mathrm{T}}) \frac{\partial \mathbf{U}}{\partial \mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I}) \frac{\partial \mathbf{V}}{\partial \mathbf{S}}$ to obtain

$$\begin{aligned}
\mathbf{S}^{n+1},_{\mathbf{S}} = (\mathbf{S}^n\mathbf{S}),_{\mathbf{S}} &= (\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}) \frac{\partial \mathbf{S}^n}{\partial \mathbf{S}} + (\mathbf{S}^n \boxtimes \mathbf{I}) \frac{\partial \mathbf{S}}{\partial \mathbf{S}} \\
&= (\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}) \sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} + (\mathbf{S}^n \boxtimes \mathbf{I})\mathbb{I} \\
&= \sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r} + (\mathbf{S}^n \boxtimes \mathbf{I})\mathbb{I} \\
&= \sum_{r=1}^{n+1} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1}
\end{aligned}$$

And hence the proof.

82. **Given two differentiable tensor valued functions** $\mathrm{U}(\mathrm{S})$ **and** $\mathrm{V}(\mathrm{S})$**, of the tensor** S **show that** $\frac{\partial(\mathrm{U}:\mathrm{V})}{\partial \mathrm{S}} = \left(\frac{\partial \mathrm{U}}{\partial \mathrm{S}}\right)^{\mathbf{T}} \mathrm{V} + \left(\frac{\partial \mathrm{V}}{\partial \mathrm{S}}\right)^{\mathbf{T}} \mathrm{U}$

In component form, let $\mathbf{U} = U_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \mathbf{V} = V_{kl}\mathbf{g}^k \otimes \mathbf{g}^l$ and

$$\mathbf{U}:\mathbf{V} = \left(U_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right):\left(V_{kl}\mathbf{g}^k \otimes \mathbf{g}^l\right)$$
$$= U_{ij}V_{kl}g^{ik}g^{jl}$$

so that,

$$\frac{\partial(\mathbf{U}:\mathbf{V})}{\partial\mathbf{S}} = \frac{\partial U_{ij}}{\partial S_{\alpha\beta}}V_{kl}g^{ik}g^{jl}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta + U_{ij}\frac{\partial V_{kl}}{\partial S_{\alpha\beta}}g^{ik}g^{jl}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= \frac{\partial U_{ij}}{\partial S_{\alpha\beta}}\left(\left(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right) \otimes \left(\mathbf{g}^i \otimes \mathbf{g}^j\right)\right)\left(V_{kl}\mathbf{g}^k \otimes \mathbf{g}^l\right)$$
$$+ \frac{\partial V_{kl}}{\partial S_{\alpha\beta}}\left(\left(\mathbf{g}_{\boldsymbol{\alpha}} \otimes \mathbf{g}_{\boldsymbol{\beta}}\right) \otimes \left(\mathbf{g}^{\boldsymbol{k}} \otimes \mathbf{g}^{\boldsymbol{l}}\right)\right)\left(U_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right)$$
$$= \left(\frac{\partial\mathbf{U}}{\partial\mathbf{S}}\right)^{\mathbf{T}}\mathbf{V} + \left(\frac{\partial\mathbf{V}}{\partial\mathbf{S}}\right)^{\mathbf{T}}\mathbf{U}$$

83. Given two differentiable tensor valued functions $\mathbf{U}(\mathbf{S})$ and $\mathbf{V}(\mathbf{S})$, of the tensor $\mathbf{S}$ show that $\frac{\partial\mathbf{UV}}{\partial\mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{V}^{\mathbf{T}})\frac{\partial\mathbf{U}}{\partial\mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I})\frac{\partial\mathbf{V}}{\partial\mathbf{S}}$

$\mathbf{U} = U_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathrm{V} = V_{kl}\mathbf{g}^k \otimes \mathbf{g}^l$ and $\mathbf{UV} = U_{ij}V^j_{.l}\mathbf{g}^i \otimes \mathbf{g}^l$ so that,

$$\frac{\partial\mathbf{UV}}{\partial\mathbf{S}} = \frac{\partial U_{ij}V^j_{.l}}{\partial S_{\alpha\beta}}\mathbf{g}^i \otimes \mathbf{g}^l \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \left(\frac{\partial U_{ij}}{\partial S_{\alpha\beta}} V^{j}_{.l} + U_{ij} \frac{\partial V^{j}_{.l}}{\partial S_{\alpha\beta}}\right) \mathbf{g}^i \otimes \mathbf{g}^l \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= g^{jk} \left(V_{kl} \frac{\partial U_{ij}}{\partial S_{\alpha\beta}} + U_{ij} \frac{\partial V_{kl}}{\partial S_{\alpha\beta}}\right) \mathbf{g}^i \otimes \mathbf{g}^l \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \left(V_{kl} \frac{\partial U_{ij}}{\partial S_{\alpha\beta}} + U_{ij} \frac{\partial V_{kl}}{\partial S_{\alpha\beta}}\right) (\mathbf{g}^i \otimes \mathbf{g}^j)(\mathbf{g}^k \otimes \mathbf{g}^l) \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \left(\mathbf{I}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)(\mathbf{g}^k \otimes \mathbf{g}^l) V_{kl}\right) \otimes \left(\frac{\partial U_{ij}}{\partial S_{\alpha\beta}} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)$$
$$+ \left(U_{ij} \mathbf{g}^i \otimes \mathbf{g}^j\right)(\mathbf{g}^k \otimes \mathbf{g}^l) \left(\frac{\partial V_{kl}}{\partial S_{\alpha\beta}} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)$$

$$= \left(\mathbf{I}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)\mathbf{V}\right) \otimes \left(\frac{\partial U_{ij}}{\partial \mathbf{S}}\right) + \mathbf{U}(\mathbf{g}^k \otimes \mathbf{g}^l) \frac{\partial V_{kl}}{\partial \mathbf{S}}$$

$$= (\mathbf{I} \boxtimes \mathbf{V}^{\mathrm{T}}) \frac{\partial \mathbf{U}}{\partial \mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I}) \frac{\partial \mathbf{V}}{\partial \mathbf{S}}$$

84. For the positive integer n, if $\mathbf{F}(\mathbf{S}) = \mathbf{S}^n$ for any differentiable tensor, show by mathematical induction that the derivative

$$\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}} = \mathbf{S}^{n-1} \boxtimes \mathbf{I} + \mathbf{S}^{n-2} \boxtimes \mathbf{S}^{\mathrm{T}} + \mathbf{S}^{n-3} \boxtimes \mathbf{S}^{2\mathrm{T}} + \cdots + \mathbf{S} \boxtimes \mathbf{S}^{(n-2)\mathrm{T}} + \mathbf{I} \boxtimes \mathbf{S}^{(n-1)\mathrm{T}}$$

$$= \sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} = \mathbf{S}^{n}{}_{,\mathbf{S}}$$

Case $n = 1$

$$\mathbf{S}_{,\mathbf{S}} = \mathbf{S}^{0} \boxtimes (\mathbf{S}^{\mathrm{T}})^{0} = \mathbf{I} \boxtimes \mathbf{I} = \mathbb{I}$$

as expected.

Case $n = 2$

$$\mathbf{S}^{2}{}_{,\mathbf{S}} = \mathbf{S} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}$$

Assume it is true for arbitrary n we use the fact that $\frac{\partial \mathbf{UV}}{\partial \mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{V}^{\mathrm{T}}) \frac{\partial \mathbf{U}}{\partial \mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I}) \frac{\partial \mathbf{V}}{\partial \mathbf{S}}$ to obtain

$$\mathbf{S}^{n+1}{}_{,\mathbf{S}} = (\mathbf{S}^{n}\mathbf{S})_{,\mathbf{S}} = (\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}) \frac{\partial \mathbf{S}^{n}}{\partial \mathbf{S}} + (\mathbf{S}^{n} \boxtimes \mathbf{I}) \frac{\partial \mathbf{S}}{\partial \mathbf{S}}$$

$$= (\mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}}) \sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} + (\mathbf{S}^{n} \boxtimes \mathbf{I})\mathbb{I}$$

$$= \sum_{r=1}^{n} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r} + (\mathbf{S}^{n} \boxtimes \mathbf{I})\mathbb{I}$$

$$= \sum_{r=1}^{n+1} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1}$$

And hence the proof.

85. For a scalar valued function ϕ and a tensor-valued function $\mathbf{F}$ of the tensor S, show that the derivative of the product,

$$\frac{\partial}{\partial \mathbf{S}}\big(\phi(\mathbf{S})\mathbf{F}(\mathbf{S})\big) = \mathbf{F} \otimes \frac{\partial \phi}{\partial \mathbf{S}} + \phi \frac{\partial \mathbf{F}}{\partial \mathbf{S}}$$

In component form, let $\mathbf{F}(\mathbf{S}) = F_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, and $\mathbf{S} = S_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$. The required derivative, in component form is,

$$\frac{\partial}{\partial \mathbf{S}}\big(\phi(\mathbf{S})\mathbf{F}(\mathbf{S})\big) = \frac{\partial}{\partial S_{\alpha\beta}}(\phi F_{ij})\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \left(\frac{\partial \phi}{\partial S_{\alpha\beta}} F_{ij} + \phi \frac{\partial F_{ij}}{\partial S_{\alpha\beta}}\right)\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \left(F_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \frac{\partial \phi}{\partial S_{\alpha\beta}} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta + \phi \frac{\partial F_{ij}}{\partial S_{\alpha\beta}} \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \mathbf{F} \otimes \frac{\partial \phi}{\partial \mathbf{S}} + \phi \frac{\partial \mathbf{F}}{\partial \mathbf{S}}$$

86. By finding the Gateaux differential of the Cayley-Hamilton equation, $\mathbf{F}(\mathbf{S}) = \mathbf{S}^3 - \operatorname{tr}\mathbf{S}\ \mathbf{S}^2 + \frac{1}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)\mathbf{S} - \mathbf{I}\det\mathbf{S} = \mathbf{0}$, establish the Rivlin's identity,

$$\left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right)\mathbf{T} = \mathbf{S}^2\mathbf{T} + \mathbf{STS} + \mathbf{TS}^2 - \mathbf{S}^2\operatorname{tr}\mathbf{T} - (\mathbf{ST} + \mathbf{TS})\operatorname{tr}\mathbf{S} + \mathbf{S}(\operatorname{tr}\mathbf{T}\operatorname{tr}\mathbf{S} - \operatorname{tr}(\mathbf{ST})) + \frac{\mathbf{T}}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2) - ((\operatorname{cof}\mathbf{S}):\mathbf{T})\mathbf{I}$$

Cayley-Hamilton states that a tensor satisfies its own characteristic equation, hence,

$$\mathbf{F}(\mathbf{S}) = \mathbf{S}^3 - \operatorname{tr}\mathbf{S}\ \mathbf{S}^2 + \frac{1}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)\mathrm{S} - \mathbf{I}\det\mathbf{S} = \mathbf{0}$$

The Gateaux differential for the function $\mathbf{F}(\mathbf{S})$ in the direction of T is,

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}\mathbf{F}(\mathbf{S} + \alpha\mathbf{T}) = \left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right)\mathbf{T} = \mathbf{0}$$

It is possible to take the Gateaux limit directly, however, in this case, the differential can be more easily found indirectly by taking the derivative, $\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}$ term by term, using two formulae: $\mathbf{S}^n{}_{,\mathbf{S}} = \sum_{r=1}^{n}\mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1}$; $(\mathbf{UV})_{,\mathrm{S}} = (\mathbf{I} \boxtimes \mathbf{V}^{\mathrm{T}})\frac{\partial \mathbf{U}}{\partial \mathbf{S}} + (\mathbf{U} \boxtimes \mathbf{I})\frac{\partial \mathbf{V}}{\partial \mathbf{S}}$

$$\mathbf{S}^3{}_{,\mathbf{S}} = \sum_{r=1}^{3} \mathbf{S}^{n-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} = \mathbf{S}^2 \boxtimes \mathbf{I} + \mathbf{S} \boxtimes \mathbf{S}^2 + \mathbf{I} \boxtimes (\mathbf{S}^{\mathrm{T}})^2$$

so that

$$\left(\frac{\partial \mathbf{S}^3}{\partial \mathbf{S}}\right)\mathbf{T} = \mathbf{S}^2\mathbf{T} + \mathbf{STS} + \mathbf{TS}^2$$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}}(I_1\mathbf{S}^2) &= \mathbf{S}^2 \otimes \frac{\partial I_1}{\partial \mathbf{S}} + I_1 \frac{\partial \mathbf{S}^2}{\partial \mathbf{S}} \\ &= \mathbf{S}^2 \otimes \mathbf{I} + I_1(\mathbf{S} \boxtimes \mathbf{I} + \mathbf{I} \boxtimes \mathbf{S}^{\mathrm{T}})\end{aligned}$$

so that,

$$\left(\frac{\partial}{\partial \mathbf{S}}(I_1\mathbf{S}^2)\right)\mathbf{T} = \mathbf{S}^2 \operatorname{tr}\mathbf{T} + (\mathbf{ST} + \mathbf{TS}) \operatorname{tr}\mathbf{S}$$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}}(I_2\mathbf{S}) &= \mathbf{S} \otimes \frac{\partial I_2}{\partial \mathbf{S}} + I_2 \frac{\partial \mathbf{S}}{\partial \mathbf{S}} \\ &= \mathbf{S}^2 \otimes (\mathbf{I} \operatorname{tr}\mathbf{S} - \mathbf{S}^{\mathrm{T}}) + \frac{1}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)\mathbb{I}\end{aligned}$$

$$\begin{aligned}\left(\frac{\partial}{\partial S}(I_2\mathbf{S})\right)\mathbf{T} &= \Big(\mathbf{S} \otimes (\mathbf{I} \operatorname{tr}\mathbf{S} - \mathbf{S}^{\mathrm{T}})\Big)\mathbf{T} + \frac{1}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)\mathbb{I}\mathbf{T} \\ &= \mathbf{S}\Big((\mathbf{I} \operatorname{tr}\mathbf{S} - \mathbf{S}^{\mathrm{T}}):\mathbf{T}\Big) + \frac{\mathbf{T}}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)\end{aligned}$$

$$= \mathbf{S}(\operatorname{tr}\mathbf{T}\operatorname{tr}\mathbf{S} - \operatorname{tr}(\mathbf{ST})) + \frac{\mathbf{T}}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2)$$

Lastly, $\frac{\partial}{\partial \mathbf{S}}(\det \mathbf{S}) = \operatorname{cof}\mathbf{S}$. We can therefore write,

$$\left(\frac{\partial \mathbf{F}(\mathbf{S})}{\partial \mathbf{S}}\right)\mathbf{T} = \mathbf{S}^2\mathbf{T} + \mathbf{STS} + \mathbf{TS}^2 - \mathbf{S}^2\operatorname{tr}\mathbf{T} - (\mathbf{ST} + \mathbf{TS})\operatorname{tr}\mathbf{S} + \mathbf{S}(\operatorname{tr}\mathbf{T}\operatorname{tr}\mathbf{S} - \operatorname{tr}(\mathbf{ST}))$$
$$+ \frac{\mathbf{T}}{2}(\operatorname{tr}^2\mathbf{S} - \operatorname{tr}\mathbf{S}^2) - \big((\operatorname{cof}\mathbf{S}):\mathbf{T}\big)\mathbf{I}$$

which is the Rivlin identity.

87. Given that $\mathrm{S}^{-n}{}_{,\mathrm{S}} = \sum_{r=-n+1}^{0} -\mathrm{S}^{-n-r} \boxtimes (\mathrm{S}^{\mathrm{T}})^{r-1}$ If $\phi(\mathrm{S}) = \operatorname{tr}\left(\mathrm{S}^{-1}\mathrm{S}^{-1}\right)$ show that $\frac{\partial}{\partial \mathrm{S}}\phi(\mathrm{S}) = -2\mathrm{S}^{-3\mathrm{T}}$

Clearly,

$$\mathbf{S}^{-2}{}_{,\mathbf{S}} = \sum_{r=-2+1}^{0} -\mathbf{S}^{-2-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} = -\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-2\mathrm{T}} - \mathbf{S}^{-2} \boxtimes \mathbf{S}^{-\mathrm{T}}$$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}}\phi(\mathbf{S}) &= \left(\frac{\partial}{\partial \mathbf{S}^{-2}}\operatorname{tr}(\mathbf{S}^{-2})\right)\left(\frac{\partial \mathbf{S}^{-2}}{\partial \mathbf{S}}\right)\\ &= -\mathbf{I}(-\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-2\mathrm{T}} - \mathbf{S}^{-2} \boxtimes \mathbf{S}^{-\mathrm{T}})\\ &= -\mathbf{S}^{-3\mathrm{T}} - \mathbf{S}^{-3\mathrm{T}} = -2\mathbf{S}^{-3\mathrm{T}}\end{aligned}$$

88. Given that $\mathrm{S}^{-n}{}_{,\mathrm{S}} = \sum_{r=-n+1}^{0} -\mathrm{S}^{-n-r} \boxtimes (\mathrm{S}^{\mathrm{T}})^{r-1}$ **If** $\boldsymbol{\phi}(\mathrm{S}) = (\mathbf{det}\,\mathrm{S})(\mathrm{S}^{-1}:\mathrm{S}^{-1})$ **show that** $\frac{\partial}{\partial \mathrm{S}}\boldsymbol{\phi}(\mathrm{S}) = (\mathbf{cof}\,\mathrm{S})\mathrm{S}^{-1}:\mathrm{S}^{-1} - \mathbf{det}\,\mathrm{S}\left(\mathrm{S}^{-1}\mathrm{S}^{-1}\mathrm{S}^{-\mathrm{T}} + \mathrm{S}^{-\mathrm{T}}\mathrm{S}^{-1}\mathrm{S}^{-\mathrm{T}}\right)$

Clearly,

$$\mathbf{S}^{-1}{}_{,\mathbf{S}} = \sum_{r=-1+1}^{0} -\mathbf{S}^{-1-r} \boxtimes (\mathbf{S}^{\mathrm{T}})^{r-1} = -\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}}$$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}}\phi(\mathbf{S}) &= \left(\frac{\partial(\det \mathbf{S})}{\partial \mathbf{S}}\mathbf{S}^{-1} + \det \mathbf{S}\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{S}}\right):\mathbf{S}^{-1} + (\det \mathbf{S})\mathbf{S}^{-1}:\frac{\partial \mathbf{S}^{-1}}{\partial \mathbf{S}} \\ &= (\mathrm{cof}\,\mathbf{S})\mathbf{S}^{-1}:\mathbf{S}^{-1} - \det \mathbf{S}\,(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}})\mathbf{S}^{-1} - (\det \mathbf{S})\mathbf{S}^{-1}(\mathbf{S}^{-1} \boxtimes \mathbf{S}^{-\mathrm{T}}) \\ &= (\mathrm{cof}\,\mathbf{S})\mathbf{S}^{-1}:\mathbf{S}^{-1} - \det \mathbf{S}\,(\mathbf{S}^{-1}\mathbf{S}^{-1}\mathbf{S}^{-1}) - (\det \mathbf{S})(\mathbf{S}^{-\mathrm{T}}\mathbf{S}^{-1}\mathbf{S}^{-\mathrm{T}})\end{aligned}$$

89. If $\boldsymbol{\phi}(\mathrm{S}) = (\mathbf{cof}\,\mathrm{S}):(\mathbf{cof}\,\mathrm{S}) = \frac{1}{2}\left((\mathrm{S}:\mathrm{S})^{2} - (\mathrm{S}^{\mathrm{T}}\mathrm{S}):(\mathrm{S}^{\mathrm{T}}\mathrm{S})\right)$ **show that** $\frac{\partial}{\partial \mathrm{S}}\boldsymbol{\phi}(\mathrm{S}) = \mathbf{2}(\mathrm{S}:\mathrm{S})\mathrm{S} + \mathbf{2}\mathrm{S}^{\mathrm{T}}\mathrm{S}\mathrm{S}^{\mathrm{T}}$

$$\frac{\partial \mathbf{S}:\mathbf{S}}{\partial \mathbf{S}} = \frac{\partial \mathbf{S}}{\partial \mathbf{S}}\mathbf{S} + \mathbf{S}\frac{\partial \mathbf{S}}{\partial \mathbf{S}} = \mathbb{I}\mathbf{S} + \mathbf{S}\mathbb{I} = 2\mathbf{S}$$

$$\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}} = \frac{\partial \mathbf{S}^{\mathrm{T}}}{\partial \mathbf{S}}\mathbf{S} + \mathbf{S}^{\mathrm{T}}\frac{\partial \mathbf{S}}{\partial \mathbf{S}} = \mathbb{T}\mathbf{S} + \mathbf{S}^{\mathrm{T}}\mathbb{I} = 2\mathbf{S}^{\mathrm{T}}$$

$$\frac{\partial}{\partial \mathbf{S}}\phi(\mathbf{S}) = \frac{1}{2}\left(\frac{\partial(\mathbf{S}:\mathbf{S})^2}{\partial \mathbf{S}:\mathbf{S}}\right)\left(\frac{\partial \mathbf{S}:\mathbf{S}}{\partial \mathbf{S}}\right) + \frac{1}{2}\left(\frac{\partial\left((\mathbf{S}^{\mathrm{T}}\mathbf{S}):(\mathbf{S}^{\mathrm{T}}\mathbf{S})\right)}{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}\right)\left(\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}}\right).$$

$$\frac{\partial\left((\mathbf{S}^{\mathrm{T}}\mathbf{S}):(\mathbf{S}^{\mathrm{T}}\mathbf{S})\right)}{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}} = \left(\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}}\right)\mathbf{S}^{\mathrm{T}}\mathbf{S} + \mathbf{S}^{\mathrm{T}}\mathbf{S}\left(\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}}\right) = 2\mathbf{S}^{\mathrm{T}}\mathbf{S}$$

Consequently,

$$\begin{aligned}\frac{\partial}{\partial \mathbf{S}}\phi(\mathbf{S}) &= \frac{1}{2}\left(\frac{\partial(\mathbf{S}:\mathbf{S})^2}{\partial \mathbf{S}:\mathbf{S}}\right)\left(\frac{\partial \mathbf{S}:\mathbf{S}}{\partial \mathbf{S}}\right) + \frac{1}{2}\left(\frac{\partial\left((\mathbf{S}^{\mathrm{T}}\mathbf{S}):(\mathbf{S}^{\mathrm{T}}\mathbf{S})\right)}{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}\right)\left(\frac{\partial \mathbf{S}^{\mathrm{T}}\mathbf{S}}{\partial \mathbf{S}}\right)\\ &= 2(\mathbf{S}:\mathbf{S})\mathbf{S} + 2\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{S}^{\mathrm{T}}\end{aligned}$$

90. By the representation theorem, every real isotropic tensor function is a function of its principal invariants. Show that for every isotropic function $\boldsymbol{f}(\mathrm{S})$ of an invertible tensor S the derivative $\frac{\partial \boldsymbol{f}(\mathrm{S})}{\partial \mathrm{S}}$ can be represented as a linear combination I, $\mathrm{S}^{\mathbf{T}}$ and $\mathrm{S}^{-\mathbf{T}}$.

By the representation theorem,

$$f(\mathbf{S}) = \phi\big(I_1(\mathbf{S}), I_2(\mathbf{S}), I_3(\mathbf{S})\big)$$

and, when $\mathbf{S}$ is invertible, $\frac{\partial I_3}{\partial \mathbf{S}} = \operatorname{cof}\mathbf{S} = I_3\mathbf{S}^{-\mathrm{T}}$ consequently,

$$\begin{aligned}\frac{\partial f(\mathbf{S})}{\partial \mathbf{S}} &= \frac{\partial f(\mathbf{S})}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{S}} + \frac{\partial f(\mathbf{S})}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{S}} \\ &= \frac{\partial f(\mathbf{S})}{\partial I_1}\mathbf{I} + \frac{\partial f(\mathbf{S})}{\partial I_2}(\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}) + \frac{\partial f(\mathbf{S})}{\partial I_3}I_3\mathbf{S}^{-\mathrm{T}} \\ &= \left(\frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2}I_1(\mathbf{S})\right)\mathbf{I} - \left(\frac{\partial f(\mathbf{S})}{\partial I_2}\right)\mathbf{S}^{\mathrm{T}} + \frac{\partial f(\mathbf{S})}{\partial I_3}I_3\mathbf{S}^{-\mathrm{T}} \\ &= \alpha_0\mathbf{I} + \alpha_1\mathbf{S}^{\mathrm{T}} + \alpha_{-1}\mathbf{S}^{-\mathrm{T}}\end{aligned}$$

where $\alpha_0 = \frac{\partial f(\mathbf{S})}{\partial I_1} + \frac{\partial f(\mathbf{S})}{\partial I_2}I_1(\mathbf{S}) + \frac{\partial f(\mathbf{S})}{\partial I_3}I_2(\mathbf{S})$, $\alpha_1 = -\frac{\partial f(\mathbf{S})}{\partial I_2} - \frac{\partial f(\mathbf{S})}{\partial I_3}I_1(\mathbf{S})$ and $\alpha_{-1} = \frac{\partial f(\mathbf{S})}{\partial I_3}I_3$ is the desired representation of the derivative.

91. Find the derivative of a vector point function $\mathrm{u}(x^1, x^2, x^3)$ using its covariant components, $u_\alpha(x^1, x^2, x^3)$, $\alpha = 1, 2, 3$

Let $\mathbf{g}^\alpha$ be the reciprocal basis vectors, Let

$$\begin{aligned}\mathbf{u} = u_\alpha\mathbf{g}^\alpha, \text{ then } d\mathbf{u} &= \frac{\partial}{\partial x^k}(u_\alpha\mathbf{g}^\alpha)dx^k \\ &= \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha\right)dx^k\end{aligned}$$

Clearly,

$$\frac{\partial \mathbf{u}}{\partial x^k} = \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha$$

And the projection of this quantity on the $\mathbf{g}^i$ direction is,

$$\frac{\partial \mathbf{u}}{\partial x^k}\cdot\mathbf{g}_i = \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha\right)\cdot\mathbf{g}_i$$

$$= \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha\cdot\mathbf{g}_i + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}\cdot\mathbf{g}_i u_\alpha$$

$$= \frac{\partial u_\alpha}{\partial x^k}\delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}\cdot\mathbf{g}_i u_\alpha$$

Now, $\mathbf{g}^i\cdot\mathbf{g}_j = \delta_i^j$ so that

$$\frac{\partial \mathbf{g}^i}{\partial x^k}\cdot\mathbf{g}_j + \mathbf{g}^i\cdot\frac{\partial \mathbf{g}_j}{\partial x^k} = 0.$$

$$\frac{\partial \mathbf{g}^i}{\partial x^k}\cdot\mathbf{g}_j = -\mathbf{g}^i\cdot\frac{\partial \mathbf{g}_j}{\partial x^k} \equiv -\begin{Bmatrix} i \\ jk \end{Bmatrix}.$$

This important quantity, necessary to quantify the derivative of a tensor in general coordinates, is called the Christoffel Symbol of the second kind.

Using this, we can now write that,

$$\frac{\partial \mathbf{u}}{\partial x^k}\cdot\mathbf{g}_i = \frac{\partial u_\alpha}{\partial x^k}\delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}\cdot\mathbf{g}_i u_\alpha$$

$$= \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} u_\alpha$$

The quantity on the RHS is the component of the derivative of vector **u** along the $\mathbf{g}_i$ direction using covariant components. It is also called the covariant derivative of **u**.

92. Find the derivative of a vector field using contravariant components

$$d\mathbf{u} = \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \frac{\partial \mathbf{g}_i}{\partial x^k} u^i \right) dx^k = \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha u^i \right) dx^k$$

So that,

$$\frac{\partial \mathbf{u}}{\partial x^k} = \frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha$$

The components of this in the direction of $\mathbf{g}_i$ can be obtained by taking a dot product as before:

$$\frac{\partial \mathbf{u}}{\partial x^k} \cdot \mathbf{g}^i = \left(\frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha \right) \cdot \mathbf{g}^i$$

$$= \frac{\partial u^\alpha}{\partial x^k} \delta_\alpha^i + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} \cdot \mathbf{g}^i u^\alpha$$

$$= \frac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} u^\alpha$$

93. Express the Christoffel symbols in terms of the metric tensor coefficients

We observe that the derivative of the covariant basis, $\mathbf{g}_i \left(= \frac{\partial \mathbf{r}}{\partial x^i}\right)$,

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial^2 \mathbf{r}}{\partial x^j \partial x^i} = \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \frac{\partial \mathbf{g}_j}{\partial x^i}$$

Taking the dot product with $\mathbf{g}_k$,

$$\begin{aligned}
\frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k &= \frac{1}{2}\left(\frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k + \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_k}{\partial x^i} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_k}{\partial x^j}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_i}{\partial x^k} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \frac{\partial}{\partial x^k}[\mathbf{g}_i \cdot \mathbf{g}_j]\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)
\end{aligned}$$

Which is the quantity defined as the *Christoffel symbols of the first kind* in the textbooks. It is therefore possible for us to write,

$$[ij, k] \equiv \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k = \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

It should be emphasized that the Christoffel symbols, even though the play a critical role in several tensor relationships, are themselves NOT tensor quantities. (Prove this). However, notice their symmetry in the i and j. The extension of this definition to the Christoffel symbols of the second kind is immediate:

Contract the above equation with the conjugate metric tensor, we have,

$$g^{k\alpha}[ij,\alpha] \equiv g^{k\alpha} \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_\alpha = g^{k\alpha} \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_\alpha = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k = \begin{Bmatrix} k \\ ij \end{Bmatrix} = -\frac{\partial \mathbf{g}^k}{\partial x^j} \cdot \mathbf{g}_i$$

Which connects the common definition of the second Christoffel symbol with the one defined in the above derivation. The relationship,

$$g^{k\alpha}[ij,\alpha] = \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

apart from defining the relationship between the Christoffel symbols of the first kind and that second kind, also highlights, once more, the index-raising property of the conjugate metric tensor.

94. Demonstrate the index raising and index lowering attributes of the metric coefficients on Christoffel symbols. (despite the fact that these are, strictly speaking, not tensors in themselves).

We contract the above equation with $g_{k\beta}$ and obtain,

$$g_{k\beta}g^{k\alpha}[ij,\alpha] = g_{k\beta}\begin{Bmatrix} k \\ ij \end{Bmatrix}\delta^{\alpha}_{\beta}\ [ij,\alpha]$$
$$= [ij,\beta] = g_{k\beta}\begin{Bmatrix} k \\ ij \end{Bmatrix}$$

so that,

$$g_{k\alpha}\begin{Bmatrix} \alpha \\ ij \end{Bmatrix} = [ij,k]$$

Showing that the metric tensor can be used to lower the contravariant index of the Christoffel symbol of the second kind to obtain the Christoffel symbol of the first kind.

95. Find the derivative of a second-order tensor field in terms of its contravariant components.

For a second-order tensor $\boldsymbol{T}$, we can express the components in dyadic form along the product basis as follows:

$$\mathbf{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T^{i}_{.j}\mathbf{g}_i \otimes \mathbf{g}^j = T_{j}^{\cdot\, i}\mathbf{g}^j \otimes \mathbf{g}_i$$

This is perfectly analogous to our expanding vectors in terms of basis and reciprocal bases. Derivatives of the tensor may therefore be expressible in any of these product bases. Consider the product covariant bases.

We have:

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}$$

Recall that, $\frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j = \begin{Bmatrix} j \\ ik \end{Bmatrix}$. It follows therefore that,

$$\begin{aligned}\frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\delta^j_\alpha &= \frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha \cdot \mathbf{g}^j \\ &= \left(\frac{\partial \mathbf{g}_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha\right) \cdot \mathbf{g}^j \\ &= 0.\end{aligned}$$

Clearly, $\frac{\partial \mathbf{g}_i}{\partial x^k} = \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha$

(Obviously since $\mathbf{g}^j$ is a basis vector it cannot vanish)

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k} \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\left(\begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha\right) \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \left(\begin{Bmatrix} \alpha \\ jk \end{Bmatrix}\mathbf{g}_\alpha\right) \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{\alpha j}\left(\begin{Bmatrix} i \\ \alpha k \end{Bmatrix}\mathbf{g}_i\right) \otimes \mathbf{g}_j + T^{i\alpha}\mathbf{g}_i \otimes \left(\begin{Bmatrix} j \\ \alpha k \end{Bmatrix}\mathbf{g}_j\right) \\ &= \left(\frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\begin{Bmatrix} i \\ \alpha k \end{Bmatrix} + T^{i\alpha}\begin{Bmatrix} j \\ \alpha k \end{Bmatrix}\right)\mathbf{g}_i \otimes \mathbf{g}_j = T^{ij}{}_{,k}\mathbf{g}_i \otimes \mathbf{g}_j\end{aligned}$$

Where

$$T^{ij}_{\ \ ,k} = \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j} \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} + T^{i\alpha} \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} \text{ or } \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\Gamma^i_{\alpha k} + T^{i\alpha}\Gamma^j_{\alpha k}$$

are the components of the covariant derivative of the tensor **T** in terms of contravariant components on the product covariant bases as shown.

96. Find the derivative of a second-order tensor field in terms of its covariant components.

In the same way, by taking the tensor expression in the dyadic form of its contravariant product bases, we can write,

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$
$$= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\Gamma^i_{\alpha k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$

Again, notice from previous derivation above, $\begin{Bmatrix} i \\ jk \end{Bmatrix} = -\frac{\partial \mathbf{g}^i}{\partial x^k} \cdot \mathbf{g}_j$ so that, $\frac{\partial \mathbf{g}^i}{\partial x^k} = -\begin{Bmatrix} i \\ \alpha k \end{Bmatrix}\mathbf{g}^\alpha = -\Gamma^i_{\alpha k}\mathbf{g}^\alpha$ Therefore,

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$
$$= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j - T_{ij}\Gamma^i_{\alpha k}\mathbf{g}^\alpha \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \Gamma^j_{\alpha k}\mathbf{g}^\alpha$$

$$= \left(\frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^{\alpha}_{ik} - T_{i\alpha}\Gamma^{\alpha}_{jk}\right)\mathbf{g}^i \otimes \mathbf{g}^j$$
$$= T_{ij,k}\mathbf{g}^i \otimes \mathbf{g}^j$$

So that

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^{\alpha}_{ik} - T_{i\alpha}\Gamma^{\alpha}_{jk}$$

97. Show that metric tensor components behave like constants under a covariant differentiation. The proof that this is so is due to Ricci:

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - \begin{Bmatrix}\alpha \\ ik\end{Bmatrix} g_{\alpha j} - \begin{Bmatrix}\beta \\ kj\end{Bmatrix} g_{i\beta}$$
$$= \frac{\partial g_{ij}}{\partial x^k} - [ik, j] - [kj, i]$$
$$= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right) - \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right) = 0.$$

The conjugate metric tensor behaves the same way as can be seen from the relationship,

$$g_{il}g^{lj} = \delta_i^j$$

The above can be differentiated covariantly with respect to x^k to obtain

$$
\begin{aligned}
g_{il},_k\, g^{lj} + g_{il} g^{lj},_k &= \delta_i^j,_k\, 0 + g_{il} g^{lj},_k \\
&= \frac{\partial \delta_i^j}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \delta_\alpha^j + \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} \delta_i^\alpha g_{il} g^{lj},_k \\
&= 0 - \begin{Bmatrix} j \\ ik \end{Bmatrix} + \begin{Bmatrix} j \\ ik \end{Bmatrix} = 0
\end{aligned}
$$

The contraction of g_{il} with $g^{lj},_k$ vanishes. Since we know that the metric tensor cannot vanish in general, we can only conclude that

$$g^{ij},_k = 0$$

98. Show that the basis vectors are constants under covariant differentiation. In other words, their covariant derivatives vanish

Recall that we can express the covariant basis vectors in terms of their dual,

$$\mathbf{g}_i = g_{ij} \mathbf{g}^j$$

By Ricci's theorem, the derivatives $g_{ij},_j$ vanish, hence for the vectors $\mathbf{g}_i$ themselves, the covariant derivatives vanish so that, $\mathbf{g}_i,_j = 0$. In the same way, $\mathbf{g}^i,_j = 0$.

99. Show that the alternating tensor, $\epsilon_{ijk},_l = 0$ that is, the Einstein tensor component is a constant under covariant differentiation.

Recall that

$$\epsilon_{ijk} = \mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k$$

Taking the covariant derivatives, we have,

$$\epsilon_{ijk,l} = \mathbf{g}_{i,l} \cdot \mathbf{g}_j \times \mathbf{g}_k + \mathbf{g}_i \cdot \mathbf{g}_{j,l} \times \mathbf{g}_k + \mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_{k,l} = 0$$

Hence the alternating tensor is a constant under covariant differentiation.

100. Let $h_1 = \sqrt{g_{11}}, h_2 = \sqrt{g_{22}}$ and $h_3 = \sqrt{g_{33}}$ in an orthogonal curvilinear system of coordinates, show that $\frac{1}{\epsilon_{123}} \frac{\partial \epsilon_{123}}{\partial x^j} = \frac{1}{h_1 h_2 h_3} \frac{\partial h_1 h_2 h_3}{\partial x^j} = \begin{Bmatrix} i \\ ij \end{Bmatrix}$

The covariant derivative,

$$\epsilon_{ijk,l} = \frac{\partial \epsilon_{ijk}}{\partial x^l} - \begin{Bmatrix} \alpha \\ il \end{Bmatrix} \epsilon_{\alpha jk} - \begin{Bmatrix} \beta \\ jl \end{Bmatrix} \epsilon_{i\beta k} - \begin{Bmatrix} \gamma \\ kl \end{Bmatrix} \epsilon_{ij\gamma} = 0$$

is valid for all admissible values of i, j and k. In particular, setting $i = 1, j = 2$ and $k = 3$, we obtain,

$$\begin{aligned} \epsilon_{123,l} &= \frac{\partial \varepsilon_{123}}{\partial x^l} - \begin{Bmatrix} \alpha \\ 1l \end{Bmatrix} \epsilon_{\alpha 23} - \begin{Bmatrix} \beta \\ 2l \end{Bmatrix} \epsilon_{1\beta 3} - \begin{Bmatrix} \gamma \\ 3l \end{Bmatrix} \epsilon_{12\gamma} = 0 \\ &= \frac{\partial \epsilon_{123}}{\partial x^l} - \begin{Bmatrix} 1 \\ 1l \end{Bmatrix} \epsilon_{123} - \begin{Bmatrix} 2 \\ 2l \end{Bmatrix} \epsilon_{123} - \begin{Bmatrix} 3 \\ 3l \end{Bmatrix} \epsilon_{123} \\ &= \epsilon_{123,l} = \frac{\partial \epsilon_{123}}{\partial x^l} - \begin{Bmatrix} i \\ il \end{Bmatrix} \epsilon_{123} = 0 \end{aligned}$$

Hence,

$$\frac{1}{h_1 h_2 h_3}\frac{\partial h_1 h_2 h_3}{\partial x^j} = \frac{1}{\epsilon_{123}}\frac{\partial \epsilon_{123}}{\partial x^j}$$

$$= \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^j} = \begin{Bmatrix} i \\ ij \end{Bmatrix}$$

$$= \frac{\partial \log \sqrt{g}}{\partial x^j} = \begin{Bmatrix} i \\ ij \end{Bmatrix}$$

101. Obtain a computational formula for the Laplacian in general coordinates

$$F^i{}_{,j} = \frac{\partial F^i}{\partial x^j} + \begin{Bmatrix} i \\ \alpha j \end{Bmatrix} F^\alpha$$

Contracting this second order tensor defines an invariant which is the divergence of F^i:

$$F^i{}_{,i} = \frac{\partial F^i}{\partial x^i} + \begin{Bmatrix} i \\ \alpha i \end{Bmatrix} F^\alpha = \frac{\partial F^i}{\partial x^i} + \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^i}$$

$$= \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}F^i\right)}{\partial x^j}$$

Let $F^i = g^{ij}\frac{\partial \Phi}{\partial x^j} = g^{ij}\phi_{,j}$ in the above equation, the we have,

$$g^{ij}\phi_{,ij} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} g^{ij} \phi_{,j}\right)}{\partial x^j}$$

102. Find the expression for $\nabla^2 \phi = 0$ in cylindrical and spherical coordinates.

In indicial form, this is the same as $g^{ij}\phi_{,ij} = 0$ or, as the above computation formula states,

$$\frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} g^{ij} \phi_{,j}\right)}{\partial x^j} = 0.$$

In Cylindrical coordinates, $g = r^2, g^{11} = 1, g^{22} = \frac{1}{r^2}, g^{33} = 1$ so that,

$$\nabla^2 \psi = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} g^{ij} \psi_{,j}\right)}{\partial x^j} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r}{r} \frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r \frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

In Spherical coordinates, $g = \rho^4 \sin^2 \theta\,, g^{11} = 1, g^{22} = \frac{1}{\rho^2}, g^{33} = \frac{1}{\rho^2 \sin \theta}$.

$$\nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} g^{ij} \psi_{,j}\right)}{\partial x^j}$$

$$= \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial\rho}\left(\rho^2 \sin\theta \frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(\frac{\rho^2 \sin\theta}{\rho^2}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho^2 \sin\theta}{\rho^2 \sin\theta}\frac{\partial\psi}{\partial z}\right)\right]$$

$$= \frac{\partial^2\psi}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\cos\theta}{\rho^2 \sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{\rho^2 \sin\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

103. The transformation equations from the Cartesian to the oblate spheroidal coordinates ξ, η and φ are: $x = f\xi\eta \sin\varphi$, $y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}$, and $z = f\xi\eta\cos\varphi$, where f is a constant representing the half the distance between the foci of a family of confocal ellipses. Find the components of the metric tensor in this system.

The metric tensor components are:

$$g_{\xi\xi} = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 + \left(\frac{\partial z}{\partial\xi}\right)^2$$

$$= (f\eta\sin\varphi)^2 + f^2\xi^2\left(\frac{1-\eta^2}{\xi^2-1}\right) + (f\eta\cos\varphi)^2 = f^2\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)$$

$$g_{\eta\eta} = \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 + \left(\frac{\partial z}{\partial\eta}\right)^2 = f^2\left(\frac{\xi^2-\eta^2}{1-\xi^2}\right)$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = (f\xi\eta)^2$$

$$g_{\xi\eta} = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial z}{\partial \xi}\right)\left(\frac{\partial z}{\partial \eta}\right)$$

$$= (f\eta \sin\varphi)(f\xi \sin\varphi) - \left(f\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\right)\left(f\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\right) + (f\eta\cos\varphi)(f\xi\cos\varphi)$$

$$= 0 = g_{\eta\varphi} = g_{\varphi\xi}$$

104. Show that the oblate spheroidal coordinate systems are orthogonal. Find an expression for the Laplacian of a scalar function in this system.

Example above shows that $g_{\xi\eta} = g_{\eta\varphi} = g_{\varphi\xi} = 0$. This is the required proof of orthogonality. Using the computation formula, $\nabla^2\psi = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}g^{ij}\psi_{,j})}{\partial x^j}$ we may write for the oblate spheroidal coordinates that,

$$\nabla^2\Phi = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{f^3\xi^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(f\xi\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(f\xi\eta\sqrt{\frac{1-\eta^2}{\xi^2-1}}\frac{\partial\Phi}{\partial\eta}\right)\right]$$

$$+ \frac{\partial}{\partial\eta}\left(\frac{f(\xi^2-\eta^2)}{\xi\eta\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial\Phi}{\partial\eta}\right)$$

105. Find the physical components of the vectors and second-order tensors in the coordinate system defined by the curvilinear system $\mathrm{r}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) = \boldsymbol{x}^{\boldsymbol{i}}(\boldsymbol{u}^1,\boldsymbol{u}^2,\boldsymbol{u}^3)\mathrm{e}_{\boldsymbol{i}} = \mathrm{r}(\boldsymbol{u}^1,\boldsymbol{u}^2,\boldsymbol{u}^3)$ above, given that the fundamental quadratic form for any system with coordinates $\boldsymbol{x}^{\boldsymbol{i}}$ is, $(\boldsymbol{ds})^2 = \boldsymbol{g}_{\boldsymbol{ij}}\, \boldsymbol{dx}^{\boldsymbol{i}}\, \boldsymbol{dx}^{\boldsymbol{j}}$

This invariant is precisely the scalar product of a small line element contravariant vector with itself. If this element lies entirely on the x^1 axis, we then find the length of an arc element whose tensor coordinates are $(dx^1, 0,0)$. This length we calculate by plugging these coordinates in

$$(ds_1)^2 = g_{11}\,(dx^1)^2$$

all other terms vanishing. It is therefore clear that the vectors with tensor coordinates,

$(dx^1, 0,0), (0, dx^2, 0)$ and $(0,0, dx^3)$ have the lengths $dx^1\sqrt{g_{11}}, dx^2\sqrt{g_{22}}$ and $dx^3\sqrt{g_{33}}$ respectively. Let $\lambda^i{}_{(1)}$, $\lambda^i{}_{(2)}$ and $\lambda^i{}_{(3)}$ be the unit vectors coinciding with the coordinate axes x^i, $i = 1,2,3$. From the above calculations, it is clear that, $\lambda^i{}_{(j)} = \frac{1}{h_j}\delta^i_j$ (no sum on j) where $h_i = (g_{ii})^{1/2}$, (no sum on i). Let $\alpha^i{}_{(j)} = h_j\, \delta^i_j$ (no sum on j) so that the nonvanishing element of the 's are inverses of those of the λ 's.

We first use the above result to calculate the angles between the coordinate curves. If θ_{12} denotes the angle between the x^1 and x^2 coordinates, then from we can write,

$$\cos\theta_{12} = g_{12}\lambda^1{}_{(1)}\lambda^2{}_{(2)} = \frac{g_{12}}{h_1 h_2} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}$$

With this same notations, we can proceed to write,

$\cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}}$ and $\cos\theta_{33} = \frac{g_{31}}{\sqrt{g_{33}g_{11}}}$

2.46

If the coordinate axes are orthogonal the cosines of the coordinate angles vanish and clearly from these equations we easily see that $g_{ij} = 0$ when $i \neq j$.

From the above discussion, we saw that the tensor coordinates do not necessarily equal projection distances on the coordinate axes. It is tempting to expect this to be the case because of our familiar Cartesian coordinate system. We discovered in the above discussion for example that the unit vector on the x^1 coordinate axes does NOT have the tensor coordinates (1,0,0) as we might have expected. Rather, its coordinates are, $\left(\frac{1}{h_1}, 0,0\right)$. When the space is Euclidean and the coordinate system is Cartesian, $h_i = 1$. The physical components of (1,0,0) on the coordinate axes (that is their actual projections on the coordinate axes) from 2.43 are $\left(\sqrt{g_{11}}, 0,0\right)$ while

that of $\left(\frac{1}{h_1}, 0,0\right)$are (1,0,0). Consider the Cartesian coordinate system formed by the unit vectors ${\lambda^i}_{(1)}$, ${\lambda^i}_{(2)}$ and ${\lambda^i}_{(3)}$. If
$A^{(1)}$ is the projection of an arbitrary contravariant vector on the ${\lambda^i}_{(1)}$ axis, then we can see that,

$$A^{(1)} = A^i {\alpha^i}_{(1)} = A^1 h_1$$

so that $A^{(2)} = A^2 h_2$ and $A^{(3)} = A^3 h_3$.
Writing $(ds)^2 = g_{ij}\, dx_i\, dx_j$ instead of eq. 2.42, we could have developed the physical components of the covariant vector by similar arguments. In this case, the physical components of the covariant vector A_i associated with A^i are:
$A_{(1)} = \frac{A_1}{h_1}, A_{(2)} = \frac{A_2}{h_2}$ and $A_{(3)} = \frac{A_3}{h_3}$.
2.48
The physical components of higher order tensors can be evaluated in a similar way. For the contravariant second-order tensor, τ^{ij} component in the ${\lambda^i}_{(1)}$ direction is,
$\tau^{(11)} = \tau^{ij} {\alpha^i}_{(1)} {\alpha^j}_{(1)} = \tau^{11}(h_1)^2$
2.49
Similarly, $\tau^{(22)} = \tau^{22}(h_2)^2$, and $\tau^{(33)} = \tau^{33}(h_3)^2$. The physical component, $\tau^{(12)}$ can be found:

$$\tau^{(12)} = \tau^{ij}\alpha^{i}{}_{(1)}\alpha^{j}{}_{(2)} = \tau^{12}h_1h_2$$
$$\tau^{(23)} = \tau^{ij}\alpha^{i}{}_{(2)}\alpha^{j}{}_{(3)} = \tau^{23}h_2h_3$$
$$\tau^{(31)} = \tau^{ij}\alpha^{i}{}_{(3)}\alpha^{j}{}_{(1)} = \tau^{31}h_3h_1$$

2.50

For a covariant second-order tensor, e_{ij}, six unique physical components are:

$\frac{e_{11}}{(h_1)^2}, \frac{e_{22}}{(h_2)^2}, \frac{e_{33}}{(h_3)^2}, \frac{e_{12}}{\mathrm{h}_1\mathrm{h}_2}, \frac{e_{23}}{\mathrm{h}_2\mathrm{h}_3}$ and $\frac{e_{31}}{\mathrm{h}_3\mathrm{h}_1}$.

2.51

We can see that for these and higher order tensors, each contravariant index leads to a multiplication by the h value corresponding to its coordinate while each covariant index requires a division by a corresponding h. The values of all the h's become unity in a Cartesian system. Hence there is no difference between the tensor components and the coordinate projections. In curvilinear systems, the h's are not equal to one. Formulae similar to the above will have to be used to find the actual (physical) components when the tensor components are known.

106. Find the physical components of the gradient of a scalar point function in Cartesian, circular cylindrical and Spherical polar coordinates.

The gradient of the invariant (absolute scalar) Φ is the covariant vector, $\frac{\partial \Phi}{\partial x^i}$. If the $\boldsymbol{x^i}$

coordinates are the Cartesian coordinates (x, y, z) then, $h_1 = h_2 = h_3 = 1$. Hence the physical coordinates are $\left\{\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right\}$.

In cylindrical polar coordinates (r, θ, z), $h_1 = 1$, $h_2 = r$ and $h_3 = 1$. This same vector has the coordinates,

$$\left\{\frac{\partial\Phi}{\partial r}, \frac{1}{r}\frac{\partial\Phi}{\partial\theta}, \frac{\partial\Phi}{\partial z}\right\}$$

while for spherical polar (ρ, θ, φ), $h_1 = 1$, $h_2 = \rho$ and $h_3 = \rho\sin\theta$. The physical components then become,

$$\left\{\frac{\partial\Phi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}, \frac{1}{\rho\sin\theta}\frac{\partial\Phi}{\partial\varphi}\right\}.$$

In all these coordinates, the tensor components have the remarkably simple form, $\frac{\partial\Phi}{\partial x^1}$, $i = 1,2,3$.