Tensor Calculus

Differentials & Directional Derivatives

The Gateaux Differential

We are presently concerned with Inner Product spaces in our treatment of the Mechanics of Continua. Consider a map,

$$\boldsymbol{F}: \mathscr{V} \to \mathscr{W}$$

This maps from the domain $\mathscr{V}$ to $\mathscr{W}$ – both of which are Euclidean vector spaces. The concepts of limit and continuity carries naturally from the real space to any Euclidean vector space.

The Gateaux Differential

Let $\boldsymbol{v}_0 \in \mathscr{V}$ and $\boldsymbol{w}_0 \in \mathscr{W}$, as usual we can say that the limit

$$\lim_{\boldsymbol{v} \to \boldsymbol{v}_0} \boldsymbol{F}(\boldsymbol{v}) = \boldsymbol{w}_0$$

if for any pre-assigned real number $\epsilon > 0$, no matter how small, we can always find a real number $\delta > 0$ such that $|\boldsymbol{F}(\boldsymbol{v}) - \boldsymbol{w}_0| \leq \epsilon$ whenever $|\boldsymbol{v} - \boldsymbol{v}_0| < \delta$.

The function is said to be continuous at $\boldsymbol{v}_0$ if $\boldsymbol{F}(\boldsymbol{v}_0)$ exists and $\boldsymbol{F}(\boldsymbol{v}_0) = \boldsymbol{w}_0$

The Gateaux Differential

Specifically, for $\alpha \in \mathcal{R}$ let this map be:

$$D\boldsymbol{F}(\boldsymbol{x},\boldsymbol{h}) \equiv \lim_{\alpha\to 0}\frac{\boldsymbol{F}(\boldsymbol{x}+\alpha\boldsymbol{h})-\boldsymbol{F}(\boldsymbol{x})}{\alpha} = \frac{d}{d\alpha}\boldsymbol{F}(\boldsymbol{x}+\alpha\boldsymbol{h})\bigg|_{\alpha=0}$$

We focus attention on the second variable $\boldsymbol{h}$ while we allow the dependency on $\boldsymbol{x}$ to be as general as possible. We shall show that while the above function can be any given function of $\boldsymbol{x}$ (linear or nonlinear), the above map is always linear in $\boldsymbol{h}$ irrespective of what kind of Euclidean space we are mapping from or into. It is called the ***Gateaux Differential***.

The Gateaux Differential

Let us make the Gateaux differential a little more familiar in real space in two steps: First, we move to the real space and allow $h \to dx$ and we obtain,

$$DF(x, dx) = \lim_{\alpha \to 0} \frac{F(x + \alpha dx) - F(x)}{\alpha} = \frac{d}{d\alpha} F(x + \alpha dx)\bigg|_{\alpha=0}$$

And let $\alpha dx \to \Delta x$, the middle term becomes,

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} dx = \frac{dF}{dx} dx$$

from which it is obvious that the Gateaux derivative is a generalization of the well-known differential from elementary calculus. The Gateaux differential helps to compute a local linear approximation of any function (linear or nonlinear).

The Gateaux Differential

It is easily shown that the Gateaux differential is linear in its second argument, ie, for $\alpha \in \mathscr{R}$

$$D\boldsymbol{F}(\boldsymbol{x}, \alpha\boldsymbol{h}) = \alpha D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h})$$

Furthermore,

$$D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g} + \boldsymbol{h}) = D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g}) + D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h})$$

and that for $\alpha, \beta \in \mathscr{R}$

$$D\boldsymbol{F}(\boldsymbol{x}, \alpha\boldsymbol{g} + \beta\boldsymbol{h}) = \alpha D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g}) + \beta D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h})$$

Linearity

$$DF(\boldsymbol{x}, \beta \boldsymbol{h}) = \lim_{\alpha \to 0} \frac{\boldsymbol{F}(\boldsymbol{x} + \alpha\beta\boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x})}{\alpha}$$

We introduce the real number $k = \alpha\beta$ so that $\alpha = \frac{k}{\beta}$,

$$DF(\boldsymbol{x}, \beta \boldsymbol{h}) = \beta \lim_{k \to 0} \frac{\boldsymbol{F}(\boldsymbol{x} + k\boldsymbol{h}) - \boldsymbol{F}(\boldsymbol{x})}{k}$$

$$= \beta D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h})$$

In a similar way,

$$D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h} + \boldsymbol{g}) = \lim_{\alpha \to 0} \frac{\boldsymbol{F}(\boldsymbol{x} + \alpha\boldsymbol{h} + \alpha\boldsymbol{g}) - \boldsymbol{F}(\boldsymbol{x})}{\alpha}$$

$$= \beta \lim_{\alpha \to 0} \frac{\boldsymbol{F}(\boldsymbol{x} + \alpha\boldsymbol{h} + \alpha\boldsymbol{g}) - \boldsymbol{F}(\boldsymbol{x} + \alpha\boldsymbol{g}) + \boldsymbol{F}(\boldsymbol{x} + \alpha\boldsymbol{g}) - \boldsymbol{F}(\boldsymbol{x})}{\alpha}$$

$$= D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{h}) + D\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g})$$

The Frechét Derivative

The Frechét derivative or gradient of a differentiable function is the linear operator, grad $\boldsymbol{F}(\boldsymbol{v})$ such that, for $d\boldsymbol{v} \in \mathscr{V}$,

$$\left[\frac{d\boldsymbol{F}(\boldsymbol{v})}{d\boldsymbol{v}}\right] d\boldsymbol{v} \equiv [\text{grad}\,\boldsymbol{F}(\boldsymbol{v})]d\boldsymbol{v} \equiv D\boldsymbol{F}(\boldsymbol{v}, d\boldsymbol{v})$$

Obviously, grad $\boldsymbol{F}(\boldsymbol{v})$ is a tensor because it is a linear transformation from one Euclidean vector space to another. Its linearity derives from the second argument of the RHS.

Differentiable Real-Valued Function

The Gradient of a Differentiable Real-Valued Function

For a real vector function, we have the map:

$$f : \mathscr{V} \rightarrow \mathscr{R}$$

The Gateaux differential in this case takes two vector arguments and maps to a real value:

$$Df : \mathscr{V} \times \mathscr{V} \rightarrow \mathscr{R}$$

This is the classical definition of the inner product so that we can define the differential as,

$$\frac{df(\boldsymbol{v})}{d\boldsymbol{v}} \cdot d\boldsymbol{v} \equiv Df(\boldsymbol{v}, d\boldsymbol{v})$$

This quantity,

$$\frac{df(\boldsymbol{v})}{d\boldsymbol{v}}$$

defined by the above equation, is clearly a vector.

Real-Valued Function

It is the gradient of the scalar valued function of the vector variable. We now proceed to find its components in general coordinates. To do this, we choose a basis $\{\mathbf{g}_i\} \subset \mathscr{V}$. On such a basis, the function,

$$\boldsymbol{v} = v^i \mathbf{g}_i$$

and,

$$f(\boldsymbol{v}) = f\left(v^i \mathbf{g}_i\right)$$

we may also express the independent vector

$$d\boldsymbol{v} = dv^i \mathbf{g}_i$$

on this basis.

Real-Valued Function

The Gateaux differential in the direction of the basis vector $\mathbf{g}_i$ is,

$$D f(\boldsymbol{v}, \mathbf{g}_i) = \lim_{\alpha \to 0} \frac{f(\boldsymbol{v} + \alpha \mathbf{g}_i) - f(\boldsymbol{v})}{\alpha} = \lim_{\alpha \to 0} \frac{f\left(v^j \mathbf{g}_j + \alpha \mathbf{g}_i\right) - f\left(v^i \mathbf{g}_i\right)}{\alpha}$$

$$= \lim_{\alpha \to 0} \frac{f\left(\left(v^j + \alpha \delta_i^j\right) \mathbf{g}_j\right) - f\left(v^i \mathbf{g}_i\right)}{\alpha}$$

As the function f does not depend on the vector basis, we can substitute the vector function by the real function of the components v^1, v^2, v^3 so that,

$$f\left(v^i \mathbf{g}_i\right) = f(v^1, v^2, v^3)$$

in which case, the above differential, similar to the real case discussed earlier becomes,

$$D f(\boldsymbol{v}, \mathbf{g}_i) = \frac{\partial f(v^1, v^2, v^3)}{\partial v^i}$$

Of course, it is now obvious that the gradient $\frac{df(v)}{dv}$ of the scalar valued vector function, expressed in its dual basis must be,

$$\frac{df(\boldsymbol{v})}{d\boldsymbol{v}} = \frac{\partial f(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}^i$$

so that we can recover,

$$Df(\boldsymbol{v}, \mathbf{g}_i) = \frac{df(\boldsymbol{v})}{d\boldsymbol{v}} \cdot d\boldsymbol{v} = \frac{\partial f(v^1, v^2, v^3)}{\partial v^j}\mathbf{g}^j \cdot \mathbf{g}_i$$
$$= \frac{\partial f(v^1, v^2, v^3)}{\partial v^j}\delta_i^j = \frac{\partial f(v^1, v^2, v^3)}{\partial v^i}$$

Hence we obtain the well-known result that

$$\text{grad } f(\boldsymbol{v}) = \frac{df(\boldsymbol{v})}{d\boldsymbol{v}} = \frac{\partial f(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}^i$$

which defines the Frechét derivative of a scalar valued function with respect to its vector argument.

Vector valued Function

* **The Gradient of a Differentiable Vector valued Function**

The definition of the Frechét clearly shows that the gradient of a vector-valued function is itself a second order tensor. We now find the components of this tensor in general coordinates. To do this, we choose a basis $\{\mathbf{g}_i\} \subset \mathscr{V}$. On such a basis, the function,

$$\boldsymbol{F}(\boldsymbol{v}) = F^k(\boldsymbol{v})\mathbf{g}_k$$

The functional dependency on the basis vectors are ignorable on account of the fact that the components themselves are fixed with respect to the basis. We can therefore write,

$$\boldsymbol{F}(\boldsymbol{v}) = F^k(v^1, v^2, v^3)\mathbf{g}_k$$

Vector Derivatives

$$\begin{aligned} D\boldsymbol{F}(\boldsymbol{v}, d\boldsymbol{v}) &= D\boldsymbol{F}\left(v^i \mathbf{g}_i, dv^i \mathbf{g}_i\right) \\ &= D\boldsymbol{F}\left(v^i \mathbf{g}_i, \mathbf{g}_i\right) dv^i \\ &= DF^k\left(v^i \mathbf{g}_i, \mathbf{g}_i\right) \mathbf{g}_k dv^i \end{aligned}$$

Again, upon noting that the functions $DF^k\left(v^i \mathbf{g}_i, \mathbf{g}_i\right)$, $k = 1,2,3$ are not functions of the vector basis, they can be written as functions of the scalar components alone so that we have, as before,

$$\begin{aligned} D\boldsymbol{F}(\boldsymbol{v}, d\boldsymbol{v}) &= DF^k\left(v^i \mathbf{g}_i, \mathbf{g}_i\right) \mathbf{g}_k dv^i \\ &= DF^k(\boldsymbol{v}, \mathbf{g}_i)\, \mathbf{g}_k dv^i \end{aligned}$$

Which we can compare to the earlier case of scalar valued function and easily obtain,

$$D\boldsymbol{F}(\boldsymbol{v}, d\boldsymbol{v}) = \left(\frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j}\mathbf{g}^j \cdot \mathbf{g}_i\right)\mathbf{g}_k dv^i$$

$$= \left(\frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j}\mathbf{g}_k \otimes \mathbf{g}^j\right)\mathbf{g}_i dv^i$$

$$= \left(\frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j}\mathbf{g}_k \otimes \mathbf{g}^j\right)d\boldsymbol{v}$$

$$= \left[\frac{d\boldsymbol{F}(\boldsymbol{v})}{d\boldsymbol{v}}\right]d\boldsymbol{v} \equiv [\text{grad } \boldsymbol{F}(\boldsymbol{v})]d\boldsymbol{v}$$

Clearly, the tensor gradient of the vector-valued vector function is,

$$\frac{d\boldsymbol{F}(\boldsymbol{v})}{d\boldsymbol{v}} = \text{grad } \boldsymbol{F}(\boldsymbol{v}) = \frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j}\mathbf{g}_k \otimes \mathbf{g}^j$$

Where due attention should be paid to the covariance of the quotient indices in contrast to the contravariance of the non quotient indices.

The Trace & Divergence

The trace of this gradient is called the divergence of the function:

$$
\begin{aligned}
\operatorname{div} \boldsymbol{F}(\boldsymbol{v}) &= \operatorname{tr}\left(\frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j} \mathbf{g}_k \otimes \mathbf{g}^j\right) \\
&= \frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j} \mathbf{g}_k \cdot \mathbf{g}^j \\
&= \frac{\partial F^k(v^1, v^2, v^3)}{\partial v^j} \delta_k^j \\
&= \frac{\partial F^j(v^1, v^2, v^3)}{\partial v^j}
\end{aligned}
$$

Tensor-Valued Vector Function

Consider the mapping,

$$\boldsymbol{T}: \mathscr{V} \to \mathscr{T}$$

Which maps from an inner product space to a tensor space. The Gateaux differential in this case now takes two vectors and produces a tensor:

$$D\boldsymbol{T}: \mathscr{V} \times \mathscr{V} \to \mathscr{T}$$

With the usual notation, we may write,

$$\begin{aligned} D\boldsymbol{T}(\boldsymbol{v}, d\boldsymbol{v}) &= D\boldsymbol{T}\left(v^i\mathbf{g}_i, dv^i\mathbf{g}_i\right) = D\boldsymbol{T}\left(v^i\mathbf{g}_i, dv^i\mathbf{g}_i\right) \\ &= \left[\frac{d\boldsymbol{T}(\boldsymbol{v})}{d\boldsymbol{v}}\right] d\boldsymbol{v} \equiv [\mathrm{grad}\, \boldsymbol{T}(\boldsymbol{v})] d\boldsymbol{v} \\ &= DT^{\alpha\beta}\left(v^i\mathbf{g}_i, dv^i\mathbf{g}_i\right)\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = DT^{\alpha\beta}(\boldsymbol{v}, \mathbf{g}_i)\mathbf{g}_\alpha \end{aligned}$$

Each of these nine functions look like the real differential of several variables we considered earlier.

The independence of the basis vectors imply, as usual, that

$$D T^{\alpha\beta}(\boldsymbol{v}, \mathbf{g}_i) = \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i}$$

so that,

$$\begin{aligned}
D\boldsymbol{T}(\boldsymbol{v}, d\boldsymbol{v}) &= D T^{\alpha\beta}(\boldsymbol{v}, \mathbf{g}_i)\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \, dv^i \\
&= \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \, dv^i \\
&= \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \left(\mathbf{g}^i \cdot d\boldsymbol{v}\right) \\
&= \left(\frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \otimes \mathbf{g}^i\right) d\boldsymbol{v} = \left[\frac{d\boldsymbol{T}(\boldsymbol{v})}{d\boldsymbol{v}}\right] d\boldsymbol{v} \\
&\equiv [\text{grad}\, \boldsymbol{T}(\boldsymbol{v})] d\boldsymbol{v}
\end{aligned}$$

Which defines the third-order tensor,

$$\frac{d\boldsymbol{T}(\boldsymbol{v})}{d\boldsymbol{v}} = \text{grad}\, \boldsymbol{T}(\boldsymbol{v}) = \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \otimes \mathbf{g}^i$$

and with no further ado, we can see that a third-order tensor transforms a vector into a second order tensor.

The Divergence of a Tensor Function

The divergence operation can be defined in several ways. Most common is achieved by the contraction of the last two basis so that,

$$\begin{aligned}\operatorname{div} \boldsymbol{T}(\boldsymbol{v}) &= \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i} \mathbf{g}_\alpha \otimes \mathbf{g}_\beta \cdot \mathbf{g}^i \\ &= \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i} \mathbf{g}_\alpha \left(\mathbf{g}_\beta \cdot \mathbf{g}^i\right) \\ &= \frac{\partial T^{\alpha\beta}(v^1, v^2, v^3)}{\partial v^i} \mathbf{g}_\alpha \delta^i_\beta\end{aligned}$$

It can easily be shown that this is the particular vector that gives, for all constant vectors $\boldsymbol{a}$,

$$(\operatorname{div} \boldsymbol{T})\boldsymbol{a} \equiv \operatorname{div}\left(\boldsymbol{T}^{\mathrm{T}}\boldsymbol{a}\right)$$

Real-Valued Tensor Functions

Two other kinds of functions are critically important in our study. These are real valued functions of tensors and tensor-valued functions of tensors. Examples in the first case are the invariants of the tensor function that we have already seen. We can express stress in terms of strains and vice versa. These are tensor-valued functions of tensors. The derivatives of such real and tensor functions arise in our analysis of continua. In this section these are shown to result from the appropriate Gateaux differentials. The gradients or Frechét derivatives will be extracted once we can obtain the Gateaux differentials.

Frechét Derivative

Consider the Map:

$$f : \mathscr{T} \to \mathscr{R}$$

The Gateaux differential in this case takes two tensor arguments and maps to a real value:

$$Df : \mathscr{T} \times \mathscr{T} \to \mathscr{R}$$

Which is, as usual, linear in the second argument. The Frechét derivative can be expressed as the first component of the following scalar product:

$$Df(\boldsymbol{T}, d\boldsymbol{T}) = \frac{df(\boldsymbol{T})}{d\boldsymbol{T}} : d\boldsymbol{T}$$

which, we recall, is the trace of the contraction of one tensor with the transpose of the other second-order tensors. This is a scalar quantity.

Guided by the previous result of the gradient of the scalar valued function of a vector, it is not difficult to see that,

$$\frac{df(\boldsymbol{T})}{d\boldsymbol{T}} = \frac{\partial f(T^{11}, T^{12}, \dots, T^{33})}{\partial T^{\alpha\beta}} \mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}$$

In the dual to the same basis, we can write,

$$d\boldsymbol{T} = dT^{ij} \mathbf{g}_i \otimes \mathbf{g}_j$$

Clearly,

$$\frac{df(\boldsymbol{T})}{d\boldsymbol{T}} : d\boldsymbol{T} = \left(\frac{\partial f(T^{11}, T^{12}, \dots, T^{33})}{\partial T^{\alpha\beta}} \mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta} \right) : \left(dT^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \right)$$

$$= \frac{\partial f(T^{11}, T^{12}, \dots, T^{33})}{\partial T^{ij}} dT^{ij}$$

Again, note the covariance of the quotient indices.

Examples

Let $\boldsymbol{S}$ be a symmetric and positive definite tensor and let $I_1(\boldsymbol{S}), I_2(\boldsymbol{S})$and $I_3(\boldsymbol{S})$ be the three principal invariants of $\mathbf{S}$ show that *(a)* $\frac{d\boldsymbol{I}_1(\mathbf{S})}{d\boldsymbol{S}} = \mathbf{I}$ the identity tensor, *(b)* $\frac{d\boldsymbol{I}_2(\mathbf{S})}{d\boldsymbol{S}} = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$ and *(c)* $\frac{dI_3(\mathbf{S})}{d\boldsymbol{S}} = I_3(\mathbf{S})\ \mathbf{S}^{-1}$

$\frac{d\boldsymbol{I}_1(\mathbf{S})}{d\boldsymbol{S}}$ can be written in the invariant component form as,

$$\frac{dI_1(\mathbf{S})}{d\mathbf{S}} = \frac{dI_1(\mathbf{S})}{dS_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

(a) Continued

Recall that $I_1(\mathbf{S}) = \mathrm{tr}(\mathbf{S}) = S_\alpha^\alpha$ hence

$$\frac{d\,\mathrm{tr}(\mathbf{S})}{d\mathbf{S}} = \frac{dI_1(\mathbf{S})}{dS_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{dS_\alpha^\alpha}{dS_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \delta_\alpha^i \delta_j^\alpha \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \delta_j^i \mathbf{g}_i \otimes \mathbf{g}^j = \mathbf{I}$$

which is the identity tensor as expected.

$(b)\ \frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$

$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}}$ in a similar way can be written in the invariant component form as,

$$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2}\frac{\partial}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j$$

where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}\left[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)\right]$. Consequently,

$$\begin{aligned}
\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} &= \frac{1}{2}\frac{\partial}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j \\
&= \frac{1}{2}\left[\delta_\alpha^i \delta_j^\alpha S_\beta^\beta + \delta_\beta^i \delta_j^\beta S_\alpha^\alpha - \delta_\beta^i \delta_j^\alpha S_\alpha^\beta - \delta_\alpha^i \delta_j^\beta S_\beta^\alpha\right]\mathbf{g}_i \otimes \mathbf{g}^j \\
&= \frac{1}{2}\left[\delta_j^i S_\beta^\beta + \delta_j^i S_\alpha^\alpha - S_i^j - S_i^j\right]\mathbf{g}_i \otimes \mathbf{g}^j \\
&= \left(\delta_j^i S_\alpha^\alpha - S_i^j\right)\mathbf{g}_i \otimes \mathbf{g}^j = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}
\end{aligned}$$

* $\frac{\partial \boldsymbol{I}_3(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial \det(\mathbf{S})}{\partial \mathbf{S}} = \mathbf{S}^{\mathrm{c}}$
* the cofactor of $\mathbf{S}$. Clearly $\mathbf{S}^{\mathrm{c}} = \det(\mathbf{S})\, \mathbf{S}^{-\mathrm{T}} = \boldsymbol{I}_3(\mathbf{S})\, \mathbf{S}^{-\mathrm{T}}$ as required. Details of this for the contravariant components of a tensor is presented below. Let

$$\det(\mathbf{S}) \equiv |\mathbf{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha \delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha \delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha \delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\mathbf{S}$

Real Tensor Functions

Consider the function $f(\boldsymbol{S}) = \mathrm{tr}\left(\boldsymbol{S}^k\right)$ where $k \in \mathcal{R}$, and $\boldsymbol{S}$ is a tensor.

$$f(\boldsymbol{S}) = \mathrm{tr}\left(\boldsymbol{S}^k\right) = \boldsymbol{S}^k : \mathbf{I}$$

To be specific, let $k = 3$.

Real Tensor Functions

The Gateaux differential in this case,

$$Df(\boldsymbol{S}, d\boldsymbol{S}) = \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\Big|_{\alpha=0} = \frac{d}{d\alpha} \operatorname{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})^3\}\Big|_{\alpha=0}$$

$$= \frac{d}{d\alpha} \operatorname{tr}\{(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\}\Big|_{\alpha=0}$$

$$= \operatorname{tr}\left[\frac{d}{d\alpha}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S})\right]\Bigg|_{\alpha=0}$$

$$= \operatorname{tr}[d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S})(\boldsymbol{S} + \alpha d\boldsymbol{S}) + (\boldsymbol{S} + \alpha d\boldsymbol{S})d\boldsymbol{S}(\boldsymbol{S} + \alpha d\boldsymbol{S})$$

Real Tensor Functions

With the last equality coming from the definition of the Inner product while noting that a circular permutation does not alter the value of the trace. It is easy to establish inductively that in the most general case, for $k > 0$, we have,

$$Df(\boldsymbol{S}, d\boldsymbol{S}) = k\left(\boldsymbol{S}^{k-1}\right)^{\mathrm{T}} : d\boldsymbol{S}$$

Clearly,

$$\frac{d}{d\boldsymbol{S}} \operatorname{tr}\left(\boldsymbol{S}^{k}\right) = k\left(\boldsymbol{S}^{k-1}\right)^{\mathrm{T}}$$

Real Tensor Functions

When $k = 1$,

$$
\begin{aligned}
Df(\boldsymbol{S}, d\boldsymbol{S}) &= \frac{d}{d\alpha} f(\boldsymbol{S} + \alpha d\boldsymbol{S})\bigg|_{\alpha=0} \\
&= \frac{d}{d\alpha} \mathrm{tr}(\boldsymbol{S} + \alpha d\boldsymbol{S})\bigg|_{\alpha=0} \\
&= \mathrm{tr}(\mathbf{I}\, d\boldsymbol{S}) = \mathbf{I}\colon d\boldsymbol{S}
\end{aligned}
$$

Or that,

$$
\frac{d}{d\boldsymbol{S}} \mathrm{tr}(\boldsymbol{S}) = \mathbf{I}.
$$

Real Tensor Functions

Derivatives of the other two invariants of the tensor $\boldsymbol{S}$ can be found as follows:

$$\frac{d}{d\boldsymbol{S}} I_2(\boldsymbol{S})$$
$$= \frac{1}{2}\frac{d}{d\boldsymbol{S}}[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)]$$
$$= \frac{1}{2}\left[2\mathrm{tr}(\boldsymbol{S})\mathbf{I} - 2\,\boldsymbol{S}^{\mathrm{T}}\right] = \mathrm{tr}(\boldsymbol{S})\mathbf{I} - \boldsymbol{S}^{\mathrm{T}}$$

.

Real Tensor Functions

To Determine the derivative of the third invariant, we begin with the trace of the Cayley-Hamilton for $\boldsymbol{S}$:

$$\mathrm{tr}(\boldsymbol{S}^3 - I_1\boldsymbol{S}^2 + I_2\boldsymbol{S} - I_3\mathbf{I}) = \mathrm{tr}(\boldsymbol{S}^3) - I_1\mathrm{tr}(\boldsymbol{S}^2) + I_2\mathrm{tr}(\boldsymbol{S}) - 3I_3 = 0$$

Therefore,

$$3I_3 = \mathrm{tr}(\boldsymbol{S}^3) - I_1\mathrm{tr}(\boldsymbol{S}^2) + I_2\mathrm{tr}(\boldsymbol{S})$$

$$I_2(\boldsymbol{S}) = \frac{1}{2}[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)]$$

.

Differentiating the Invariants

We can therefore write,

$$3I_3(\boldsymbol{S})$$
$$= \mathrm{tr}(\boldsymbol{S}^3) - \mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + \left(\frac{1}{2}[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)]\right)\mathrm{tr}(\boldsymbol{S})$$

so that, in terms of traces only,

$$I_3(\boldsymbol{S}) = \frac{1}{6}[\mathrm{tr}^3(\boldsymbol{S}) - 3\mathrm{tr}(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}^2) + 2\mathrm{tr}(\boldsymbol{S}^3)]$$

Real Tensor Functions

Clearly,

$$\frac{dI_3(\boldsymbol{S})}{d\boldsymbol{S}} = \frac{1}{6}\left[3\mathrm{tr}^2(\boldsymbol{S})\mathbf{I} - 3\mathrm{tr}(\boldsymbol{S}^2) - 3\mathrm{tr}(\boldsymbol{S})2\boldsymbol{S}^T + 2 \times 3(\boldsymbol{S}^2)^T\right]$$

$$= I_2\mathbf{I} - \mathrm{tr}(\boldsymbol{S})\boldsymbol{S}^T + \boldsymbol{S}^{2T}$$

The Euclidean Point Space

In Classical Theory, the world is a Euclidean Point Space of dimension three. We shall define this concept now and consequently give specific meanings to related concepts such as

- Frames of Reference,
- Coordinate Systems and
- Global Charts

The Euclidean Point Space

- Scalars, Vectors and Tensors we will deal with in general vary from point to point in the material. They are therefore to be regarded as functions of position in the physical space occupied.
- Such functions, associated with positions in the Euclidean point space, are called fields.
- We will therefore be dealing with scalar, vector and tensor fields.

The Euclidean Point Space

* The Euclidean Point Space $\mathcal{E}$ is a set of elements called points. For each pair of points $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$, $\exists\, \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \in \mathscr{E}$ with the following two properties:

1. $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) + \boldsymbol{u}(\boldsymbol{z}, \boldsymbol{y}) \qquad \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{E}$
2. $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) \iff \boldsymbol{y} = \boldsymbol{z}$

Based on these two, we proceed to show that,

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{0}$$

And that,

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) = -\boldsymbol{u}(\boldsymbol{z}, \boldsymbol{x})$$

The Euclidean Point Space

From property 1, let $\boldsymbol{y} \to \boldsymbol{x}$ it is clear that,

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) + \boldsymbol{u}(\boldsymbol{z}, \boldsymbol{x})$$

And it we further allow $\boldsymbol{z} \to \boldsymbol{x}$, we find that,

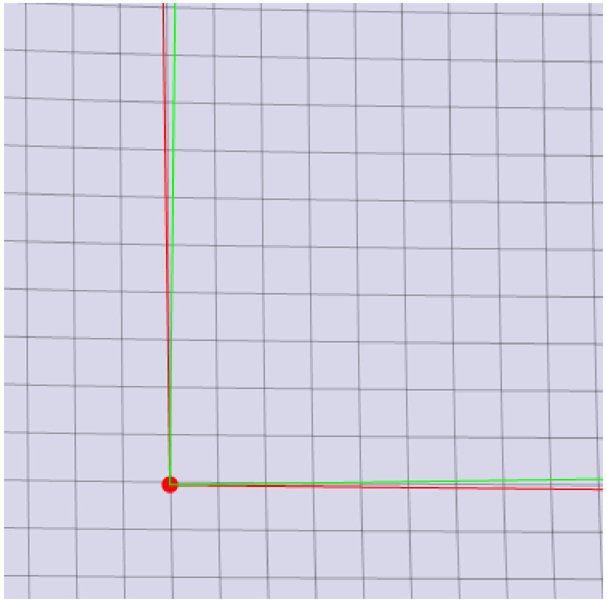
$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) + \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = 2\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x})$$

Which clearly shows that $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{o}$ the zero vector.

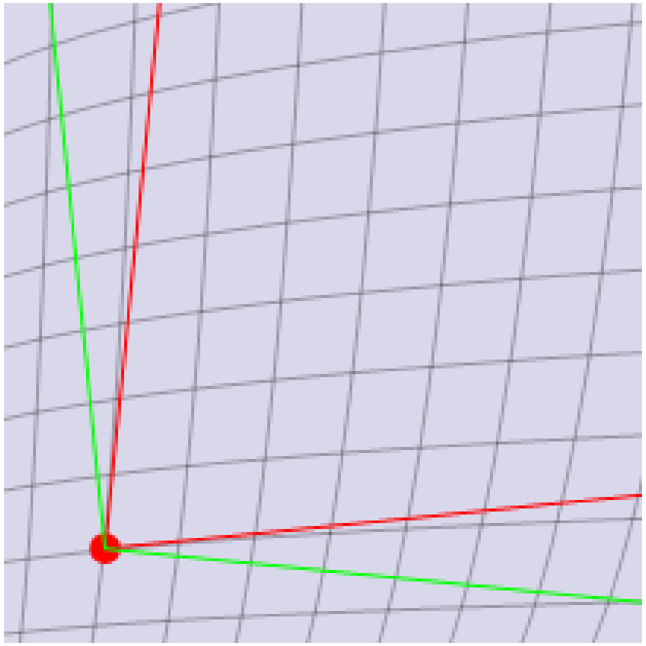
Similarly, from the above, we find that,

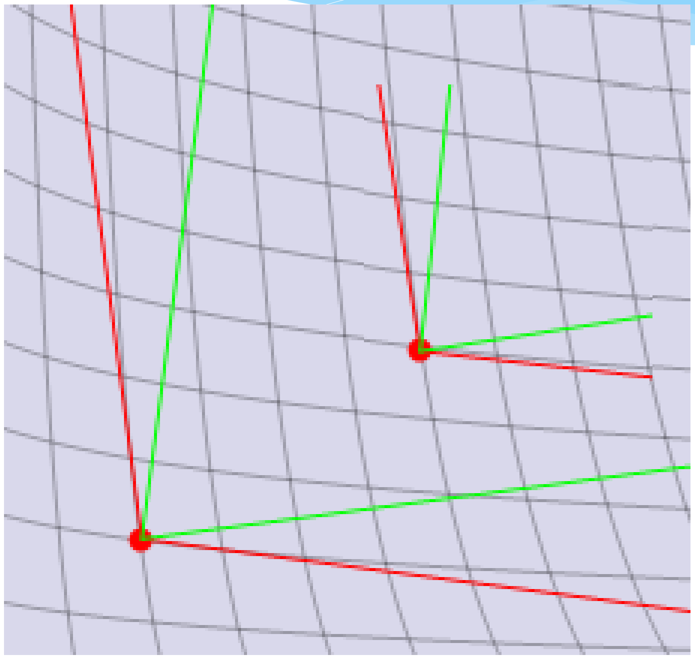
$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{o} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) + \boldsymbol{u}(\boldsymbol{z}, \boldsymbol{x})$$

So that $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{z}) = -\boldsymbol{u}(\boldsymbol{z}, \boldsymbol{x})$

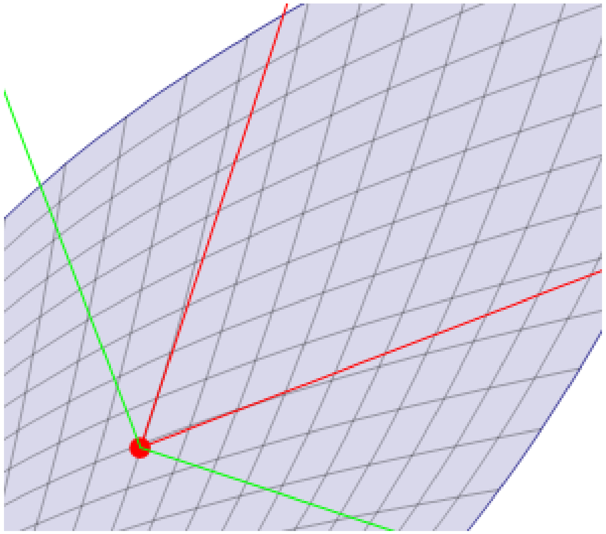


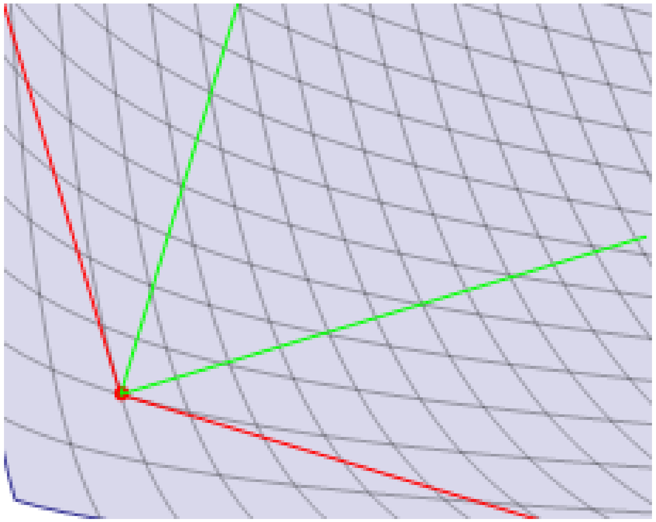
Coordinate System





Coordinate System





Position Vectors

Note that $\mathcal{E}$ is NOT a vector space. In our discussion, the vector space $\mathscr{E}$ to which $\boldsymbol{u}$ belongs, is associated as we have shown. It is customary to oscillate between these two spaces. When we are talking about the vectors, they are in $\mathscr{E}$ while the points are in $\mathcal{E}$.

We adopt the convention that $\boldsymbol{x}(\boldsymbol{y}) \equiv \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ referring to the vector $\boldsymbol{u}$. If therefore we choose an arbitrarily fixed point $\boldsymbol{0} \in \mathcal{E}$, we are associating $\boldsymbol{x}(\boldsymbol{0})$, $\boldsymbol{y}(\boldsymbol{0})$ and $\boldsymbol{z}(\boldsymbol{0})$ respectively with $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{0})$, $\boldsymbol{u}(\boldsymbol{y}, \boldsymbol{0})$ and $\boldsymbol{u}(\boldsymbol{z}, \boldsymbol{0})$.

These are vectors based on the points $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ with reference to the origin chosen. To emphasize the association with both points of $\mathcal{E}$ as well as members of $\mathscr{E}$ they are called **Position Vectors**.

Length in the Point Space

Recall that by property 1,

$$\boldsymbol{x}(\boldsymbol{y}) \equiv \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(\boldsymbol{x}, \mathbf{0}) + \boldsymbol{u}(\mathbf{0}, \boldsymbol{y})$$

Furthermore, we have deduced that $\boldsymbol{u}(\mathbf{0}, \boldsymbol{y}) = -\boldsymbol{u}(\boldsymbol{y}, \mathbf{0})$

We may therefore write that,

$$\boldsymbol{x}(\boldsymbol{y}) = \boldsymbol{x}(\mathbf{0}) - \boldsymbol{y}(\mathbf{0})$$

Which, when there is no ambiguity concerning the chosen origin, we can write as,

$$\boldsymbol{x}(\boldsymbol{y}) = \boldsymbol{x} - \boldsymbol{y}$$

And the distance between the two is,

$$\boldsymbol{d}\big(\boldsymbol{x}(\boldsymbol{y})\big) = \boldsymbol{d}(\boldsymbol{x} - \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$$

Metric Properties

* The norm of a vector is the square root of the inner product of the vector with itself. If the coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ on a set of independent vectors are x^i and y^i, then the distance we seek is,

$$\boldsymbol{d}(\boldsymbol{x}-\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\| = \sqrt{g_{ij}(x^i - y^i)(x^j - y^j)}$$

The more familiar Pythagorean form occurring only when $g_{ij} = 1$.

Coordinate Transformations

- Consider a Cartesian coordinate system x, y, z or y^1, y^2 and $y^3$with an orthogonal basis. Let us now have the possibility of transforming to another coordinate system of an arbitrary nature: x^1, x^2, x^3. We can represent the transformation and its inverse in the equations:
- $y^i = y^i(x^1, x^2, x^3), \;\; x^i = x^i(y^1, y^2, y^3)$
- And if the Jacobian of transformation,

Jacobian Determinant

$$\left|\frac{\partial y^i}{\partial x^j}\right| \equiv \left|\frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}\right| \equiv \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} & \frac{\partial y^1}{\partial x^3} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^2}{\partial x^3} \\ \frac{\partial y^3}{\partial x^1} & \frac{\partial y^3}{\partial x^2} & \frac{\partial y^3}{\partial x^3} \end{vmatrix}$$

does not vanish, then the inverse transformation will exist. So that,

$$x^i = x^i(y^1, y^2, y^3)$$

Given a position vector

$$\mathbf{r} = \mathbf{r}(x^1, x^2, x^3)$$

- In the new coordinate system, we can form a set of three vectors,

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}, i = 1,2,3$$

Which represent vectors along the tangents to the coordinate lines. (This is easily established for the Cartesian system and it is true in all systems. $\mathbf{r} = \mathbf{r}(x^1, x^2, x^3) = y^1\mathbf{i} + y^2\mathbf{j} + y^3\mathbf{k}$

So that

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial y^1}, \mathbf{j} = \frac{\partial \mathbf{r}}{\partial y^2} \text{ and } \mathbf{k} = \frac{\partial \mathbf{r}}{\partial y^3}$$

The fact that this is true in the general case is easily seen when we consider that along those lines, only the variable we are differentiating with respect to varies. Consider the general case where,

$$\mathbf{r} = \mathbf{r}(x^1, x^2, x^3)$$

The total differential of $\mathbf{r}$ is simply,

Natural Dual Bases

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^1} dx^1 + \frac{\partial \mathbf{r}}{\partial x^2} dx^2 + \frac{\partial \mathbf{r}}{\partial x^3} dx^3$$
$$\equiv dx^1 \mathbf{g}_1 + dx^2 \mathbf{g}_2 + dx^3 \mathbf{g}_3$$

With $\mathbf{g}_i(x^1, x^2, x^3), i = 1,2,3$ now depending in general on x^1, x^2 and x^3 now forming a basis on which we can describe other vectors in the coordinate system. We have no guarantees that this vectors are unit in length nor that they are orthogonal to one another. In the Cartesian case, $\mathbf{g}_i, i = 1,2,3$ are constants, normalized and orthogonal. They are our familiar

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial y^1}, \mathbf{j} = \frac{\partial \mathbf{r}}{\partial y^2} \text{ and } \mathbf{k} = \frac{\partial \mathbf{r}}{\partial y^3}.$$

We now proceed to show that the set of basis vectors,

$$\mathbf{g}^i \equiv \nabla x^i$$

is reciprocal to $\mathbf{g}_i$. The total differential $d\mathbf{r}$ can be written in terms of partial differentials as,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial y^i} dy^i$$

Which when expressed in component form yields,

$$dx^1 = \frac{\partial x^1}{\partial y^1} dy^1 + \frac{\partial x^1}{\partial y^2} dy^2 + \frac{\partial x^1}{\partial y^3} dy^3$$
$$dx^2 = \frac{\partial x^2}{\partial y^1} dy^1 + \frac{\partial x^2}{\partial y^2} dy^2 + \frac{\partial x^2}{\partial y^3} dy^3$$
$$dx^3 = \frac{\partial x^3}{\partial y^1} dy^1 + \frac{\partial x^3}{\partial y^2} dy^2 + \frac{\partial x^3}{\partial y^3} dy^3$$

Or, more compactly that,

$$dx^j = \frac{\partial x^j}{\partial y^i} dy^i = \nabla x^j \cdot d\mathbf{r}$$

In Cartesian coordinates, $y^i, i = 1, .., 3$ for any scalar field $\phi\,(y^1, y^2, y^3)$,

$$\frac{\partial\phi}{\partial y^i}dy^i = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = \text{grad}\,\phi\cdot d\mathbf{r}$$

We are treating each curvilinear coordinate as a scalar field, $x^j = x^j(y^1, y^2, y^3)$.

$$\begin{aligned} dx^j &= \nabla x^j\cdot\frac{\partial\mathbf{r}}{\partial x^m}dx^m \\ &= \mathbf{g}^j\cdot\mathbf{g}_m dx^m \\ &= \delta^j_m dx^m. \end{aligned}$$

The last equality arises from the fact that this is the only way one component of the coordinate differential can equal another.

$$\Rightarrow \mathbf{g}^j\cdot\mathbf{g}_m = \delta^j_m$$

Which recovers for us the reciprocity relationship and shows that $\mathbf{g}^j$ and $\mathbf{g}_m$ are reciprocal systems.

The position vector in Cartesian coordinates is $\boldsymbol{r} = x_i \boldsymbol{e}_i$. Show that (a) div $\boldsymbol{r} = 3$, (b) div $(\boldsymbol{r} \otimes \boldsymbol{r}) = 4\boldsymbol{r}$, (c) div $\boldsymbol{r} = 3$, and (d) grad $\boldsymbol{r} = \mathbf{1}$ and (e) curl $(\boldsymbol{r} \otimes \boldsymbol{r}) = -\boldsymbol{r}\times$

$$\begin{aligned}
\operatorname{grad} \boldsymbol{r} &= x_{i,j}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= \delta_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \mathbf{1}
\end{aligned}$$

$$\begin{aligned}
\operatorname{div} \boldsymbol{r} &= x_{i,j}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j \\
&= \delta_{ij}\delta_{ij} = \delta_{jj} = 3.
\end{aligned}$$

$$\boldsymbol{r} \otimes \boldsymbol{r} = x_i \boldsymbol{e}_i \otimes x_j \boldsymbol{e}_j = x_i x_j \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

$$\operatorname{grad}(\boldsymbol{r} \otimes \boldsymbol{r}) = \left(x_i x_j\right)_{,k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

$$\operatorname{div}(\boldsymbol{r} \otimes \boldsymbol{r}) = \left(x_{i,k}\, x_j + x_i x_{j,k}\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k$$

$$= \left(\delta_{ik} \boldsymbol{x}_j + \boldsymbol{x}_i \delta_{jk}\right) \delta_{jk} \boldsymbol{e}_i = \left(\delta_{ik} \boldsymbol{x}_k + \boldsymbol{x}_i \delta_{jj}\right) \boldsymbol{e}_i$$

$$= 4 x_i \boldsymbol{e}_i = 4\boldsymbol{r}$$

$$\begin{aligned}
\operatorname{curl}(\boldsymbol{r} \otimes \boldsymbol{r}) &= \epsilon_{\alpha\beta\gamma} \left(x_i x_\gamma\right)_{,\beta} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i \\
&= \epsilon_{\alpha\beta\gamma} \left(x_{i,\beta}\, x_\gamma + x_i x_{\gamma,\beta}\right) \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i \\
&= \epsilon_{\alpha\beta\gamma} \left(\delta_{i\beta} x_\gamma + x_i \delta_{\gamma\beta}\right) \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i \\
&= \epsilon_{\alpha i\gamma}\, x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i + \epsilon_{\alpha\beta\beta} x_i \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i
\end{aligned}$$

$$= -\epsilon_{\alpha\gamma i}\, x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i = -\boldsymbol{r}\times$$

For a scalar field ϕ and a tensor field T show that grad $(\boldsymbol{\phi}\mathbf{T}) = \boldsymbol{\phi}\mathbf{grad}\ \mathbf{T} + \mathbf{T} \otimes \mathbf{grad}\boldsymbol{\phi}$. Also show that div $(\boldsymbol{\phi}\mathbf{T}) = \boldsymbol{\phi}\ \mathbf{div}\ \mathbf{T} + \mathbf{Tgrad}\boldsymbol{\phi}$.

$$\begin{aligned} \mathrm{grad}(\phi\mathbf{T}) &= \left(\phi T^{ij}\right)_{,k} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\ &= \mathbf{T} \otimes \mathrm{grad}\phi + \phi\mathrm{grad}\ \mathbf{T} \end{aligned}$$

Furthermore, we can contract the last two bases and obtain,

$$\begin{aligned} \mathrm{div}(\phi\mathbf{T}) &= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^k \\ &= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \delta_j^k \\ &= T^{ik} \phi_{,k} \mathbf{g}_i + \phi T^{ik}{}_{,k} \mathbf{g}_i \\ &= \mathbf{T}\mathrm{grad}\phi + \phi\ \mathrm{div}\ \mathbf{T} \end{aligned}$$

* **For two arbitrary vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, show that** $\mathbf{grad}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{u}\times)\mathbf{grad}\boldsymbol{v} - (\boldsymbol{v}\times)\mathbf{grad}\boldsymbol{u}$
* $$\begin{aligned} \operatorname{grad}(\boldsymbol{u} \otimes \boldsymbol{v}) &= \left(\epsilon^{ijk} u_j v_k\right)_{,l} \mathbf{g}_i \otimes \mathbf{g}^l \\ &= \left(\epsilon^{ijk} u_{j,l}\, v_k + \epsilon^{ijk} u_j v_{k,l}\right) \mathbf{g}_i \otimes \mathbf{g}^l \\ &= \left(u_{j,l}\, \epsilon^{ijk} v_k + v_{k,l}\, \epsilon^{ijk} u_j\right) \mathbf{g}_i \otimes \mathbf{g}^l \\ &= -(\boldsymbol{v}\times)\operatorname{grad}\boldsymbol{u} + (\boldsymbol{u}\times)\operatorname{grad}\boldsymbol{v} \end{aligned}$$

For any two tensor fields $\mathbf{u}$ and $\mathbf{v}$, Show that,

$$(\mathbf{u}\times):\mathrm{grad}\,\mathbf{v}=\mathbf{u}\cdot\mathrm{curl}\,\mathbf{v}$$

$$\begin{aligned}(\mathbf{u}\times):\mathrm{grad}\,\mathbf{v} &= \left(\epsilon^{ijk}u_j\mathbf{g}_i\otimes\mathbf{g}_k\right):\left(v_{\alpha,\beta}\,\mathbf{g}^{\alpha}\otimes\mathbf{g}^{\beta}\right)\\ &= \epsilon^{ijk}u_jv_{\alpha,\beta}\,(\mathbf{g}_i\cdot\mathbf{g}^{\alpha})\left(\mathbf{g}_k\cdot\mathbf{g}^{\beta}\right)\\ &= \epsilon^{ijk}u_jv_{\alpha,\beta}\,\delta_i^{\alpha}\delta_k^{\beta}=\epsilon^{ijk}u_jv_{i,k}\\ &= \mathbf{u}\cdot\mathrm{curl}\,\mathbf{v}\end{aligned}$$

For a vector field $\boldsymbol{u}$, show that $\mathbf{grad}(\boldsymbol{u}\times)$ is a third ranked tensor. Hence or otherwise show that $\mathbf{div}(\boldsymbol{u}\times) = -\mathbf{curl}\,\boldsymbol{u}$.

The second–order tensor $(\boldsymbol{u}\times)$ is defined as $\epsilon^{ijk}u_j\mathbf{g}_i \otimes \mathbf{g}_k$. Taking the covariant derivative with an independent base, we have

$$\text{grad}(\boldsymbol{u}\times) = \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}^l$$

This gives a third order tensor as we have seen. Contracting on the last two bases,

$$\begin{aligned}\text{div}(\boldsymbol{u}\times) &= \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \cdot \mathbf{g}^l \\ &= \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i\delta_k^l \\ &= \epsilon^{ijk}u_{j,k}\,\mathbf{g}_i \\ &= -\text{curl}\,\boldsymbol{u}\end{aligned}$$

Show that curl $(\phi \mathbf{I}) = (\text{ grad } \phi) \times$

Note that $\phi\mathbf{I} = (\phi g_{\alpha\beta})\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$, and that curl $\boldsymbol{T} = \epsilon^{ijk} T_{\alpha k}{}_{,j} \; \mathbf{g}_i \otimes \mathbf{g}^\alpha$ so that,

$$\begin{aligned}\text{curl } (\phi\mathbf{I}) &= \epsilon^{ijk}(\phi g_{\alpha k})_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha \\ &= \epsilon^{ijk}(\phi_{,j}\, g_{\alpha k})\mathbf{g}_i \otimes \mathbf{g}^\alpha = \epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \\ &= (\text{ grad } \phi) \times\end{aligned}$$

Show that curl $(\boldsymbol{v}\times) = (\text{div } \boldsymbol{v})\mathbf{I} - \text{ grad } \boldsymbol{v}$

$$(\boldsymbol{v}\times) = \epsilon^{\alpha\beta k} v_\beta\, \mathbf{g}_\alpha \otimes \mathbf{g}_k$$

$$\text{curl } \mathbf{T} = \epsilon^{ijk} T_{\alpha k}{}_{,j} \; \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

so that

$$\begin{aligned}\text{curl } (\boldsymbol{v}\times) &= \epsilon^{ijk}\epsilon^{\alpha\beta k} v_{\beta}{}_{,j} \; \mathbf{g}_i \otimes \mathbf{g}_\alpha \\ &= \left(g^{i\alpha}g^{j\beta} - g^{i\beta}g^{j\alpha}\right) v_{\beta}{}_{,j} \; \mathbf{g}_i \otimes \mathbf{g}_\alpha \\ &= v^j{}_{,j}\, \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha - v^i{}_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^j \\ &= (\text{div } \mathbf{v})\mathbf{I} - \text{ grad } \mathbf{v}\end{aligned}$$

Show that div $(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{u} - \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v}$

$$\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) = \left(\epsilon^{ijk} u_j v_k\right)_{,i}$$

Noting that the tensor ϵ^{ijk} behaves as a constant under a covariant differentiation, we can write,

$$\begin{aligned}\operatorname{div}(\boldsymbol{u} \times \boldsymbol{v}) &= \left(\epsilon^{ijk} u_j v_k\right)_{,i} \\ &= \epsilon^{ijk} u_{j,i}\, v_k + \epsilon^{ijk} u_j v_{k,i} \\ &= \boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{u} - \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v}\end{aligned}$$

Given a scalar point function ϕ and a vector field v, show that curl $(\phi \mathbf{v}) = \phi \operatorname{curl} \mathbf{v} + (\operatorname{grad}\phi) \times \mathbf{v}$.

$$\begin{aligned}\operatorname{curl}(\phi \mathbf{v}) &= \epsilon^{ijk} (\phi v_k)_{,j}\, \mathbf{g}_i \\ &= \epsilon^{ijk} \left(\phi_{,j}\, v_k + \phi v_{k,j}\right) \mathbf{g}_i \\ &= \epsilon^{ijk} \phi_{,j}\, v_k \mathbf{g}_i + \epsilon^{ijk} \phi v_{k,j}\, \mathbf{g}_i \\ &= (\operatorname{grad}\phi) \times \mathbf{v} + \phi \operatorname{curl} \mathbf{v}\end{aligned}$$

Given that $\varphi(t) = |\mathbf{A}(t)|$, Show that $\dot{\varphi}(t) = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$

$$\varphi^2 \equiv \mathbf{A} : \mathbf{A}$$

Now,

$$\frac{d}{dt}(\varphi^2) = 2\varphi \frac{d\varphi}{dt} = \frac{d\mathbf{A}}{dt} : \mathbf{A} + \mathbf{A} : \frac{d\mathbf{A}}{dt} = 2\mathbf{A} : \frac{d\mathbf{A}}{dt}$$

as inner product is commutative. We can therefore write that

$$\frac{d\varphi}{dt} = \frac{\mathbf{A}}{\varphi} : \frac{d\mathbf{A}}{dt} = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$$

as required.

Given a tensor field $\boldsymbol{T}$, obtain the vector $\boldsymbol{w} \equiv \boldsymbol{T}^{\mathrm{T}}\mathbf{v}$ *and show that its divergence* is $\boldsymbol{T}: (\nabla\mathbf{v}) + \mathbf{v} \cdot \mathbf{div}\, \boldsymbol{T}$

The divergence of $\boldsymbol{w}$ is the scalar sum ,$\left(T_{ji}v^j\right)_{,i}$.
Expanding the product covariant derivative we obtain,

$$\begin{aligned}\operatorname{div}\left(\boldsymbol{T}^{\mathrm{T}}\mathbf{v}\right) = \left(T_{ji}v^j\right)_{,i} &= T_{ji,i}\, v^j + T_{ji}v^j{}_{,i} \\ &= (\operatorname{div}\boldsymbol{T}) \cdot \mathbf{v} + \operatorname{tr}\left(\boldsymbol{T}^{\mathrm{T}}\operatorname{grad}\mathbf{v}\right) \\ &= (\operatorname{div}\boldsymbol{T}) \cdot \mathbf{v} + \boldsymbol{T}: (\operatorname{grad}\mathbf{v})\end{aligned}$$

Recall that scalar product of two vectors is commutative so that

$$\operatorname{div}\left(\boldsymbol{T}^{\mathrm{T}}\mathbf{v}\right) = \boldsymbol{T}: (\operatorname{grad}\mathbf{v}) + \mathbf{v} \cdot \operatorname{div}\boldsymbol{T}$$

For a second-order tensor $\boldsymbol{T}$ define $\operatorname{curl}\boldsymbol{T} \equiv \epsilon^{ijk}\boldsymbol{T}_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$ show that for any constant vector $\boldsymbol{a}$, $(\operatorname{curl}\boldsymbol{T})\,\boldsymbol{a} = \operatorname{curl}\left(\boldsymbol{T}^T\boldsymbol{a}\right)$

Express vector $\boldsymbol{a}$ in the invariant form with contravariant components as $\boldsymbol{a} = a^\beta \mathbf{g}_\beta$. It follows that

$$\begin{aligned}(\operatorname{curl}\boldsymbol{T})\,\boldsymbol{a} &= \epsilon^{ijk}T_{\alpha k,j}\ (\mathbf{g}_i \otimes \mathbf{g}^\alpha)\boldsymbol{a}\\ &= \epsilon^{ijk}T_{\alpha k,j}\ a^\beta(\mathbf{g}_i \otimes \mathbf{g}^\alpha)\mathbf{g}_\beta\\ &= \epsilon^{ijk}T_{\alpha k,j}\ a^\beta\mathbf{g}_i\delta^\alpha_\beta\\ &= \epsilon^{ijk}(T_{\alpha k})_{,j}\ \mathbf{g}_i a^\alpha\\ &= \epsilon^{ijk}(T_{\alpha k}a^\alpha)_{,j}\ \mathbf{g}_i\end{aligned}$$

The last equality resulting from the fact that vector $\boldsymbol{a}$ is a constant vector. Clearly,

$$(\operatorname{curl}\boldsymbol{T})\,\boldsymbol{a} = \operatorname{curl}\left(\boldsymbol{T}^T\boldsymbol{a}\right)$$

For any two vectors u and v, show that $\text{curl}\,(\mathbf{u}\otimes\mathbf{v}) = [(\text{grad}\,\mathbf{u})\mathbf{v}\times]^{T} + (\text{curl}\,\mathbf{v})\otimes\boldsymbol{u}$ **where** $\mathbf{v}\times$ **is the skew tensor** $\epsilon^{ikj}v_k\,\mathbf{g}_i\otimes\mathbf{g}_j$**.**

Recall that the curl of a tensor $\boldsymbol{T}$ is defined by curl $\boldsymbol{T} \equiv \epsilon^{ijk}T_{\alpha k,j}\;\mathbf{g}_i\otimes\mathbf{g}^{\alpha}$. Clearly therefore,

$$
\begin{aligned}
\text{curl}\,(\boldsymbol{u}\otimes\boldsymbol{v}) &= \epsilon^{ijk}(u_\alpha v_k)_{,j}\;\mathbf{g}_i\otimes\mathbf{g}^{\alpha} \\
&= \epsilon^{ijk}\left(u_{\alpha,j}\,v_k + u_\alpha v_{k,j}\right)\mathbf{g}_i\otimes\mathbf{g}^{\alpha} \\
&= \epsilon^{ijk}u_{\alpha,j}\,v_k\,\mathbf{g}_i\otimes\mathbf{g}^{\alpha} + \epsilon^{ijk}u_\alpha v_{k,j}\;\mathbf{g}_i\otimes\mathbf{g}^{\alpha} \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\right)\otimes\left(u_{\alpha,j}\,\mathbf{g}^{\alpha}\right) + \left(\epsilon^{ijk}v_{k,j}\;\mathbf{g}_i\right)\otimes\left(u_\alpha\mathbf{g}^{\alpha}\right) \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\otimes\mathbf{g}_j\right)\left(u_{\alpha,\beta}\,\mathbf{g}^{\beta}\otimes\mathbf{g}^{\alpha}\right) + \left(\epsilon^{ijk}v_{k,j}\;\mathbf{g}_i\right) \\
&\quad\otimes\left(u_\alpha\mathbf{g}^{\alpha}\right) = -(\boldsymbol{v}\times)(\text{grad}\,\boldsymbol{u})^{T} + (\text{curl}\,\boldsymbol{v})\otimes\boldsymbol{u} \\
&= [(\text{grad}\,\boldsymbol{u})\boldsymbol{v}\times]^{T} + (\text{curl}\,\boldsymbol{v})\otimes\boldsymbol{u}
\end{aligned}
$$

upon noting that the vector cross is a skew tensor.

Show that $\operatorname{curl}(\boldsymbol{u}\times\boldsymbol{v}) = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v}-\boldsymbol{v}\otimes\boldsymbol{u})$

The vector $\boldsymbol{w}\equiv\boldsymbol{u}\times\boldsymbol{v}=w_{\boldsymbol{k}}\mathbf{g}^{\boldsymbol{k}}=\epsilon_{k\alpha\beta}u^{\alpha}v^{\beta}\mathbf{g}^{\boldsymbol{k}}$ and $\operatorname{curl}\boldsymbol{w}=\epsilon^{ijk}w_{k,j}\,\mathbf{g}_i$. Therefore,

$$
\begin{aligned}
\operatorname{curl}(\boldsymbol{u}\times\boldsymbol{v}) &= \epsilon^{ijk}w_{k,j}\,\mathbf{g}_i\\
&= \epsilon^{ijk}\epsilon_{k\alpha\beta}\left(u^{\alpha}v^{\beta}\right)_{,j}\mathbf{g}_i\\
&= \left(\delta^i_\alpha\delta^j_\beta-\delta^i_\beta\delta^j_\alpha\right)\left(u^{\alpha}v^{\beta}\right)_{,j}\mathbf{g}_i\\
&= \left(\delta^i_\alpha\delta^j_\beta-\delta^i_\beta\delta^j_\alpha\right)\left(u^{\alpha}{}_{,j}\,v^{\beta}\right.\\
&= \left[u^i{}_{,j}\,v^j+u^iv^j{}_{,j}-\left(u^j{}_{,j}\,v^i\right.\right.\\
&= \left[\left(u^iv^j\right)_{,j}-\left(u^jv^i\right)_{,j}\right]\mathbf{g}_i\\
&= \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v}-\boldsymbol{v}\otimes\boldsymbol{u})
\end{aligned}
$$

since $\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{v})=\left(u^iv^j\right)_{,\alpha}\mathbf{g}_i\otimes\mathbf{g}_j\cdot\mathbf{g}^{\alpha}=\left(u^iv^j\right)_{,j}\mathbf{g}_i$.

Given a scalar point function ϕ and a second-order tensor field $\boldsymbol{T}$, show that curl $(\phi \boldsymbol{T}) = \phi$ curl $\boldsymbol{T} + \big((\text{grad}\phi)\,\times\big)\boldsymbol{T}^T$ where $[(\text{grad}\phi)\,\times]$ is the skew tensor $\epsilon^{ijk}\phi_{,j}\,\mathbf{g}_i \otimes \mathbf{g}_k$

$$
\begin{aligned}
\text{curl}\,(\phi \boldsymbol{T}) &\equiv \epsilon^{ijk}(\phi T_{\alpha k})_{,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\big(\phi_{,j}\,T_{\alpha k} + \phi T_{\alpha k,j}\big)\,\mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \epsilon^{ijk}\phi_{,j}\,T_{\alpha k}\,\mathbf{g}_i \otimes \mathbf{g}^\alpha + \phi\epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \\
&= \big(\epsilon^{ijk}\phi_{,j}\,\mathbf{g}_i \otimes \mathbf{g}_k\big)\big(T_{\alpha\beta}\mathbf{g}^\beta \otimes \mathbf{g}^\alpha\big) + \phi\epsilon^{ijk}T_{\alpha k,j}\ \mathbf{g}_i \\
&= \phi\ \text{curl}\,\boldsymbol{T} + \big((\text{grad}\phi)\,\times\big)\boldsymbol{T}^T
\end{aligned}
$$

For a second-order tensor field $\boldsymbol{T}$, show that $\text{div}(\text{curl}\,\boldsymbol{T}) = \text{curl}\left(\text{div}\,\boldsymbol{T}^{\text{T}}\right)$

Define the second order tensor S as

$$\text{curl}\,\boldsymbol{T} \equiv \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha = S^i_{.\alpha} \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

The gradient of $\boldsymbol{S}$ is $S^i_{.\alpha,\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta = \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

Clearly,

$$\begin{aligned} \text{div}(\text{curl}\,\boldsymbol{T}) &= \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \cdot \mathbf{g}^\beta \\ &= \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i\ g^{\alpha\beta} \\ &= \epsilon^{ijk} T^\beta{}_{k,j\beta}\ \mathbf{g}_i = \text{curl}\left(\text{div}\,\boldsymbol{T}^{\text{T}}\right) \end{aligned}$$

If the volume V is enclosed by the surface S, the position vector $\boldsymbol{r} = x^i \mathbf{g}_i$ and $\boldsymbol{n}$ is the external unit normal to each surface element, show that $\frac{1}{6}\int_S \nabla(\boldsymbol{r}\cdot\boldsymbol{r})\cdot\boldsymbol{n}dS$ equals the volume contained in V.

$$\boldsymbol{r}\cdot\boldsymbol{r} = x^i x^j \mathbf{g}_i \cdot \mathbf{g}_j = x^i x^j g_{ij}$$

By the Divergence Theorem,

$$\int_S \nabla(\boldsymbol{r}\cdot\boldsymbol{r})\cdot\boldsymbol{n}dS = \int_V \nabla\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{r})]dV$$

$$= \int_V \partial_l\left[\partial_k\left(x^i x^j g_{ij}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\, dV$$

$$= \int_V \partial_l\left[g_{ij}\left(x^i{}_{,k}\, x^j + x^i x^j{}_{,k}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\, dV$$

$$= \int_V g_{ij} g^{lk}\left(\delta^i_k x^j + x^i \delta^j_k\right)_{,l}\, dV = \int_V 2 g_{ik} g^{lk} x^i{}_{,l}\, dV = \int_V 2\delta^l_i \delta^i_l\, dV$$

$$= 6\int_V dV$$

For any Euclidean coordinate system, show that $\text{div}\,\mathbf{u}\times\mathbf{v} = \mathbf{v}\,\text{curl}\,\mathbf{u} - \mathbf{u}\,\text{curl}\,\mathbf{v}$

Given the contravariant vector u^i and v^i with their associated vectors u_i and v_i, the contravariant component of the above cross product is $\epsilon^{ijk}u_j v_k$ .The required divergence is simply the contraction of the covariant x^i derivative of this quantity:

$$\left(\epsilon^{ijk}u_j v_k\right)_{,i} = \epsilon^{ijk}u_{j,i}v_k + \epsilon^{ijk}u_j v_{k,i}$$

where we have treated the tensor ϵ^{ijk} as a constant under the covariant derivative.

Cyclically rearranging the RHS we obtain,

$$\left(\epsilon^{ijk}u_j v_k\right)_{,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jki}v_{k,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jik}v_{k,i}$$

where we have used the anti-symmetric property of the tensor ϵ^{ijk}. The last expression shows clearly that

$$\text{div}\,\mathbf{u}\times\mathbf{v} = \mathbf{v}\,\text{curl}\,\mathbf{u} - \mathbf{u}\,\text{curl}\,\mathbf{v}$$

as required.

Liouville Formula

For a scalar variable α, if the tensor $\mathbf{T} = \mathbf{T}(\alpha)$ and $\dot{\mathbf{T}} \equiv \frac{d\mathbf{T}}{d\alpha}$, Show that $\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\operatorname{tr}(\dot{\mathbf{T}}\mathbf{T}^{-1})$

Proof:

Let $\mathbf{A} \equiv \dot{\mathbf{T}}\mathbf{T}^{-1}$ so that, $\dot{\mathbf{T}} = \mathbf{AT}.$ In component form, we have $\dot{T}_j^i = A_m^i T_j^m$. Therefore,

$$\frac{d}{d\alpha}\det(\mathbf{T}) = \frac{d}{d\alpha}\left(\epsilon^{ijk}T_i^1 T_j^2 T_k^3\right) = \epsilon^{ijk}\left(\dot{T}_i^1 T_j^2 T_k^3 + T_i^1 \dot{T}_j^2 T_k^3 + T_i^1 T_j^2 \dot{T}_k^3\right)$$

$$= \epsilon^{ijk}\left(A_l^1 T_i^l T_j^2 T_k^3 + T_i^1 A_m^2 T_j^m T_k^3 + T_i^1 T_j^2 A_n^3 T_k^n\right)$$

$$= \epsilon^{ijk}\left[\left(A_1^1 T_i^1 + \boxed{A_2^1 T_i^2} + \boxed{A_3^1 T_i^3}\right) T_j^2 T_k^3\right.$$

*

Liouville Theorem Cont'd

(For example, the first boxed term yields, $\epsilon^{ijk} A_2^1 T_i^2 T_j^2 T_k^3$

Which is symmetric as well as antisymmetric in i and j. It therefore vanishes. The same is true for all other such terms.)

$$\begin{aligned}\frac{d}{d\alpha}\det(\mathbf{T}) &= \epsilon^{ijk}\left[\left(A_1^1 T_i^1\right) T_j^2 T_k^3 + T_i^1\left(A_2^2 T_j^2\right) T_k^3 + T_i^1 T_j^2\left(A_3^3 T_k^3\right)\right] \\ &= A_m^m\, \epsilon^{ijk} T_i^1 T_j^2 T_k^3 \\ &= \mathrm{tr}\left(\dot{\mathbf{T}}\mathbf{T}^{-1}\right)\det(\mathbf{T})\end{aligned}$$

as required.

For a general tensor field T show that, $\text{curl}(\text{curl}\, T) = [\nabla^2(\text{tr}\, T) -$

When $\boldsymbol{T}$ is symmetric, show that $\operatorname{tr}(\operatorname{curl} \boldsymbol{T})$ vanishes.

$$\operatorname{curl} \boldsymbol{T} = \boldsymbol{\epsilon}^{ijk} T_{\beta k,j}\, \mathbf{g}_i \otimes \mathbf{g}^\beta$$

$$\operatorname{tr}(\operatorname{curl} \boldsymbol{T}) = \epsilon^{ijk} T_{\beta k,j}\, \mathbf{g}_i \cdot \mathbf{g}^\beta$$

$$= \epsilon^{ijk} T_{\beta k,j}\, \delta_i^\beta = \epsilon^{ijk} T_{ik,j}$$

which obviously vanishes on account of the symmetry and antisymmetry in i and k. In this case,

$$\operatorname{curl}(\operatorname{curl} \boldsymbol{T})$$
$$= [\nabla^2(\operatorname{tr} \boldsymbol{T}) - \operatorname{div}(\operatorname{div} \boldsymbol{T})]\boldsymbol{I} - \operatorname{grad}\big(\operatorname{grad}(\operatorname{tr}\boldsymbol{T})\big)$$
$$+ 2\big(\operatorname{grad}(\operatorname{div} \boldsymbol{T})\big) - \nabla^2 \boldsymbol{T}$$

as $\big(\operatorname{grad}(\operatorname{div} \boldsymbol{T})\big)^{\mathrm{T}} = \operatorname{grad}(\operatorname{div} \boldsymbol{T})$ if the order of differentiation is immaterial and T is symmetric.

For a scalar function Φ and a vector v^i show that the divergence of the vector $v^i\Phi$ is equal to, $\mathbf{v} \cdot \mathrm{grad}\Phi + \Phi\ \mathrm{div}\ \mathbf{v}$

$$\left(v^i\Phi\right)_{,i} = \Phi v^i{}_{,i} + v^i\Phi_{,\mathrm{i}}$$

Hence the result.

Show that $\boldsymbol{curl}\ \mathbf{u} \times \mathbf{v} = (\mathbf{v} \cdot \nabla \mathbf{u}) + (\mathbf{u} \cdot \operatorname{div} \mathbf{v}) - (\mathbf{v} \cdot \operatorname{div} \mathbf{u}) - (\mathbf{u} \cdot \nabla\, \mathbf{v})$

Taking the associated (covariant) vector of the expression for the cross product in the last example, it is straightforward to see that the LHS in indicial notation is,

$$\epsilon^{lmi}\left(\epsilon_{ijk} u^j v^k\right)_{,m}$$

Expanding in the usual way, noting the relation between the alternating tensors and the Kronecker deltas,

$$\epsilon^{lmi}\left(\epsilon_{ijk} u^j v^k\right)_{,m} = \delta^{lmi}_{jki}\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right)$$

$$= \delta^{lm}_{jk}\ \left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right) = \begin{vmatrix} \delta^l_j & \delta^m_j \\ \delta^l_k & \delta^m_k \end{vmatrix} \left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right)$$

$$= \left(\delta^l_j \delta^m_k - \delta^l_k \delta^m_j\right)\left(u^j{}_{,m} v^k - u^j v^k{}_{,m}\right)$$

$$= \delta^l_j \delta^m_k u^j{}_{,m} v^k - \delta^l_j \delta^m_k u^j v^k{}_{,m} + \delta^l_k \delta^m_j u^j{}_{,m} v^k$$

$$- \delta^l_k \delta^m_j u^j v^k{}_{,m}$$

$$= u^l{}_{,m} v^m - u^m{}_{,m} v^l + u^l v^m{}_{,m} - u^m v^l{}_{,m}$$

Which is the result we seek in indicial notation.

Executive Summary

* Here are the names of your tormentors:
  * Kronecker, Einstein, Gateaux, Frechét, Christoffel, Ricci, Riemann

Gateaux gave us a powerful way to express the change, or differential of any function of any order in a linear fashion no matter the functional relationship constituting the function.

$$DF(\mathbf{v}, \mathbf{dv}) \equiv \lim_{\alpha \to 0} \frac{F(\mathbf{v} + \alpha \mathbf{v}) - F(\mathbf{v})}{\alpha} \equiv \frac{d}{d\alpha} F(\mathbf{v} + \alpha d\mathbf{v}) \Big|_{\alpha \to 0}$$

This is the Gateaux Differential. We showed this as a powerful super differential as your old concept of differential is contained in it and can be considered a special case of Gateaux

Here comes Mr Frechét!

* And, what did he say?
* He defines a kind of derivative with respect to the variable whether that variable itself is a scalar, vector or a tensor. Given the Gateaux differential, we write

$$DF(\mathbf{v}, \mathrm{d}\mathbf{v}) \equiv \frac{\partial F(\mathbf{v})}{\partial \mathbf{v}}\, \mathrm{d}\mathbf{v}$$

As the defining equation for the Frechét derivative. The exact kind of product we have on the RHS as well as the kind of quantity in the product is to be found.

Three Cases for Vector Variables

1. F is a scalar function of a vector variable.

Here, the Gateaux differential is obviously a scalar and the product is certainly a scalar product so that $\frac{\partial F}{\partial \mathbf{v}}$ is necessarily a vector quantity. If we express $\mathbf{v} = v^i \mathbf{g}_i$ the result of all earlier proofs is simply that,

$$\frac{\partial F}{\partial \mathbf{v}} = \frac{\partial F}{\partial v^i} \mathbf{g}^i$$

With the covariance of the index arising from the quotient occurrence of a vector. This is the real definition of Grad or Frechet derivative for a scalar function of a vector.

2. F is a vector function

The Gateaux here is certainly a vector quantity! You now have that $\frac{\partial \mathbf{F}(\mathbf{v})}{\partial \mathbf{v}}$ is operating on the vector d$\mathbf{v}$ to produce a vector! We conclude that $\frac{\partial \mathbf{F}(\mathbf{v})}{\partial \mathbf{v}}$ is nothing but a second-order tensor! This Frechét derivative, or the gradient of a vector was what your teachers all along could never tell you about! If $\mathbf{F} = F^j \mathbf{g}_j$ and as before, we write $\mathbf{v} = v^i \mathbf{g}_i$, then we can say that,

$$\frac{\partial \mathbf{F}}{\partial \mathbf{v}} = \frac{\partial F^j}{\partial v^i} \mathbf{g}_j \otimes \mathbf{g}^i$$

3. Gradient of a Tensor Function

* You have already seen that the gradient increases the order of any function by adding an extra basis at the end. We can simply go on without any further and simply say that for a tensor function of a vector, we have

$$\frac{\partial \mathbf{T}}{\partial \mathbf{v}} = \frac{\partial T^{ij}}{\partial v^k} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k$$

Differentiation of Fields

Given a vector point function $\boldsymbol{u}(x^1, x^2, x^3)$ and its covariant components, $u_\alpha(x^1, x^2, x^3), \alpha = 1,2,3$, with $\mathbf{g}^\alpha$ as the reciprocal basis vectors, then

$$\boldsymbol{u} = u_\alpha \mathbf{g}^\alpha, \text{so that } d\boldsymbol{u} = \frac{\partial}{\partial x^k}(u_\alpha \mathbf{g}^\alpha) dx^k = \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} u_\alpha\right) dx^k$$

Clearly,

$$\frac{\partial \boldsymbol{u}}{\partial x^k} = \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} u_\alpha$$

And the projection of this quantity on the $\mathbf{g}^i$ direction is,

$$\frac{\partial \boldsymbol{u}}{\partial x^k} \cdot \mathbf{g}_i = \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} u_\alpha\right) \cdot \mathbf{g}_i = \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha \cdot \mathbf{g}_i + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} \cdot \mathbf{g}_i u_\alpha$$

$$= \frac{\partial u_\alpha}{\partial x^k}\delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} \cdot \mathbf{g}_i u_\alpha$$

Christoffel Symbols

* Now, $\mathbf{g}^i \cdot \mathbf{g}_j = \delta_i^j$ so that

$$\frac{\partial \mathbf{g}^i}{\partial x^k} \cdot \mathbf{g}_j + \mathbf{g}^i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} = 0.$$

$$\frac{\partial \mathbf{g}^i}{\partial x^k} \cdot \mathbf{g}_j = -\mathbf{g}^i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} \equiv -\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}.$$

This important quantity, necessary to quantify the derivative of a tensor in general coordinates, is called the **Christoffel Symbol** of the second kind.

Derivatives in General Coordinates

Using this, we can now write that,

$$\frac{\partial \boldsymbol{u}}{\partial x^k} \cdot \mathbf{g}_i = \frac{\partial u_\alpha}{\partial x^k} \delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} \cdot \mathbf{g}_i u_\alpha = \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} u_\alpha$$

* The quantity on the RHS is the component of the derivative of vector $\boldsymbol{u}$ along the $\mathbf{g}_i$ direction using covariant components. It is the covariant derivative of $\boldsymbol{u}$. Using contravariant components, we could write,

$$d\boldsymbol{u} = \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \frac{\partial \mathbf{g}_i}{\partial x^k} u^i \right) dx^k = \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha u^i \right) dx^k$$

* So that,

$$\frac{\partial \boldsymbol{u}}{\partial x^k} = \frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha$$

* The components of this in the direction of $\mathbf{g}_i$ can be obtained by taking a dot product as before:

$$\frac{\partial \boldsymbol{u}}{\partial x^k} \cdot \mathbf{g}^i = \left(\frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha \right) \cdot \mathbf{g}^i = \frac{\partial u^\alpha}{\partial x^k} \delta_\alpha^i + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} \cdot \mathbf{g}^i u^\alpha = \frac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} u^\alpha$$

Covariant Derivatives

The two results above are represented symbolically as,

$$u_{i,k} = \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} u_\alpha \text{ and } u^i_{,k} = \frac{\partial u^\alpha}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} u^\alpha$$

- Which are the components of the covariant derivatives in terms of the covariant and contravariant components respectively.
- It now becomes useful to establish the fact that our definition of the Christoffel symbols here conforms to the definition you find in the books using the transformation rules to define the tensor quantities.

Error Discussion

Assume we do not know the components of a given vector $\boldsymbol{F}$ and we want to find it. First consider the base vectors, $\mathbf{g}^1, \mathbf{g}^2$and $\mathbf{g}^3$ or $\mathbf{g}^i, i = 1, \dots, 3$. Let us write,

$$\boldsymbol{F} = \alpha\mathbf{g}^1 + \beta\mathbf{g}^2 + \gamma\mathbf{g}^3$$

So that we now try to find what these components are. Recall the standard procedure is to take an inner product of the equation with a dual base vector:

$$\boldsymbol{F} \cdot \mathbf{g}_1 = \alpha\mathbf{g}^1 \cdot \mathbf{g}_1 + \beta\mathbf{g}^2 \cdot \mathbf{g}_1 + \gamma\mathbf{g}^3 \cdot \mathbf{g}_1$$

From which we can immediately see that,

$$\alpha = \boldsymbol{F} \cdot \mathbf{g}_1 \equiv F_1$$

A similar argument shows that $\beta = \boldsymbol{F} \cdot \mathbf{g}_2 \equiv F_2$ and $\gamma = \boldsymbol{F} \cdot \mathbf{g}_3 \equiv F_3$

This enables us to write, $\boldsymbol{F} = \alpha\mathbf{g}^1 + \beta\mathbf{g}^2 + \gamma\mathbf{g}^3 = F_i\mathbf{g}^i$. You can now see that the dotting by $\mathbf{g}_i$ gave us the coefficients along $\mathbf{g}^i$ contrary to what I had claimed in the notes! I am sorry for the misleading information!

Christoffel Symbols

* We observe that the derivative of the covariant basis, $\mathbf{g}_i \left(= \frac{\partial \mathbf{r}}{\partial x^i}\right)$,

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial^2 \mathbf{r}}{\partial x^j \partial x^i} = \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \frac{\partial \mathbf{g}_j}{\partial x^i}$$

* Taking the dot product with $\mathbf{g}_k$,

$$\frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k = \frac{1}{2}\left(\frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k + \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_k}{\partial x^i} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_k}{\partial x^j}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_i}{\partial x^k} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}[\mathbf{g}_j \cdot \mathbf{g}_k] + \frac{\partial}{\partial x^j}[\mathbf{g}_i \cdot \mathbf{g}_k] - \frac{\partial}{\partial x^k}[\mathbf{g}_i \cdot \mathbf{g}_j]\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

The First Kind

- Which is the quantity defined as the ***Christoffel symbols of the first kind*** in the textbooks. It is therefore possible for us to write,

$$[ij,k] \equiv \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k = \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

It should be emphasized that the Christoffel symbols, even though the play a critical role in several tensor relationships, are themselves NOT tensor quantities. (Prove this). However, notice their symmetry in the i and j. The extension of this definition to the Christoffel symbols of the second kind is immediate:

The Second Kind

* Contract the above equation with the conjugate metric tensor, we have,

$$g^{k\alpha}[ij,\alpha] \equiv g^{k\alpha}\ \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_\alpha = g^{k\alpha} \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_\alpha = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$$

$$= -\frac{\partial \mathbf{g}^k}{\partial x^j} \cdot \mathbf{g}_i$$

Which connects the common definition of the second Christoffel symbol with the one defined in the above derivation. The relationship,

$$g^{k\alpha}[ij,\alpha] = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\}$$

apart from defining the relationship between the Christoffel symbols of the first kind and that second kind, also highlights, once more, the index-raising property of the conjugate metric tensor.

Two Christoffel Symbols Related

We contract the above equation with $g_{k\beta}$ and obtain,

$$g_{k\beta} g^{k\alpha} [ij, \alpha] = g_{k\beta} \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

$$\delta_\beta^\alpha \, [ij, \alpha] = [ij, \beta] = g_{k\beta} \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

so that,

$$g_{k\alpha} \begin{Bmatrix} \alpha \\ ij \end{Bmatrix} = [ij, k]$$

Showing that the metric tensor can be used to lower the contravariant index of the Christoffel symbol of the second kind to obtain the Christoffel symbol of the first kind.

Higher Order Tensors

We are now in a position to express the derivatives of higher order tensor fields in terms of the Christoffel symbols.

For a second-order tensor $\mathbf{T}$, we can express the components in dyadic form along the product basis as follows:

$$\mathbf{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T^{i}_{.j}\mathbf{g}_i \otimes \mathbf{g}^j = T_j^{.i}\mathbf{g}^j \otimes \mathbf{g}_i$$

This is perfectly analogous to our expanding vectors in terms of basis and reciprocal bases. Derivatives of the tensor may therefore be expressible in any of these product bases. As an example, take the product covariant bases.

Higher Order Tensors

We have:

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}$$

Recall that, $\frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j = \begin{Bmatrix} j \\ ik \end{Bmatrix}$. It follows therefore that,

$$\frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \delta_\alpha^j = \frac{\partial \mathbf{g}_i}{\partial x^k} \cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha \cdot \mathbf{g}^j$$

$$= \left(\frac{\partial \mathbf{g}_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha \right) \cdot \mathbf{g}^j = 0.$$

Clearly, $\frac{\partial \mathbf{g}_i}{\partial x^k} = \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha$

(Obviously since $\mathbf{g}^j$ is a basis vector it cannot vanish)

Higher Order Tensors

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k} \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\left(\begin{Bmatrix}\alpha \\ ik\end{Bmatrix}\mathbf{g}_\alpha\right) \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \left(\begin{Bmatrix}\alpha \\ jk\end{Bmatrix}\mathbf{g}_\alpha\right) \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{\alpha j}\left(\begin{Bmatrix}i \\ \alpha k\end{Bmatrix}\mathbf{g}_i\right) \otimes \mathbf{g}_j + T^{i\alpha}\mathbf{g}_i \otimes \left(\begin{Bmatrix}j \\ \alpha k\end{Bmatrix}\mathbf{g}_j\right) \\ &= \left(\frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\begin{Bmatrix}i \\ \alpha k\end{Bmatrix} + T^{i\alpha}\begin{Bmatrix}j \\ \alpha k\end{Bmatrix}\right)\mathbf{g}_i \otimes \mathbf{g}_j = T^{ij}_{\ ,k}\mathbf{g}_i \otimes \mathbf{g}_j\end{aligned}$$

Where

$$T^{ij}_{\ ,k} = \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\begin{Bmatrix}i \\ \alpha k\end{Bmatrix} + T^{i\alpha}\begin{Bmatrix}j \\ \alpha k\end{Bmatrix} \text{ or } \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\,\Gamma^i_{\alpha k} + T^{i\alpha}\Gamma^j_{\alpha k}$$

are the components of the covariant derivative of the tensor **T** in terms of contravariant components on the product covariant bases as shown.

Higher Order Tensors

In the same way, by taking the tensor expression in the dyadic form of its contravariant product bases, we can write,

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$
$$= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\Gamma^i_{\alpha k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$

Again, notice from previous derivation above, $\left\{ {i \atop jk} \right\} = -\frac{\partial \mathbf{g}^i}{\partial x^k} \cdot \mathbf{g}_j$ so that, $\frac{\partial \mathbf{g}^i}{\partial x^k} = -\left\{ {i \atop \alpha k} \right\}\mathbf{g}^\alpha = -\Gamma^i_{\alpha k}\mathbf{g}^\alpha$ Therefore,

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$
$$= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j - T_{ij}\Gamma^i_{\alpha k}\mathbf{g}^\alpha \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \Gamma^j_{\alpha k}\mathbf{g}^\alpha$$
$$= \left(\frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^\alpha_{ik} - T_{i\alpha}\Gamma^\alpha_{jk}\right)\mathbf{g}^i \otimes \mathbf{g}^j = T_{ij,k}\mathbf{g}^i \otimes \mathbf{g}^j$$

So that

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^\alpha_{ik} - T_{i\alpha}\Gamma^\alpha_{jk}$$

Higher Order Tensors

Two other expressions can be found for the covariant derivative components in terms of the mixed tensor components using the mixed product bases defined above. It is a good exercise to derive these.

The formula for covariant differentiation of higher order tensors follow the same kind of logic as the above definitions. Each covariant index will produce an additional term similar to that in 3 with a dummy index supplanting the appropriate covariant index. In the same way, each contravariant index produces an additional term like that in 3 with a dummy index supplanting an appropriate contravariant index.

Higher Order Mixed Tensors

* The covariant derivative of the mixed tensor, $A_{i_1, i_2, \dots, i_n}^{j_1, j_2, \dots, j_m}$ is the most general case for the covariant derivative of an absolute tensor:

$$A_{i_1, i_2, \dots, i_n \, , j}^{j_1, j_2, \dots, j_m} = \frac{\partial A_{i_1, i_2, \dots, i_n}^{j_1, j_2, \dots, j_m}}{\partial x^j} - \begin{Bmatrix} \alpha \\ i_1 j \end{Bmatrix} A_{\alpha, i_2, \dots, i_n}^{j_1, j_2, \dots, j_m} - \begin{Bmatrix} \alpha \\ i_2 j \end{Bmatrix} A_{i_1, \alpha, \dots, i_n}^{j_1, j_2, \dots, j_m} - \cdots - \begin{Bmatrix} \alpha \\ i_n j \end{Bmatrix} A_{i_1, i_2, \dots, \alpha}^{j_1, j_2, \dots, j_m} + \begin{Bmatrix} j_1 \\ \beta j \end{Bmatrix} A_{i_1, i_2, \dots, i_n}^{\beta, j_2, \dots, j_m} + \begin{Bmatrix} j_2 \\ \beta j \end{Bmatrix} A_{i_1, i_2, \dots, i_n}^{j_1, \beta, \dots, j_m} + \cdots$$

Ricci's Theorem

The metric tensor components behave like constants under a covariant differentiation. The proof that this is so is due to Ricci:

$$
\begin{aligned}
g_{ij,k} &= \frac{\partial g_{ij}}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} g_{\alpha j} - \begin{Bmatrix} \beta \\ kj \end{Bmatrix} g_{i\beta} \\
&= \frac{\partial g_{ij}}{\partial x^k} - [ik, j] - [kj, i] \\
&= \frac{\partial g_{ij}}{\partial x^k} - \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j}\right) - \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^j} + \frac{\partial g_{ik}}{\partial x^j}\right) \\
&= 0.
\end{aligned}
$$

Ricci's Theorem

The conjugate metric tensor behaves the same way as can be seen from the relationship,

$$g_{il}g^{lj} = \delta_i^j$$

The above can be differentiated covariantly with respect to x^k to obtain

$$g_{il},_k\, g^{lj} + g_{il}g^{lj},_k = \delta_i^j,_k$$

$$0 + g_{il}g^{lj},_k = \frac{\partial \delta_i^j}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \delta_\alpha^j + \begin{Bmatrix} j \\ \alpha k \end{Bmatrix} \delta_i^\alpha$$

$$g_{il}g^{lj},_k = 0 - \begin{Bmatrix} j \\ ik \end{Bmatrix} + \begin{Bmatrix} j \\ ik \end{Bmatrix} = 0$$

The contraction of g_{il} with $g^{lj},_k$ vanishes. Since we know that the metric tensor cannot vanish in general, we can only conclude that

$$g^{ij},_k = 0$$

* Which is the second part of the Ricci theorem. With these two, we can treat the metric tensor as well as its conjugate *as constants in covariant differentiation*. Notice that along with this proof, we also obtained the result that the Kronecker Delta vanishes under a partial derivative (an obvious fact since it is a constant), as well as under the covariant derivative. We summarize these results as follows:
* *The metric tensors* g_{ij}*and* g^{ij} *as well as the alternating tensors* $\epsilon_{ijk}, \epsilon^{ijk}$ *are all constants under a covariant differention. The Kronecker delta* δ_i^j *is a constant under both covariant as well as the regular partial differentiation*

Integral Theorems

The divergence theorem is central to several other results in Continuum Mechanics. We present here a generalized form [Ogden] which states that,

Gauss Divergence Theorem

For a tensor field $\mathbf{\Xi}$, The volume integral in the region $\Omega \subset \mathcal{E}$,

$$\int_{\Omega} (\text{grad}\, \mathbf{\Xi})\, dv = \int_{\partial\Omega} \mathbf{\Xi} \otimes \mathbf{n}\, ds$$

where $\mathbf{n}$ is the outward drawn normal to $\partial\Omega$ – the boundary of Ω.

Special Cases: Vector Field

Vector field. Replacing the tensor with the vector field $\mathbf{f}$ and contracting, we have,

$$\int_{\Omega} (\operatorname{div} \mathbf{f})\, dv = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n}\, ds$$

Which is the usual form of the Gauss theorem.

Scalar Field

For a scalar field ϕ, the divergence becomes a gradient and the scalar product on the RHS becomes a simple multiplication. Hence the divergence theorem becomes,

$$\int_{\Omega} (\text{grad}\,\phi)\,dv = \int_{\partial\Omega} \phi\mathbf{n}\,ds$$

The procedure here is valid and will become obvious if we write, $\mathbf{f} = \phi\boldsymbol{a}$ where $\boldsymbol{a}$ is an arbitrary **constant** vector.

$$\int_{\Omega} (\mathrm{div}[\phi \boldsymbol{a}])\, dv = \int_{\partial\Omega} \phi \boldsymbol{a} \cdot \mathbf{n}\, ds = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

For the LHS, note that, $\mathrm{div}[\phi \boldsymbol{a}] = \mathrm{tr}(\mathrm{grad}[\phi \boldsymbol{a}])$

$$\begin{aligned} \mathrm{grad}[\phi \boldsymbol{a}] &= \left(\phi a^i\right)_{,j} \mathbf{g}_i \otimes \mathbf{g}^j \\ &= a^i \phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}^j \end{aligned}$$

The trace of which is,

$$\begin{aligned} a^i \phi_{,j}\, \mathbf{g}_i \cdot \mathbf{g}^j &= a^i \phi_{,j}\, \delta_i^j \\ &= a^i \phi_{,i} = \boldsymbol{a} \cdot \mathrm{grad}\, \phi \end{aligned}$$

For the arbitrary constant vector $\boldsymbol{a}$, we therefore have that,

$$\int_{\Omega} (\mathrm{div}[\phi \boldsymbol{a}])\, dv = \boldsymbol{a} \cdot \int_{\Omega} \mathrm{grad}\, \phi \;\, dv = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

$$\int_{\Omega} \mathrm{grad}\, \phi \;\, dv = \int_{\partial\Omega} \phi \mathbf{n}\, ds$$

Second-Order Tensor field

For a second-order tensor $\mathbf{T}$, the Gauss Theorem becomes,

$$\int_{\Omega} (\mathrm{div}\mathbf{T})\, dv = \int_{\partial\Omega} \mathbf{T}\mathbf{n}\, ds$$

The original outer product under the integral can be expressed in dyadic form:

$$\int_{\Omega} (\mathrm{grad}\ \mathbf{T})\, dv = \int_{\Omega} T^{ij}{}_{,k}\, \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k\, dv$$

$$= \int_{\partial\Omega} \mathbf{T} \otimes \mathbf{n}\, ds$$

$$= \int_{\partial\Omega} T^{ij} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \left(n_k \mathbf{g}^k\right) ds$$

Second-Order Tensor field

Or

$$\int_{\Omega} T^{ij},_k \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k dv = \int_{\partial\Omega} T^{ij} n_k \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \, ds$$

Contracting, we have

$$\int_{\Omega} T^{ij},_k (\mathbf{g}_i \otimes \mathbf{g}_j)\mathbf{g}^k dv = \int_{\partial\Omega} T^{ij} n_k (\mathbf{g}_i \otimes \mathbf{g}_j)\mathbf{g}^k \, ds$$

$$\int_{\Omega} T^{ij},_k \delta_j^k \mathbf{g}_i \, dv = \int_{\partial\Omega} T^{ij} n_k \delta_j^k \mathbf{g}_i \, ds$$

$$\int_{\Omega} T^{ij},_j \mathbf{g}_i \, dv = \int_{\partial\Omega} T^{ij} n_j \mathbf{g}_i \, ds$$

Which is the same as,

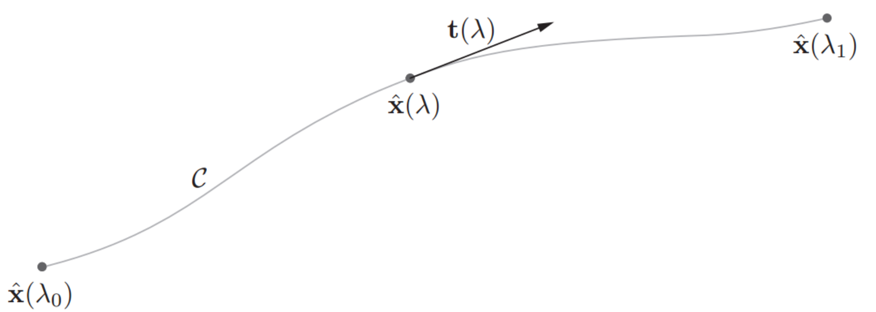
$$\int_{\Omega} (\mathrm{div}\mathbf{T}) \, dv = \int_{\partial\Omega} \mathbf{T}\mathbf{n} \, ds$$

Stokes Theorem

Consider the Euclidean Point Space $\mathcal{E}$. A curve $\mathcal{C}$ is defined by the parametrization (Gurtin):

$\mathbf{x} = \hat{\mathbf{x}}(\lambda)$ where $\lambda_0 \leq \lambda \in \mathcal{R} \leq \lambda_1$

$\mathcal{C}$ is said to be a closed curve if $\hat{\mathbf{x}}(\lambda_0) = \hat{\mathbf{x}}(\lambda_1)$



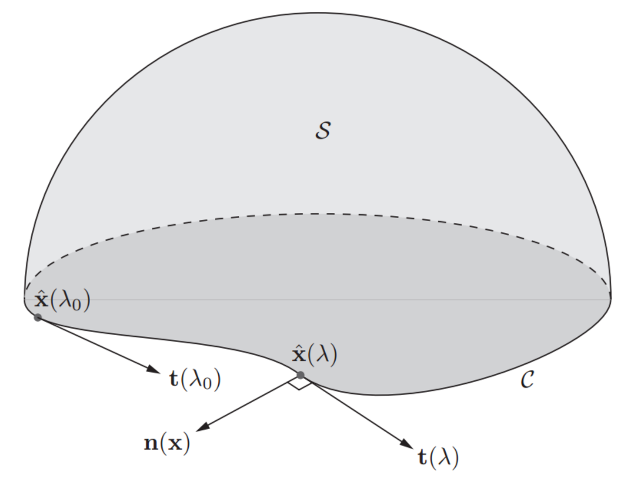
Define $\mathbf{t}(\lambda) \equiv \frac{d\hat{\mathbf{x}}(\lambda)}{d\lambda}$.

For any vector point function defined everywhere along $\mathcal{C}$, the line integral,

$$\int_{\mathcal{C}} \mathbf{v} \cdot \boldsymbol{d}\mathbf{x} = \int_{\lambda_0}^{\lambda_1} \mathbf{v}\big(\hat{\mathbf{x}}(\lambda_0)\big) \cdot \frac{d\hat{\mathbf{x}}(\lambda)}{d\lambda} d\lambda = \int_{\lambda_0}^{\lambda_1} \mathbf{v}\big(\hat{\mathbf{x}}(\lambda_0)\big) \cdot \mathbf{t}(\lambda) d\lambda$$

Stokes Theorem

Let ϕ be a scalar field on $\mathcal{E}$, the Chain rule immediately implies that,



$$\int_{\lambda_0}^{\lambda_1} \operatorname{grad} \phi \cdot \boldsymbol{dx} = \int_{\lambda_0}^{\lambda_1} \operatorname{grad} \phi \cdot \frac{d\hat{\mathbf{x}}(\lambda)}{d\lambda} d\lambda$$

$$= \int_{\lambda_0}^{\lambda_1} \frac{\partial \phi\left(\hat{\mathbf{x}}(\lambda)\right)}{\partial \lambda} d\lambda = \phi\left(\hat{\mathbf{x}}(\lambda_1)\right) - \phi\left(\hat{\mathbf{x}}(\lambda_0)\right)$$

So that for a close curve $\int_{\mathcal{C}} \operatorname{grad} \phi \cdot \boldsymbol{dx} = 0$

For a positively oriented surface bounded by a closed curve $\mathcal{C}$ (Gurtin), Stokes theorem is stated as follows:

Stokes Theorem

Stokes' Theorem Let ϕ, $\mathbf{v}$, and $\mathbf{T}$ be scalar, vector, and tensor fields with common domain a region $\mathcal{R}$. Then given any positively oriented surface $\mathcal{S}$, with boundary $\mathcal{C}$ closed curve, in $\mathcal{R}$

$$\int_{\mathcal{C}} \phi d\mathbf{x} = \int_{\mathcal{S}} \mathbf{n} \times \operatorname{grad} \phi \, da$$

$$\int_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{x} = \int_{\mathcal{S}} \mathbf{n} \times \operatorname{curl} \mathbf{v} \, da$$

$$\int_{\mathcal{C}} \mathbf{T} d\mathbf{x} = \int_{\mathcal{S}} (\operatorname{curl} \mathbf{T})^{\mathrm{T}} da$$

Physical Components of Tensors

* The components of a vector or tensor in a Cartesian system are projections of the vector on directions that have no dimensions and a value of unity.
* These components therefore have the same units as the vectors themselves.
* It is natural therefore to expect that the components of a tensor have the same dimensions.
* In general, this is not so. In curvilinear coordinates, components of tensors do not necessarily have a direct physical meaning. This comes from the fact that base vectors are not guaranteed to have unit values ($h_i \neq 1$ in general).

Physical Components

* They may not be dimensionless. For example, in orthogonal spherical polar, the base vectors are $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$.
* These can be expressed in terms of dimensionless unit vectors as, $\rho \mathbf{e}_\rho, \rho \sin\phi\, \mathbf{e}_\theta$, and $\mathbf{e}_\phi$ since the magnitudes of the basis vectors are $\rho, \rho \sin\phi$, and 1 or $\left(\sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}\right)$ respectively.
* As an example consider a force with the contravariant components F^1, F^2 and F^3,

$$\begin{aligned}\mathbf{F} &= F^1\mathbf{g}_1 + F^2\mathbf{g}_2 + F^3\,\mathbf{g}_3\\ &= \rho F^1\mathbf{e}_\rho + \rho\sin\phi\, F^2\mathbf{e}_\theta + F^3\mathbf{e}_\phi\end{aligned}$$

Physical Components

* Which may also be expressed in terms of physical components,

$$\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi.$$

* While these physical components $\{F_\rho, F_\theta, F_\phi\}$ have the dimensions of force, for the contravariant components normalized by the in terms of the unit vectors along these axes to be consistent, $\{\rho F^1, \rho \sin\theta\, F^2, F^3\}$ must each be in the units of a force. Hence, F^1, F^2 and F^3 may not themselves be in force units. The consistency requirement implies,

$$F_\rho = \rho F^1, \qquad F_\theta = \rho \sin\theta\, F^2, \qquad \text{and } F_\phi = F^3$$

Physical Components

* For the same reasons, if we had used covariant components, the relationships,

$$F_\rho = \frac{F_1}{\rho}, \qquad F_\theta = \frac{F_2}{\rho \sin\theta}, \qquad \text{and } F_\phi = F_3$$

* The magnitudes of the reciprocal base vectors are $\frac{1}{\rho}, \frac{1}{\rho \sin\phi}$, and 1. While the physical components have the dimensions of force, $F_1 = \rho F_\rho$ and $F_2 = \rho \sin\phi\, F_\theta$ have the dimensions of moment, while $F^1 = \frac{F_\rho}{\rho}$ and $F^2 = \frac{F_\theta}{\rho \sin\phi}$ are in dimensions of force per unit length. Only the third components in both cases are given in the dimensions of force.

* The physical components of a vector or tensor are components that have physically meaningful units and magnitudes. Often it is convenient to derive the governing equations for a problem in terms of the tensor components but to solve the problem in physical components.
* In an **orthogonal system** of coordinates, to obtain a physical component from a tensor component we must divide by the magnitude of the relevant coordinate for each covariant index and multiply by each contravariant index. To illustrate this point, consider the evaluation of the physical components of the symmetric tensor components τ^{ij} or τ^i_j or τ_{ij} in spherical polar coordinates. Here, as we have seen, the magnitudes of the base vectors $\sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}$ or h_1, h_2 and h_3 are $\rho, \rho \sin\phi$ and 1. Using the rule specified above, the table below computes the physical components from the three associated tensors as follows:

Physical Component	Contravariant	Covariant	Mixed
$\tau(\rho\rho)$	$\tau^{11}h_1h_1 = \tau^{11}\rho^2$	$\frac{\tau_{11}}{h_1\,h_1} = \frac{\tau_{11}}{\rho^2}$	$\frac{\tau_1^1}{h_1}h_1 = \tau_1^1$
$\tau(\rho\theta)$	$\tau^{12}h_1h_2 = \tau^{12}\rho^2\sin\phi$	$\frac{\tau_{12}}{h_1\,h_1} = \frac{\tau_{12}}{\rho^2\sin\phi}$	$\frac{\tau_2^1}{h_2}h_1 = \frac{\tau_2^1}{\sin\phi}$
$\tau(\theta\theta)$	$\tau^{22}h_2h_2 = \tau^{22}\rho^2\sin^2\phi$	$\frac{\tau_{22}}{h_2\,h_2} = \frac{\tau_{22}}{\rho^2\sin^2\phi}$	$\frac{\tau_2^2}{h_2}h_2 = \tau_2^2$
$\tau(\theta\phi)$	$\tau^{23}h_2h_3 = \tau^{23}\rho\sin\phi$	$\frac{\tau_{23}}{h_2\,h_3} = \frac{\tau_{23}}{\rho\sin\phi}$	$\frac{\tau_3^2}{h_3}h_2 = \tau_3^2\rho\sin\phi$
$\tau(\phi\phi)$	$\tau^{33}h_3h_3 = \tau^{33}$	$\frac{\tau_{33}}{h_3\,h_3} = \tau_{33}$	$\frac{\tau_3^3}{h_3}h_3 = \tau_3^3$
$\tau(\rho\phi)$	$\tau^{31}h_3h_1 = \tau^{31}\rho$	$\frac{\tau_{31}}{h_3\,h_1} = \frac{\tau_{31}}{\rho}$	$\frac{\tau_1^3}{h_1}h_3 = \frac{\tau_1^3}{\rho}$

17. The transformation equations from the Cartesian to the oblate spheroidal coordinates ξ, η and φ are: $x = f\xi\eta \sin\varphi$, $y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}$, and $z = f\xi\eta\cos\varphi$, where f is a constant representing the half the distance between the foci of a family of confocal ellipses. Find the components of the metric tensor in this system.

The metric tensor components are:

$$g_{\xi\xi} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2$$

$$= (f\eta \sin\varphi)^2 + f^2\xi^2\left(\frac{1-\eta^2}{\xi^2 - 1}\right) + (f\eta\cos\varphi)^2 = f^2\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)$$

$$g_{\eta\eta} = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 = f^2\left(\frac{\xi^2 - \eta^2}{1 - \xi^2}\right)$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = (f\xi\eta)^2$$

$$g_{\xi\eta} = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial z}{\partial \xi}\right)\left(\frac{\partial z}{\partial \eta}\right)$$

$$= (f\eta\sin\varphi)(f\xi\sin\varphi) - \left(f\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\right)\left(f\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\right) + (f\eta\cos\varphi)(f\xi\cos\varphi)$$

$$= 0 = g_{\eta\varphi} = g_{\varphi\xi}$$

Find an expression for the divergence of a vector in orthogonal curvilinear coordinates.

The gradient of a vector $\boldsymbol{F} = F^i \mathbf{g}_i$ is $\nabla \otimes \boldsymbol{F} = \mathbf{g}^j \partial_j \otimes \left(F^i \mathbf{g}_i\right) = F^i{}_{,j} \mathbf{g}^j \otimes \mathbf{g}_i$. The divergence is the contraction of the gradient. While we may use this to evaluate the divergence directly it is often easier to use the computation formula in equation Ex 15:

$$\nabla \cdot \mathbf{F} = F^i{}_{,j} \mathbf{g}^j \cdot \mathbf{g}_i = F^i{}_{,i} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g} F^i\right)}{\partial x^i}$$

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1} (h_1 h_2 h_3 F^1) + \frac{\partial}{\partial x^2} (h_1 h_2 h_3 F^2) + \frac{\partial}{\partial x^3} (h_1 h_2 h_3 F^3)\right]$$

Recall that the physical (components having the same units as the tensor in question) components of a contravariant tensor are not equal to the tensor components unless the coordinate system is Cartesian. The physical component $F(i) = F^i h_i$ (no sum on i). In terms of the physical components therefore, the divergence becomes,

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial x^1} (h_2 h_3 F(1)) + \frac{\partial}{\partial x^2} (h_1 h_3 F(2)) + \frac{\partial}{\partial x^3} (h_1 h_2 F(3))\right]$$

Find an expression for the Laplacian operator in Orthogonal coordinates.

For the contravariant component of a vector, F^j,

$$F^j{}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,F^j\right)}{\partial x^j}.$$

Now the contravariant component of gradient $F^j = g^{ij}\varphi_{,i}$. Using this in place of the vector F^j, we can write,

$$g^{ij}\varphi_{,ji} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,g^{ij}\varphi_{,j}\right)}{\partial x^i}$$

given scalar φ, the Laplacian $\nabla^2\varphi$ is defined as, $g^{ij}\varphi_{,ji}$ so that,

$$\nabla^2\varphi = g^{ij}\varphi_{,ji} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial\varphi}{\partial x^j}\right)$$

When coordinates are orthogonal, $g_{ij} = g^{ij} = 0$ whenever $i \neq j$. Expanding the computation formula therefore, we can write,

$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_1h_2h_3}{h_1}\frac{\partial\varphi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1h_2h_3}{h_2}\frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2h_3}{h_3}\frac{\partial\varphi}{\partial x^3}\right)\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(h_2h_3\frac{\partial\varphi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(h_1h_3\frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(h_1h_2\frac{\partial\varphi}{\partial x^3}\right)\right]$$

Show that the oblate spheroidal coordinate systems are orthogonal. Find an expression for the Laplacian of a scalar function in this system.

Example above shows that $g_{\xi\eta} = g_{\eta\varphi} = g_{\varphi\xi} = 0$. This is the required proof of orthogonality. Using the computation formula in example 11, we may write for the oblate spheroidal coordinates that,

$$\nabla^2\Phi = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{f^3\xi^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(f\xi\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\frac{\partial\Phi}{\partial\xi}\right)\right.$$

For a vector field $\boldsymbol{u}$, show that $\mathrm{grad}(\boldsymbol{u}\times)$ is a third ranked tensor. Hence or otherwise show that $\mathrm{div}(\boldsymbol{u}\times) = -\mathrm{curl}\,\boldsymbol{u}$.

The second–order tensor $(\boldsymbol{u}\times)$ is defined as $\epsilon^{ijk}u_j\mathbf{g}_i \otimes \mathbf{g}_k$. Taking the covariant derivative with an independent base, we have

$$\mathrm{grad}(\boldsymbol{u}\times) = \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}^l$$

This gives a third order tensor as we have seen. Contracting on the last two bases,

$$\begin{aligned}\mathrm{div}(\boldsymbol{u}\times) &= \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i \otimes \mathbf{g}_k \cdot \mathbf{g}^l\\ &= \epsilon^{ijk}u_{j,l}\,\mathbf{g}_i\delta_k^l\\ &= \epsilon^{ijk}u_{j,k}\,\mathbf{g}_i\\ &= -\mathrm{curl}\,\boldsymbol{u}\end{aligned}$$

Coordinate transformations

Begin with our familiar Cartesian system of coordinates. We can represent the position of a point (position vector) with three coordinates x, y, z ($\in \mathscr{R}$) such that,

$$\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

* That is, the choice of any three scalars can be used to locate a point. We now introduce a transformation (called a polar transformation) of $\{x, y\} \to \{r, \phi\}$ such that, $x = r\cos\phi$, and $y = r\sin\phi$. Note also that this transformation is invertible: $r = \sqrt{x^2 + y^2}$,and $\phi = \tan^{-1}\frac{y}{x}$

Curvilinear Coordinates

* With such a transformation, we can locate any point in the 3-D space with three scalars $\{r, \phi, z\}$ instead of our previous set $\{x, y, z\}$. Our position vector is now,

$$\mathbf{r} = r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j} + z\boldsymbol{k} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$$

* where we define $\boldsymbol{e}_r \equiv \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}$, $\boldsymbol{e}_z$ is no different from $\boldsymbol{k}$. In order to complete our triad of basis vectors, we need a third vector, e_ϕ. In selecting $\boldsymbol{e}_\phi$, we want it to be such that $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ can form an orthonormal basis.

Cylindrical Polar coordinate system

Let

$$\boldsymbol{e}_\phi = \xi\boldsymbol{i} + \eta\boldsymbol{j}$$

* To satisfy our conditions,

$\boldsymbol{e}_\phi \cdot \boldsymbol{e}_r = 0$, $\boldsymbol{e}_\phi \cdot \boldsymbol{e}_z = 0$, and $\sqrt{\xi^2 + \eta^2} = 1$.

* It is easy to see that $\boldsymbol{e}_\phi \equiv -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}$ satisfies these requirements. $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ forms an orthonormal (that is, *each member has unit magnitude and they are pairwise orthogonal*) triad just like $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The transformation we have just described can be given a geometric interpretation. In either case, it is the definition of the ***Cylindrical Polar coordinate system***.

* Unlike our Cartesian system, we note that $\{\boldsymbol{e}_r(\phi), \boldsymbol{e}_\phi(\phi), \boldsymbol{e}_z\}$ as the first two of these are not constants but vary with angular orientation. $\boldsymbol{e}_z$ remains a constant vector as in the Cartesian case.

Spherical Polar

* Continuing further with our transformation, we may again introduce two new scalars such that $\{r, z\} \to \{\rho, \theta\}$ in such a way that the position vector,

$$\mathbf{r} = r\boldsymbol{e}_r + z\boldsymbol{e}_z = \rho \sin\theta\, \boldsymbol{e}_r + \rho\cos\theta\, \boldsymbol{e}_z \equiv \rho\boldsymbol{e}_\rho$$

* Here, $r = \rho \sin\theta$, $z = \rho\cos\theta$. As before, we can use three scalars, $\{\rho, \theta, \phi\}$ instead of $\{r, \phi, z\}$. In comparison to the original Cartesian system we began with, we have that,

$$\begin{aligned}\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = &\ \rho\sin\theta\, \boldsymbol{e}_r + \rho\cos\theta\, \boldsymbol{e}_z \\ &= \rho\sin\theta\,(\cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}) + \rho\cos\theta\, \boldsymbol{k} \\ &= \rho\sin\theta\cos\phi\, \boldsymbol{i} + \rho\sin\theta\sin\phi\, \boldsymbol{j} + \rho\cos\theta\, \boldsymbol{k} \\ &\equiv \rho\boldsymbol{e}_\rho\end{aligned}$$

from which it is clear that the unit vector $\boldsymbol{e}_\rho \equiv \sin\theta\cos\phi\, \boldsymbol{i} + \sin\theta\sin\phi\, \boldsymbol{j} + \cos\theta\, \boldsymbol{k}$.

Spherical Polar

* Again, we introduce the unit vector, $\boldsymbol{e}_\theta \equiv \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}$ and retain $\boldsymbol{e}_\phi = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}$ as before.
* It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho, \theta, \phi\}$ directly by the transformation equations, $x = \rho\sin\theta\cos\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\theta$.
* The orthonormal set of basis for the $\{\rho, \theta, \phi\}$ system is $\left\{\boldsymbol{e}_\rho(\theta, \phi), \boldsymbol{e}_\theta\,(\theta, \phi), \boldsymbol{e}_\phi\,(\phi)\right\}$.
* This is the ***Spherical Polar Coordinate System***.

Variable Bases

* There two main points to note in transforming from Cartesian to Cylindrical or Spherical polar coordinate systems. The latter two (also called curvilinear systems) have unit basis vector sets that are dependent on location. Explicitly, we may write,

$$\boldsymbol{e}_r(r, \phi, z) = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}$$
$$\boldsymbol{e}_\phi(r, \phi, z) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}$$
$$\boldsymbol{e}_z(r, \phi, z) = \boldsymbol{k}$$

For Cylindrical Polar, and for Spherical Polar,

* $\boldsymbol{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$
  $\boldsymbol{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}$
  $\boldsymbol{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}$.
* Of course, there are specific cases in which some of these basis vectors are constants as we can see. It is instructive to note that curvilinear (so called because coordinate lines are now curves rather than straight lines) coordinates generally have basis vectors that depend on the coordinate variables.
* Unlike the Cartesian system, we will no longer be able to assume that the derivatives of the basis vectors vanish.

* The second point to note is more subtle. While it is true that a vector referred to a curvilinear basis will have coordinate projections in each of the basis so that, a typical vector

$$\mathbf{v} = v_1\mathbf{g}_1 + v_2\mathbf{g}_2 + v_3\mathbf{g}_3$$

* when referred to the basis vectors $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$, for position vectors, in order that the transformation refers to the same position vector we started with in the Cartesian system

We have for Cylindrical Polar, the vector,

$$\mathbf{r}(r, \phi, z) = r\boldsymbol{e}_r(\phi) + z\boldsymbol{e}_z$$

and for Spherical Polar,

$$\mathbf{r}(r, \theta, \phi) = \rho\boldsymbol{e}_\rho(\theta, \phi)$$

* Expressing position vectors in Curvilinear coordinates must be done carefully. We do well to take into consideration the fact that in curvilinear systems, the basis vectors are not fully specified until they are expressly specified with their locational dependencies.

* It is only in this situation that we can express the position vector of a specific location Spherical and Cylindrical Polar coordinates are two more commonly used curvilinear systems. Others will be introduced as necessary in due course.

Natural Bases

* Beginning from the position vector, given a system with coordinate variables $(\alpha_1, \alpha_2, \alpha_3)$, it is easy to prove that the set of vectors,

$$\left\{\frac{\partial \mathbf{r}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_1}, \frac{\partial \mathbf{r}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_1}, \frac{\partial \mathbf{r}(\alpha_1, \alpha_2, \alpha_3)}{\partial \alpha_1}\right\}$$

Which we can write more compactly as

$$\mathbf{g}^i \equiv \frac{\partial \mathbf{r}}{\partial \alpha_i}, i = 1,2,3$$

constitute an independent set. This set, with its dual set of vectors, $\mathbf{g}_j$ such that $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, constitute the "Natural bases" for the coordinate system.

Orthonormality of Natural Bases

* We can only guarantee the independence of the bases for any arbitrary system. They may NOT be normalized neither are they guaranteed of mutual orthogonality.

HW. Show that the natural bases for Spherical and Cylindrical polar systems are orthogonal but not normalized. And that the dual natural bases for the Cartesian system coincide.