1. Given that vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore,

$$\det \mathbf{T} = \frac{[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \neq 0$$

Consequently, $[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}] \neq 0$. Which shows that $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

2. Given that vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, if $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly dependent, then $\exists \alpha, \beta$ and γ all real such that $\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{o}$. But $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent. This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \neq \mathbf{o}$.

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = \boldsymbol{o}.$$

This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!

3. Given that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

If $\mathbf{T}$ is not singular, if $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly dependent, then $\exists\alpha$, and β both real such that $\alpha\mathbf{Tu} + \beta\mathbf{Tv} = \mathbf{o}$. But $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. This means that $\alpha\mathbf{u} + \beta\mathbf{v} \neq \mathbf{o}$.

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} = \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) = \boldsymbol{o}.$$

This means that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!

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4. Given that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore the cofactor, $\mathbf{T}^{\mathrm{c}} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T} \neq 0$ also exists and is non-zero. The linear independence of $\mathbf{u}$ and $\mathbf{v}$ means that the parallelogram formed by them has a non-trivial area $\mathbf{u} \times \mathbf{v} \neq 0$. Now, the parallelogram formed by $\mathbf{Tu}$ and $\mathbf{Tv}$ is also non zero because,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v}) \neq 0$$

Hence $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.

5. Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$

$$
\begin{aligned}
(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} - [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{w}]\mathbf{v} \\
&= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} \\
&= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{w} \\
&= (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] &= \mathbf{e} \cdot [(\mathbf{e} \times \mathbf{u}) \times (\mathbf{e} \times \mathbf{v})] \\
&= \mathbf{e} \cdot [(\mathbf{e} \otimes \mathbf{e})(\mathbf{u} \times \mathbf{v})] \\
&= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{e} \otimes \mathbf{e})\mathbf{e} \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e} = [\mathbf{e}, \mathbf{u}, \mathbf{v}]
\end{aligned}
$$

making use of the symmetry of $(\mathbf{e} \otimes \mathbf{e})$.

6. For any tensor $\boldsymbol{A}$, define $\left(\mathrm{Sym}(\boldsymbol{A})\right)_{ij} = \frac{1}{2}\left(A_{ij} + A_{ji}\right)$. Show that $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A}$

Clearly, $\mathrm{Sym}(\boldsymbol{S}) = \frac{1}{2}(\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}})$

It also follows that,

$$\boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A} = \frac{1}{2}\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{S} + \boldsymbol{S}^{\mathrm{T}})\boldsymbol{A}$$

$$= \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{A})$$

But $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{A})$.

Hence $\mathrm{Sym}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{A}) = \boldsymbol{A}^{\mathrm{T}}\mathrm{Sym}(\boldsymbol{S})\boldsymbol{A}$

7. Given that the trace of a dyad $\mathbf{a} \otimes \mathbf{b}$, $\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$. By expressing the tensors $\mathbf{T}$ and $\mathbf{S}$ in component form, show that $\mathrm{tr}(\mathbf{ST}) = \mathrm{tr}(\mathbf{TS}) = \mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$

In component form, $\mathbf{S} = S_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \mathbf{T} = T_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$.

$$\begin{aligned}
\mathbf{ST} &= S_{ij}T_{\alpha\beta}(\mathbf{g}^i \otimes \mathbf{g}^j)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}T_{\alpha\beta}(\mathbf{g}^i \otimes \mathbf{g}^j)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta) \\
&= S_{ij}T_{\alpha\beta}\mathbf{g}^i \otimes \mathbf{g}^\beta g^{j\alpha} \\
\mathrm{tr}(\mathbf{ST}) &= S_{ij}T_{\alpha\beta}\mathbf{g}^i \cdot \mathbf{g}^\beta g^{j\alpha} = S_{ij}T_{\alpha\beta}g^{i\beta}g^{j\alpha} \\
&= S_{ij}T^{ji}
\end{aligned}$$

$$\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = S_{ij}T_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\beta \otimes \mathbf{g}^\alpha) = S_{ij}T_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\alpha g^{i\beta}$$

so that

$$\begin{aligned}
\mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) &= S_{ij}T_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\alpha g^{i\beta} = S_{ij}T_{\alpha\beta}g^{j\alpha}g^{i\beta} \\
&= S_{ij}T^{ji}
\end{aligned}$$

Similar computations lead to the conclusion that

$$\mathrm{tr}(\mathbf{ST}) = \mathrm{tr}(\mathbf{TS}) = \mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$$

8. Given an arbitrary tensor $\mathbf{T}$ a skew tensor $\mathbf{W}$ and a symmetric tensor $\mathbf{S}$. Show that

$$\mathbf{S}:\mathbf{T} = \mathbf{S}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\mathrm{sym}\,\mathbf{T}$$

$$\mathbf{W}:\mathbf{T} = -\mathbf{W}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\mathrm{skw}\,\mathbf{T}$$

$$\mathbf{S}:\mathbf{W} = 0$$

Note that $\mathbf{T} = \mathrm{sym}\,\mathbf{T} + \mathrm{skw}\,\mathbf{T}$, and $\mathbf{T}^{\mathrm{T}} = \mathrm{sym}\,\mathbf{T} - \mathrm{skw}\,\mathbf{T}$. Also note that the inner product between a skew and a symmetric tensor vanishes. Consequently,

$$\begin{aligned}
\mathbf{S}:\mathbf{T} &= \mathbf{S}:(\mathrm{sym}\,\mathbf{T} + \mathrm{skw}\,\mathbf{T}) \\
&= \mathbf{S}:\mathrm{sym}\,\mathbf{T} + \mathbf{S}:\mathrm{skw}\,\mathbf{T} \\
&= \mathbf{S}:\mathrm{sym}\,\mathbf{T} \\
&= \mathbf{S}:\mathbf{T}^{\mathrm{T}} \\
\mathbf{W}:\mathbf{T} &= \mathbf{W}:(\mathrm{sym}\,\mathbf{T} + \mathrm{skw}\,\mathbf{T}) \\
&= \mathbf{W}:\mathrm{sym}\,\mathbf{T} + \mathbf{W}:\mathrm{skw}\,\mathbf{T} \\
&= \mathbf{W}:\mathrm{skw}\,\mathbf{T} \\
&= -\mathbf{S}:\mathbf{T}^{\mathrm{T}}
\end{aligned}$$

To show that $\mathbf{S}:\mathbf{W} = 0$. Observe that, in component form, $\mathbf{S} = S_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, $\mathbf{W} = W_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$.

$$\mathbf{S}^{\mathrm{T}}\mathbf{W} = S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$$

$$= S_{ij} W_{\alpha\beta} (\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$$
$$= S_{ij} W_{\alpha\beta} \mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha}$$
$$\mathbf{S} : \mathbf{W} = \operatorname{tr} \mathbf{S}^{\mathrm{T}} \mathbf{W}$$
$$= S_{ij} W_{\alpha\beta} \mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij} W_{\alpha\beta} g^{j\beta} g^{i\alpha}$$
$$= S_{ij} W^{ij} = -S_{ij} W^{ji}$$
$$= -S_{ji} W^{ji} = -S_{ij} W^{ij}$$

Which vanishes because it is equal to the negative of itself.

9. Show that if for every skew tensor $\mathbf{W}$, the inner product $\mathbf{S} : \mathbf{W} = 0$, it must follow that $\mathbf{S}$ is symmetric.

$$\mathbf{S}^{\mathrm{T}} \mathbf{W} = S_{ij} W_{\alpha\beta} (\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$$
$$= S_{ij} W_{\alpha\beta} (\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)$$
$$= S_{ij} W_{\alpha\beta} \mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha}$$
$$\mathbf{S} : \mathbf{W} = \operatorname{tr} \mathbf{S}^{\mathrm{T}} \mathbf{W}$$
$$= S_{ij} W_{\alpha\beta} \mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij} W_{\alpha\beta} g^{j\beta} g^{i\alpha}$$
$$= S_{ij} W^{ij} = -S_{ij} W^{ji} = 0 = S_{ij} W^{ji}$$

Since all inner products $\mathbf{S} : \mathbf{W} = 0$. But,

$$S_{ij} W^{ij} = S_{ij} W^{ji} = S_{ji} W^{ij}$$

So that $(S_{ij} - S_{ji})W^{ij} = 0 \Rightarrow S_{ij} = S_{ji}$ Hence, $\mathbf{S}$ is symmetric.

10. Show that if for every symmetric tensor $\mathbf{S}$, the inner product $\mathbf{S}:\mathbf{W} = 0$, it must follow that $\mathbf{W}$ is anti-symmetric.

$$\begin{aligned}\mathbf{S}^{\mathrm{T}}\mathbf{W} &= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)\\ &= S_{ij}W_{\alpha\beta}(\mathbf{g}^j \otimes \mathbf{g}^i)(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta)\\ &= S_{ij}W_{\alpha\beta}\mathbf{g}^j \otimes \mathbf{g}^\beta g^{i\alpha}\\ \mathbf{S}:\mathbf{W} &= \operatorname{tr}\mathbf{S}^{\mathrm{T}}\mathbf{W}\\ &= S_{ij}W_{\alpha\beta}\mathbf{g}^j \cdot \mathbf{g}^\beta g^{i\alpha} = S_{ij}W_{\alpha\beta}g^{j\beta}g^{i\alpha}\\ &= S_{ij}W^{ij} = S_{ji}W^{ij} = 0 = -S_{ji}W^{ji}\end{aligned}$$

Since all inner products $\mathbf{S}:\mathbf{W} = 0$. But,

$$S_{ij}W^{ij} = S_{ji}W^{ji} = -S_{ji}W^{ji}$$

So that $S_{ij}(W^{ij} + W^{ji}) = 0 \Rightarrow W^{ij} = -W^{ji}$ Hence, $\mathbf{W}$ is anti-symmetric

11. If we can find α, β and γ unit tensor, $\mathbf{I} = \alpha\mathbf{a} \otimes \mathbf{b} + \beta\mathbf{b} \otimes \mathbf{c} + \gamma\mathbf{c} \otimes \mathbf{a}$, show that unless $\mathbf{b} \cdot \mathbf{a} = \alpha^{-1}$ then $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ cannot be linearly independent.

In the expression,

$$\mathbf{I} = \alpha\mathbf{a} \otimes \mathbf{b} + \beta\mathbf{b} \otimes \mathbf{c} + \gamma\mathbf{c} \otimes \mathbf{a}$$

Take a product on the right with vector **a,**

$$\mathbf{Ia} = \alpha\mathbf{a}(\mathbf{b}\cdot\ \mathbf{a}) + \ \beta\mathbf{b}(\mathbf{c}\cdot\ \mathbf{a}) + \gamma\mathbf{c}(\mathbf{a}\cdot\ \mathbf{a})$$

$$\Rightarrow \mathbf{a}\big(\mathbf{I} - \alpha(\mathbf{b}\cdot\ \mathbf{a})\big) = \ \beta\mathbf{b}(\mathbf{c}\cdot\ \mathbf{a}) + \gamma\mathbf{c}(\mathbf{a}\cdot\ \mathbf{a})$$

$$\mathbf{a} = \ \frac{\beta\mathbf{b}(\mathbf{c}\cdot\ \mathbf{a})}{1 - \alpha(\mathbf{b}\cdot\ \mathbf{a})} + \frac{\gamma\mathbf{c}(\mathbf{a}\cdot\ \mathbf{a})}{\big(1 - \alpha(\mathbf{b}\cdot\ \mathbf{a})\big)}$$

So that this expression enables us to write **a** in terms of **b** and **c.**

12. Show that the dyad $\mathbf{u}\otimes\mathbf{v}$ is NOT, in general symmetric: $\mathbf{u}\otimes\mathbf{v} = \mathbf{v}\otimes\mathbf{u} - (\mathbf{u}\times\mathbf{v})\times$

$$\mathbf{u}\times\mathbf{v} = \epsilon^{ijk}u_j v_k\,\mathbf{g}_i$$

$$\big((\mathbf{u}\times\mathbf{v})\times\big) = \epsilon_{\alpha i\beta}\epsilon^{ijk}u_j v_k\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$

$$= -\left(\delta^j_\alpha\delta^k_\beta - \delta^k_\alpha\delta^j_\beta\right)u_j v_k\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$

$$= \left(-u_\alpha v_\beta + u_\beta v_\alpha\right)\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$

$$= \ \mathbf{v}\otimes\mathbf{u} - \ \mathbf{u}\otimes\mathbf{v}$$

13. Given that, $\delta^{rsk}_{ijk} = \delta^r_i\delta^s_j - \delta^s_i\delta^r_j$, show that $\delta^{rjk}_{ijk} = 2\delta^r_i$.

Contracting one more index, we have:

$$e_{ijk}e^{rjk} = \delta^{rjk}_{ijk} = \delta^r_i\delta^j_j - \delta^j_i\delta^r_j = 3\delta^r_i - \delta^r_i = 2\delta^r_i$$

These results are useful in several situations.

14. Show that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$

Note that

$$g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g_{\gamma i}\begin{vmatrix} g^{i\alpha} & g^{i\beta} & g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix} = \begin{vmatrix} g_{\gamma i}g^{i\alpha} & g_{\gamma i}g^{i\beta} & g_{\gamma i}g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \begin{vmatrix} \delta^\alpha_\gamma & \delta^\beta_\gamma & \delta^\gamma_\gamma \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \delta^\alpha_\gamma \begin{vmatrix} g^{j\beta} & g^{j\gamma} \\ g^{k\beta} & g^{k\gamma} \end{vmatrix} - \delta^\beta_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\gamma} \\ g^{k\alpha} & g^{k\gamma} \end{vmatrix} + \delta^\gamma_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= \begin{vmatrix} g^{j\beta} & g^{j\alpha} \\ g^{k\beta} & g^{k\alpha} \end{vmatrix} - \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} + 3\begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} = \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$$

15. For vectors $\mathbf{u}$ and $\mathbf{v}$ and tensor $\mathbf{T}$show that $(\mathbf{u} \otimes \mathbf{v})\mathbf{T} = \left(\mathbf{u} \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{v})\right)$

Operate both sides on the vector $\mathbf{z}$ and let $\mathbf{Tz} = \mathbf{w}$. On the LHS,

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{Tz} = (\mathbf{u} \otimes \mathbf{v})\mathbf{w}$$

On the RHS, we have:

$$\left(\mathbf{u} \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{v})\right)\mathbf{z} = \mathbf{u}\left((\mathbf{T}^{\mathrm{T}}\mathbf{v}) \cdot \mathbf{z}\right) = \mathbf{u}\left(\mathbf{z} \cdot (\mathbf{T}^{\mathrm{T}}\mathbf{v})\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\mathbf{u}\left(\mathbf{z} \cdot (\mathbf{T}^{\mathrm{T}}\mathbf{v})\right) = \mathbf{u}(\mathbf{v} \cdot (\mathbf{Tz}))$$

follows from the definition of transposition. Hence,

$$\left(\mathbf{u} \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{v})\right)\mathbf{z} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) = (\mathbf{u} \otimes \mathbf{v})\mathbf{w}$$

Hence the result.

16. For vectors $\mathbf{u}$ and $\mathbf{v}$ and tensor $\mathbf{T}$ show that $\mathbf{T}(\mathbf{u} \otimes \mathbf{v}) = \left((\mathbf{Tu}) \otimes (\mathbf{v})\right)$

Operate both sides on a vector $\mathbf{w}$. On the LHS,

$$\mathbf{T}(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{Tu}(\mathbf{v} \cdot \mathbf{w})$$

And, on the RHS,

$$[(\mathbf{Tu}) \otimes \mathbf{v}]\mathbf{w} = \mathbf{Tu}(\mathbf{v} \cdot \mathbf{w})$$

17. Show that the transpose of a dyad is a simple reversal of its elements

Recall that for $\mathbf{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\mathbf{A}^{\mathrm{T}}$ satisfying

$$\mathbf{w} \cdot (\mathbf{A}^{\mathrm{T}}\mathbf{v}) = \mathbf{v} \cdot (\mathbf{A}\mathbf{w})$$

Is called the transpose of $\mathbf{A}$. Now let $\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$ a dyad.

$$\begin{aligned}\mathbf{v} \cdot (\mathbf{A}\mathbf{w}) &= \mathbf{v} \cdot [(\mathbf{a} \otimes \mathbf{b})\mathbf{w}] = \mathbf{v} \cdot [\mathbf{a}(\mathbf{b} \cdot \mathbf{w})] = (\mathbf{v} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{w}) = (\mathbf{w} \cdot \mathbf{b})(\mathbf{v} \cdot \mathbf{a}) \\ &= \mathbf{w} \cdot (\mathbf{b} \otimes \mathbf{a})\mathbf{v}\end{aligned}$$

So that $(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$

Showing that the transpose of a dyad is simply a reversal of its factors.

18. For $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathscr{V}$, show that the dyad composition, $(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \otimes \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$

The proof is to show that both sides produce the same result when they act on the same vector. Let $\mathbf{y} \in \mathscr{V}$, then the LHS on $\mathbf{y}$ yields:

$$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x})\mathbf{y} = (\mathbf{u} \otimes \mathbf{v})[\mathbf{w}(\mathbf{x} \cdot \mathbf{y})] = \mathbf{u}(\mathbf{v} \cdot \mathbf{w})(\mathbf{x} \cdot \mathbf{y})$$

Which is obviously the result from the RHS also.

19. For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{v}\otimes(\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times$.

The tensor $(\mathbf{u}\times) = -\epsilon_{lmn}u^n\mathbf{g}^l\otimes\mathbf{g}^m$ similarly, $(\mathbf{v}\times) = -\epsilon^{\alpha\beta\gamma}v_\gamma\mathbf{g}_\alpha\otimes\mathbf{g}_\beta$ and $(\mathbf{w}\times) = -\epsilon^{ijk}w_k\mathbf{g}_i\otimes\mathbf{g}_j$. Clearly,

$$\begin{aligned}
(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) &= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}^l\otimes\mathbf{g}^m)(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= -\epsilon^{\alpha\beta\gamma}\epsilon_{lmn}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}^l\otimes\mathbf{g}_j)\delta^m_\alpha g_{\beta i}\\
&= \epsilon^{m\beta\gamma}\epsilon_{mln}\epsilon^{ijk}g_{\beta i}u^n v_\gamma w_k(\mathbf{g}^l\otimes\mathbf{g}_j)\\
&= \left(\delta^\beta_l\delta^\gamma_n - \delta^\beta_n\delta^\gamma_l\right)g_{\beta i}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}^l\otimes\mathbf{g}_j)\\
&= g_{li}\epsilon^{ijk}u^n v_n w_k(\mathbf{g}^l\otimes\mathbf{g}_j) - g_{ni}\epsilon^{ijk}u^n v_l w_k(\mathbf{g}^l\otimes\mathbf{g}_j)\\
&= \epsilon^{ijk}u^n v_n w_k(\mathbf{g}_i\otimes\mathbf{g}_j) - \epsilon^{ijk}u_i v_l w_k(\mathbf{g}^l\otimes\mathbf{g}_j)\\
&= (v_l\mathbf{g}^l)\otimes\left(\epsilon^{ikj}u_i w_k\mathbf{g}_j\right) - (u^n v_n)\epsilon^{ikj}w_k(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= [\mathbf{v}\otimes(\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]
\end{aligned}$$

20. Show that $[\mathbf{u},\mathbf{v},\mathbf{w}] = -\mathrm{tr}[(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)]$

In the above we have shown that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = [\mathbf{v}\otimes(\mathbf{u}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]$ Because the vector cross is traceless, the trace of $[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times] = 0$. The trace of the first term, $\mathbf{v}\otimes(\mathbf{u}\times\mathbf{w})$ is obviously the same as $-[\mathbf{u},\mathbf{v},\mathbf{w}]$ which completes the proof.

21. Show that $(\mathbf{u}\times)(\mathbf{v}\times) = \mathbf{v}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{I}$ and that $\text{tr}[(\mathbf{u}\times)(\mathbf{v}\times)] = -2(\mathbf{u}\cdot\mathbf{v})$

$$
\begin{aligned}
(\mathbf{u}\times)(\mathbf{v}\times) &= \epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}^l\otimes\mathbf{g}^m)(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)\\
&= \epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}^l\otimes\mathbf{g}_\beta)\delta_\alpha^m\\
&= \epsilon_{lmn}\epsilon^{m\beta\gamma}u^n v_\gamma(\mathbf{g}^l\otimes\mathbf{g}_\beta)\\
&= \left[\delta_l^\gamma\delta_n^\beta - \delta_n^\gamma\delta_l^\beta\right]u^n v_\gamma(\mathbf{g}^l\otimes\mathbf{g}_\beta)\\
&= u^\beta v_l(\mathbf{g}^l\otimes\mathbf{g}_\beta) - u^\gamma v_\gamma(\mathbf{g}^l\otimes\mathbf{g}_l)\\
&= \mathbf{v}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{I}
\end{aligned}
$$

Obviously, the trace of this tensor is $-2(\mathbf{u}\cdot\mathbf{v})$

22. If $\mathbf{u}$ is perpendicular to $\mathbf{v}$ show that all the eigenvalues of the dyad $\mathbf{u}\otimes\mathbf{v}$ are zero.

For this tensor, $I_1 = \mathbf{u}\cdot\mathbf{v} = 0$ on account of $\mathbf{u}$ being perpendicular to $\mathbf{v}$. We now examine the other two invariants:

$$I_2 = \frac{[(\mathbf{u}\otimes\mathbf{v})\mathbf{a},(\mathbf{u}\otimes\mathbf{v})\mathbf{b},\mathbf{c}] + [\mathbf{a},(\mathbf{u}\otimes\mathbf{v})\mathbf{b},(\mathbf{u}\otimes\mathbf{v})\mathbf{c}] + [(\mathbf{u}\otimes\mathbf{v})\mathbf{a},\mathbf{b},(\mathbf{u}\otimes\mathbf{v})\mathbf{c}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

For linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Clearly,

$$I_2 = \frac{[(\mathbf{v}\cdot\mathbf{a})\mathbf{u},(\mathbf{v}\cdot\mathbf{b})\mathbf{u},\mathbf{c}] + [a,(\mathbf{v}\cdot\mathbf{b})\mathbf{u},(\mathbf{v}\cdot\mathbf{c})\mathbf{u}] + [(\mathbf{v}\cdot\mathbf{a})\mathbf{u},\mathbf{b},(\mathbf{v}\cdot\mathbf{c})\mathbf{u}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = 0$$

on the collinearity of two vectors in each triple product.

$$I_3 = \frac{[(\mathbf{u} \otimes \mathbf{v})\mathbf{a}, (\mathbf{u} \otimes \mathbf{v})\mathbf{b}, (\mathbf{u} \otimes \mathbf{v})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[(\mathbf{v} \cdot \mathbf{a})\mathbf{u}, (\mathbf{v} \cdot \mathbf{b})\mathbf{u}, (\mathbf{v} \cdot \mathbf{c})\mathbf{u}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 0$$

The latter being the triple product of three parallel vectors. Hence we have a case of a tensor with three principal invariants vanishing. The characteristic equation becomes,

$$\lambda^3 - I_1\lambda^2 + I_2\lambda + I_3 = \lambda^3 = 0$$

Yielding three equal roots of zero. $\mathbf{u} \otimes \mathbf{v}$ is thus a non-zero tensor with zero eigenvalues.

23. Tensors $\mathbf{S}$ and $\mathbf{T}$ are said to be similar if the invertible tensor exists such that $\mathbf{S} = \mathbf{BTB}^{-1}$. Show that $\mathbf{S}$ and $\mathbf{T}$ have the same eigenvalues as well as principal invariants.

The characteristic equation for $\mathbf{S}$ is,

$$\mathbf{Sv} = \lambda\mathbf{v}$$

where λ and is the eigenvalue and $\mathbf{v}$ the eigenvector. But $\mathbf{S} = \mathbf{BTB}^{-1}$ substituting, we have,

$$\mathbf{BTB}^{-1}\mathbf{v} = \lambda\mathbf{v}$$

so that

$$\mathbf{TB}^{-1}\mathbf{v} = \lambda\mathbf{B}^{-1}\mathbf{v}$$

If we define $\mathbf{v}_1 \equiv \mathbf{B}^{-1}\mathbf{v}$, we obtain,

$$\mathbf{T}\mathbf{v}_1 = \lambda\mathbf{v}_1$$

yielding the exactly same characteristic equation as well as eigenvalues and principal invariants as $\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$.

24. For a tensor $\mathbf{A}$ with three eigenvalues λ_i, if $\boldsymbol{\gamma}_i$ are the corresponding eigenvectors, Find a spectral form for the tensor $\mathbf{A}$

Consider the dual basis to the eigenbasis, $\{\boldsymbol{\gamma}^i\}$ such that $\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}^i = \delta_i^j$. The mixed components of $\mathbf{A}$ are evaluated as,

$$A^{j}_{.\,i} = \boldsymbol{\gamma}^j \cdot (\mathbf{A}\boldsymbol{\gamma}_i) = \lambda_i \boldsymbol{\gamma}^j \cdot \boldsymbol{\gamma}_i = \sum_{i=1}^{3} \lambda_i \delta_i^j = \boldsymbol{\gamma}_i \cdot \left(\mathbf{A}^{\mathrm{T}} \boldsymbol{\gamma}^j\right)$$

We can therefore write

$$\mathbf{A} = A^{j}_{.\,i} \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i = \sum_{i=1}^{3} \lambda_i \delta_i^j \, \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^j = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i$$

in which all the off-diagonal terms vanish.

25. Show that the vector cross of $\mathbf{u} \times \mathbf{v}$ is the skew tensor $\mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = \epsilon^{ijk} u_j v_k \, \mathbf{g}_i$$

$$\left((\mathbf{u} \times \mathbf{v}) \times\right) = \epsilon_{\alpha i \beta} \epsilon^{ijk} u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$$

$$= -\left(\delta^j_\alpha \delta^k_\beta - \delta^k_\alpha \delta^j_\beta\right) u_j v_k \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$$
$$= (-u_\alpha v_\beta + u_\beta v_\alpha)\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$$
$$= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$$

? all the vector cross problems to be checked Jan 21, 2016.

26. Given that, $(\mathbf{u}\times)(\mathbf{v}\times) = \mathbf{u} \otimes \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{I}$. If $\boldsymbol{\omega}_1$and $\boldsymbol{\omega}_2$ are the vector cross of the skew tensors $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively, show that $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_1 - \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2$, $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = (\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)\times$ and that $\mathbf{W}_1\mathbf{W}_2 + \mathbf{W}_2\mathbf{W}_1 =$

Clearly, $(\boldsymbol{\omega}_1\times)(\boldsymbol{\omega}_2\times) - (\boldsymbol{\omega}_2\times)(\boldsymbol{\omega}_1\times)$
$$= (\boldsymbol{\omega}_2 \cdot \boldsymbol{\omega}_1)\mathbf{I} - \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_1 - (\boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2)\mathbf{I} + \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2$$
$$= \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2 - \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_1$$

Clearly, $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_1 - \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2$.

The skew tensor,

$$((\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2)\times) = \epsilon^{\alpha i \beta}\epsilon_{ijk} w_1^i w_2^j\, \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= \left(\delta^\beta_j \delta^\alpha_k - \delta^\beta_k \delta^\alpha_j\right) w_1^i w_2^j \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= w_1^\beta w_2^\alpha \mathbf{g}_\alpha \otimes \mathbf{g}_\beta - w_1^\alpha w_2^\beta \mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= \boldsymbol{\omega}_2 \otimes \boldsymbol{\omega}_1 - \boldsymbol{\omega}_1 \otimes \boldsymbol{\omega}_2$$
$$= \mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1$$

27. For arbitrary, mutually orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\mathbf{u}\cdot\mathbf{T}\mathbf{v}=0$ if and only if $\mathbf{T}=\lambda\mathbf{I}$.

If we write $\mathbf{T}=T_{\alpha\beta}\mathbf{g}^\alpha\otimes\mathbf{g}^\beta, \mathbf{u}=u_i\mathbf{g}^i$ and $\mathbf{v}=u_j\mathbf{g}^j$ then

$$\mathbf{u}\cdot\mathbf{T}\mathbf{v}=\left(u_i\mathbf{g}^i\right)\cdot\left(T_{\alpha\beta}\mathbf{g}^\alpha\otimes\mathbf{g}^\beta\right)v_j\mathbf{g}^j$$
$$=\left(u_i\mathbf{g}^i\right)\cdot\left(T_{\alpha\beta}v^\beta\mathbf{g}^\alpha\right)=T_{\alpha\beta}u^\alpha v^\beta$$

If $\mathbf{T}=\lambda\mathbf{I}$ then we can write $\mathbf{T}=T_{\alpha\beta}\mathbf{g}^\alpha\otimes\mathbf{g}^\beta=\lambda g_{\alpha\beta}\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$ so that $T_{\alpha\beta}=\lambda g_{\alpha\beta}$. In this case, $\mathbf{u}\cdot\mathbf{T}\mathbf{v}=\lambda g_{\alpha\beta}u^\alpha v^\beta=\lambda\,(\mathbf{u}\cdot\mathbf{v})=0$ when $\mathbf{u}$ and $\mathbf{v}$ are mutually orthogonal.

Conversely, when $\mathbf{u}$ and $\mathbf{v}$ are mutually orthogonal, $T_{\alpha\beta}u^\alpha v^\beta=0$. A situation which can only be satisfied when $T_{\alpha\beta}=\lambda g_{\alpha\beta}$ or that $\mathbf{T}=\lambda\mathbf{I}$.

28. Show that the vector cross of a cross product is not equal to the product of the two vector crosses; that is, $\left((\mathbf{u}\times\mathbf{v})\times\right)\neq(\mathbf{u}\times)(\mathbf{v}\times)$

$$\mathbf{u}\times\mathbf{v}=\epsilon^{ijk}u_jv_k\,\mathbf{g}_i$$
$$\left((\mathbf{u}\times\mathbf{v})\times\right)=\epsilon_{\alpha i\beta}\epsilon^{ijk}u_jv_k\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$
$$=-\left(\delta_\alpha^j\delta_\beta^k-\delta_\alpha^k\delta_\beta^j\right)u_jv_k\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$
$$=\left(-u_\alpha v_\beta+u_\beta v_\alpha\right)\mathbf{g}^\alpha\otimes\mathbf{g}^\beta$$

$$= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$$

$$(\mathbf{u}\times)(\mathbf{v}\times) = -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta)(\mathbf{g}^l \otimes \mathbf{g}^m)$$
$$= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha \otimes \mathbf{g}^m)\delta_\beta^l$$
$$= -\epsilon_{\beta mn}\epsilon^{\beta\gamma\alpha}u^n v_\gamma(\mathbf{g}_\alpha \otimes \mathbf{g}^m)$$
$$= [\delta_n^\gamma \delta_m^\alpha - \delta_m^\gamma \delta_n^\alpha]u^n v_\gamma(\mathbf{g}_\alpha \otimes \mathbf{g}^m)$$
$$= u^n v_n(\mathbf{g}_\alpha \otimes \mathbf{g}^\alpha) - u^n v_m(\mathbf{g}_n \otimes \mathbf{g}^m) = (\mathbf{u}\cdot\mathbf{v})\mathbf{I} - \mathbf{u} \otimes \mathbf{v}$$

The two tensors are not equal.

29. The vector cross $\mathbf{w}$ of the skew tensor $\mathbf{W} \equiv (\mathbf{w}\times) = \epsilon^{i\alpha j}w_\alpha \mathbf{g}_i \otimes \mathbf{g}_j$. $\mathbf{w}$ is called the axial vector of $\mathbf{W}$. Show that the components of the axial vector can be obtained from its vector cross by the formula, $w_k = -\frac{1}{2}\epsilon_{ijk}W^{ij}$.

Clearly, $W^{ij} = \epsilon^{i\alpha j}w_\alpha$. Contracting with ϵ_{ijk} we obtain,

$$\epsilon_{ijk}W^{ij}\mathbf{g}^k = \epsilon_{ijk}\epsilon^{i\alpha j}w_\alpha \mathbf{g}^k = \left(\delta_j^\alpha \delta_k^j - \delta_k^\alpha \delta_j^j\right)w_\alpha \mathbf{g}^k$$
$$-2\delta_k^\alpha w_\alpha \mathbf{g}^k = -2w_k \mathbf{g}^k$$

so that $w_k = -\frac{1}{2}\epsilon_{ijk}W^{ij}$.

30. Given that the unit vector $\mathbf{e} = x_i \mathbf{g}^i$, Find the vector cross, $\mathbf{W} \equiv (\mathbf{e}\times)$.

The vector cross is traceless; only non-trivial components are off-diagonal. Using the formula, $\mathbf{W} \equiv (\mathbf{e}\times) = \epsilon^{i\alpha j}x_\alpha \mathbf{g}_i \otimes \mathbf{g}_j$.

$W^{12} = \epsilon^{132}x_3 = -\frac{x_3}{\sqrt{g}} = -W^{21}$, $W^{23} = \epsilon^{213}x_1 = -\frac{x_1}{\sqrt{g}} = -W^{32}$, $W^{13} = \epsilon^{123}x_2 = \frac{x_2}{\sqrt{g}} = -W^{31}$. So that the vector cross

$$\mathbf{W} = \mathbf{e}\times = \frac{1}{\sqrt{g}}\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

If the reference basis is orthonormal, then the vector cross is,

$$\mathbf{W} = \mathbf{e}\times = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

in this case, $\sqrt{g} = 1$

31. Consider a unit $\mathbf{w}$ vector lying on a plane containing $\mathbf{e}_3$, cutting the $\mathbf{e}_1 - \mathbf{e}_2$ at angle α inclined to $\mathbf{e}_3$at an angle β. Find an expression for $\mathbf{W} \equiv (\mathbf{w}\times)$ and for $\mathbf{W}^2$.

The unit vector $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. The vector cross for this axis is,

$$\mathbf{W} = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

$$\mathbf{W}^2 = \begin{pmatrix} -\sin^2\alpha \sin^2\beta - \cos^2\beta & \sin\alpha\cos\alpha\ \sin^2\beta & \cos\alpha\ \sin\beta\cos\beta \\ \sin\alpha\cos\alpha\sin^2\beta & -\cos^2\alpha\sin^2\beta - \cos^2\beta & \sin\alpha\sin\beta\cos\beta \\ \cos\alpha\sin\beta\cos\beta & \sin\alpha\sin\beta\cos\beta & -\sin^2\alpha\sin^2\beta - \cos^2\alpha\sin^2\beta \end{pmatrix}$$

32. Given that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ are two orthonormal bases, show that the spectral form, $\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$.

To prove that $\mathbf{Q}$ is a rotation, first observe that,

$$\begin{aligned}\mathbf{Q}\mathbf{Q}^{\mathrm{T}} &= (\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3)(\mathbf{e}_1 \otimes \boldsymbol{\xi}_1 + \mathbf{e}_2 \otimes \boldsymbol{\xi}_2 + +\mathbf{e}_3 \otimes \boldsymbol{\xi}_3) \\ &= \boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_3 = \mathbf{I}\end{aligned}$$

Furthermore,

$$\det \mathbf{Q} = [\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3] = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = 1$$

since the set $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ is orthonormal.

We have already seen that each coordinate vector in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ rotates to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$respectively because, $\mathbf{Q}\mathbf{e}_1 = (\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3)\mathbf{e}_1 = \boldsymbol{\xi}_1$, similarly, $\mathbf{Q}\mathbf{e}_2 = \boldsymbol{\xi}_2$ and $\mathbf{Q}\mathbf{e}_3 = \boldsymbol{\xi}_3$.

33. For the tensor $\mathbf{T}$, given that $\boldsymbol{YT} = \boldsymbol{I}$, show that $\boldsymbol{TY} = \boldsymbol{YT} = \boldsymbol{I}$.

Consider $\boldsymbol{TYTu}$ where $\boldsymbol{u}$ is a vector. Since $\boldsymbol{YT} = \boldsymbol{I}$, it follows that $\boldsymbol{TYTu} = \boldsymbol{TIu} =$

$\boldsymbol{Tu} \equiv \boldsymbol{v}$ where $\boldsymbol{v}$ is also a vector. Clearly,

$$\boldsymbol{TYTu} = \boldsymbol{TYv} = \boldsymbol{v}$$

which immediately shows that $\boldsymbol{TY} = \boldsymbol{I}$ as required to be shown.

34. Given the basis vectors $\mathbf{g}_i, i = 1,2,3$ show that for any tensor $\mathbf{T}$, the product

$$\mathbf{T}\mathbf{g}_i = T_{\alpha i}\mathbf{g}^\alpha$$

In component form, let $\mathbf{T} = T_{\alpha\beta}(\mathbf{g}^\alpha \otimes \mathbf{g}^\alpha)$ so that

$$\begin{aligned}\mathbf{T}\mathbf{g}_i &= T_{\alpha\beta}\left(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\right)\mathbf{g}_i \\ &= T_{\alpha\beta}\left(\mathbf{g}^\beta \cdot \mathbf{g}_i\right)\mathbf{g}^\alpha = T_{\alpha\beta}\delta_i^\beta\mathbf{g}^\alpha \\ &= T_{\alpha i}\mathbf{g}^\alpha\end{aligned}$$

35. Show that the trace of a tensor $\mathbf{T}$, in component form, can be written as $T_{i}^{.i} = T_{ij}g^{ij} = T^{ij}g_{ij}$.

Observe that any three linearly independent vectors can be treated as a basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$\mathrm{tr}\,(\mathbf{T}) \equiv I_1(\mathbf{T}) = \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$
$$= \frac{[(T_{i1}\mathbf{g}^i), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}\mathbf{g}^j), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}\mathbf{g}^k)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$
$$= \frac{[(T_{i1}g^{\alpha i}\mathbf{g}_\alpha), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}g^{\beta j}\mathbf{g}_\beta), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}g^{\gamma k}\mathbf{g}_\gamma)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$
$$= \frac{T_1^\alpha[\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_3] + T_2^\beta[\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_3] + T_3^\gamma[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_\gamma]}{\epsilon_{123}}$$
$$= \frac{T_1^\alpha\epsilon_{\alpha 23} + T_2^\beta\epsilon_{1\beta 3} + T_3^\gamma\epsilon_{12\gamma}}{\epsilon_{123}}$$
$$= \frac{T_1^1\epsilon_{123} + T_2^2\epsilon_{123} + T_3^3\epsilon_{123}}{\epsilon_{123}} = T_i^{\ i} = T_{ij}g^{ij} = T^{ij}g_{ij}$$

36. Show that $[\mathbf{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{T}\mathbf{g}_k] = T_\alpha^\alpha[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]$

Note that,

$$[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]$$
$$= [(T_{i1}\mathbf{g}^i), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}\mathbf{g}^j), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}\mathbf{g}^k)]$$
$$= [(T_{i1}g^{\alpha i}\mathbf{g}_\alpha), \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, (T_{j2}g^{\beta j}\mathbf{g}_\beta), \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, (T_{k3}g^{\gamma k}\mathbf{g}_\gamma)]$$
$$= T_1^\alpha[\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_3] + T_2^\beta[\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_3] + T_3^\gamma[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_\gamma]$$

$$= T_1^\alpha \epsilon_{\alpha 23} + T_2^\beta \epsilon_{1\beta 3} + T_3^\gamma \epsilon_{12\gamma}$$
$$= T_1^1 \epsilon_{123} + T_2^2 \epsilon_{123} + T_3^3 \epsilon_{123}$$
$$= T_\alpha^\alpha \epsilon_{123} = T_\alpha^\alpha [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

Swapping $\mathbf{g}_2$ and $\mathbf{g}_3$, it is clear that,

$$[\mathbf{Tg}_1, \mathbf{g}_3, \mathbf{g}_2] + [\mathbf{g}_1, \mathbf{Tg}_3, \mathbf{g}_2] + [\mathbf{g}_1, \mathbf{g}_3, \mathbf{Tg}_2]$$
$$= -[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{g}_3] - [\mathbf{g}_1, \mathbf{g}_2, \mathbf{Tg}_3]$$
$$= T_\alpha^\alpha \epsilon_{132} = T_\alpha^\alpha [\mathbf{g}_1, \mathbf{g}_3, \mathbf{g}_2]$$

Continuing, we have that

$$[\mathbf{Tg}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{Tg}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{Tg}_k] = T_\alpha^\alpha [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] = T_\alpha^\alpha \epsilon_{ijk}$$

37. Show that $I_1(\boldsymbol{T}) \equiv \text{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{Ta},\boldsymbol{b},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{Tb},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{b},\boldsymbol{Tc}]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$ is independent of the choice of the linearly independent vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

Let us refer each vector to a covariant basis so that, $\boldsymbol{a} = a^i \mathbf{g}_i$, $\boldsymbol{b} = b^j \mathbf{g}_j$, and $\boldsymbol{c} = c^k \mathbf{g}_k$. Hence,

$$I_1(\boldsymbol{T}) \equiv \text{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}(a^i \mathbf{g}_i), b^j \mathbf{g}_j, c^k \mathbf{g}_k] + [a^i \mathbf{g}_i, \boldsymbol{T}(b^j \mathbf{g}_j), c^k \mathbf{g}_k] + [a^i \mathbf{g}_i, b^j \mathbf{g}_j, \boldsymbol{T}(c^k \mathbf{g}_k)]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$
$$= \frac{a^i b^j c^k ([\boldsymbol{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \boldsymbol{T}\mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \boldsymbol{T}\mathbf{g}_k])}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

But $[\mathbf{Tg}_i, \mathbf{g}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{Tg}_j, \mathbf{g}_k] + [\mathbf{g}_i, \mathbf{g}_j, \mathbf{Tg}_k] = T^\alpha_\alpha [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]$. We have that

$$I_1(\boldsymbol{T}) = \frac{a^i b^j c^k T^\alpha_\alpha [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]}{\epsilon_{ijk} a^i b^j c^k} = \frac{\epsilon_{ijk} a^i b^j c^k}{\epsilon_{ijk} a^i b^j c^k} T^\alpha_\alpha = T^\alpha_\alpha$$

Which is obviously independent of the choice of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

38. Write the second tensor invariant in terms of components

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$I_2(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

The first of the numerator terms can be simplified as,

$$[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] = [(T_{i1}\mathbf{g}^i), (T_{j2}\mathbf{g}^j), \mathbf{g}_3] = [(T_{i1} g^{\alpha i}\mathbf{g}_\alpha), (T_{j2} g^{\beta j}\mathbf{g}_\beta), \mathbf{g}_3] = T^\alpha_1 T^\beta_2 [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_3]$$

The other terms are similarly simplified. Clearly,

$$
\begin{aligned}
I_2(\mathbf{T}) &= \frac{T_1^\alpha T_2^\beta [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_3] + T_2^\beta T_3^\gamma [\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_\gamma] + T_1^\alpha T_3^\gamma [\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_\gamma]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{T_1^\alpha T_2^\beta \epsilon_{\alpha\beta 3} + T_2^\beta T_3^\gamma \epsilon_{1\beta\gamma} + T_1^\alpha T_3^\gamma \epsilon_{\alpha 2\gamma}}{\epsilon_{123}} \\
&= \frac{[(T_1^1 T_2^2 - T_1^2 T_2^1) + (T_2^2 T_3^3 - T_2^3 T_3^2) + (T_3^3 T_1^1 - T_3^1 T_1^3)]\epsilon_{123}}{\epsilon_{123}} \\
&= \frac{1}{2}\left(T_\alpha^\alpha T_\beta^\beta - T_\beta^\alpha T_\alpha^\beta\right)
\end{aligned}
$$

39. Using direct notation, show that the second invariant of a tensor is the trace of its cofactor.

Given three basis vectors, $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$

$$
\begin{aligned}
I_2(\mathbf{T}) &= \frac{[\mathbf{T}\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{T}\mathbf{g}_2, \mathbf{T}\mathbf{g}_3] + [\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{\mathbf{T}^{\mathrm{c}}(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 + \mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3) + [\mathbf{T}^{\mathrm{c}}(\mathbf{g}_3 \times \mathbf{g}_1) \cdot \mathbf{g}_2]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{T}^{\mathrm{cT}} \mathbf{g}_3 + (\mathbf{g}_2 \times \mathbf{g}_3) \cdot \mathbf{T}^{\mathrm{cT}} \mathbf{g}_1 + (\mathbf{g}_3 \times \mathbf{g}_1) \cdot \mathbf{T}^{\mathrm{cT}} \mathbf{g}_2}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{T}^{\mathrm{cT}} \mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}} \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_3, \mathbf{g}_1, \mathbf{T}^{\mathrm{cT}} \mathbf{g}_2]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}
\end{aligned}
$$

$$= I_1(\mathbf{T}^{c\mathrm{T}}) = I_1(\mathbf{T}^{c})$$

Which is the trace of its cofactor as required.

40. Express the third tensor invariant in terms of its components.

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken for granted. Consequently, for any tensor $\mathbf{T}$,

$$I_3(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} = \frac{\left[\left(T_{i1}\mathbf{g}^i\right), \left(T_{j2}\mathbf{g}^j\right), \left(T_{k3}\mathbf{g}^k\right)\right]}{\epsilon_{123}}$$

$$= \frac{\left[\left(T_{i1}g^{\alpha i}\mathbf{g}_\alpha\right), \left(T_{j2}g^{\beta j}\mathbf{g}_\beta\right), \left(T_{k3}g^{\gamma k}\mathbf{g}_\gamma\right)\right]}{\epsilon_{123}} = \frac{T_1^\alpha T_2^\beta T_3^\gamma [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_\gamma]}{\epsilon_{123}}$$

$$= \frac{T_1^\alpha T_2^\beta T_3^\gamma \epsilon_{\alpha\beta\gamma}}{\epsilon_{123}} = e_{\alpha\beta\gamma} T_1^\alpha T_2^\beta T_3^\gamma = \det \mathbf{T}$$

41. In component form, the third tensor invariant of a tensor $\mathbf{T}, I_3(\mathbf{T}) = e_{\alpha\beta\gamma} T_1^\alpha T_2^\beta T_3^\gamma = \det \mathbf{T}$. Show that $e_{ijk} T_\alpha^i T_\beta^j T_\gamma^k = e_{\alpha\beta\gamma} \det \mathbf{T}$.

We do this by first establishing the fact that the LHS is completely antisymmetric in α, β and γ. We note that the indices i, j and k are dummy and therefore,

$$e_{ijk} T_\alpha^i T_\beta^j T_\gamma^k = e_{kji} T_\alpha^k T_\beta^j T_\gamma^i = e_{kji} T_\gamma^i T_\alpha^k T_\beta^j = -e_{ijk} T_\gamma^i T_\beta^j T_\alpha^k$$

Showing that a simple swap of α and γ changes the sign. This is similarly true for the other pairs in the lower symbols. Thus we establish anti-symmetry in α, β and γ. Noting that both sides take the same values when α, β and γ are equal to $1, 2$ and 3 respectively. The arrangement of the indices makes this value positive or negative in the same antisymmetric way. This completes the proof. Similarly we can write,

$$e^{ijk}T_i^\alpha T_j^\beta T_k^\gamma = e^{ijk}T_i^1 T_j^2 T_k^3 e^{\alpha\beta\gamma} = e^{\alpha\beta\gamma}\det \mathbf{T}$$

42. Given the basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and their dual, $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$, Show that for any other basis pair, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ and their dual, $\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3$, the relationship, $(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = \delta_\alpha^\beta$ holds.

By simply reversing the step, it is immediately obvious that,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta$$

Observe that the expression in the parentheses is the unit tensor – having no effect on a vector; it follows that,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta = \mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta_\alpha^\beta$$

43. Show that the first invariant has the same value in every coordinate system.

We have shown elsewhere that for any tensor $\boldsymbol{T}$ the first invariant, $I_1(\boldsymbol{T})=$

$$\mathrm{tr}(\boldsymbol{T}) \equiv \frac{[\{\mathbf{Tg}_1\},\ \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{\mathbf{Tg}_2\}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, \{\mathbf{Tg}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]} = T_{ij}g^{ij} = T_i^{.i}$$

We proceed to show that this quantity has the same value in any other independent set of basis vectors. Let $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)$ be another arbitrary set of linearly independent vectors. They therefore form a basis of a coordinate system. Let $(\boldsymbol{\gamma}^1,\ \boldsymbol{\gamma}^2,\ \boldsymbol{\gamma}^3)$ be the dual to the set. It is easy to establish that the new set will be related to the old in;

$$\boldsymbol{\gamma}_j = (\boldsymbol{\gamma}_j \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha, \text{ and } \boldsymbol{\gamma}^i = (\boldsymbol{\gamma}^i \cdot \mathbf{g}_\beta)\mathbf{g}^\beta$$

Let us write the $(i,\ j)^{th}$ components in the $\boldsymbol{\gamma}$ – bases as $\tilde{T}_i^{.j}$.

Clearly,

$$\begin{aligned}\tilde{T}_i^{.j} &\equiv \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^j = (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha \cdot \left[\mathbf{T}(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta)\mathbf{g}^\beta\right] \\ &= (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta)[\mathbf{g}_\alpha \cdot \mathbf{Tg}^\beta] \\ &= [(\boldsymbol{\gamma}^j \otimes \boldsymbol{\gamma}_i)\mathbf{g}^\alpha] \cdot \mathbf{g}_\beta \left[T_\alpha^{.\beta}\right]\end{aligned}$$

Contracting i with j, we obtain the sum,

$$\begin{aligned}\tilde{T}_i^{.i} = \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^i &= [(\boldsymbol{\gamma}^i \otimes \boldsymbol{\gamma}_i)\mathbf{g}^\alpha] \cdot \mathbf{g}_\beta \left[T_\alpha^{.\beta}\right] \\ &= \mathbf{g}^\alpha \cdot \mathbf{g}_\beta T_\alpha^{.\beta} = \delta_\beta^\alpha T_\alpha^{.\beta} = T_\alpha^{.\alpha}\end{aligned}$$

So that the trace has the same value in any arbitrarily chosen coordinate system including curvilinear ones whether orthogonal or not.

44. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, find the principal invariants of the dyad $\mathbf{u} \otimes \mathbf{v}$

The first principal invariant $I_1(\mathbf{u} \otimes \mathbf{v})$ is the trace, defined as,

$$\operatorname{tr}(\mathbf{u} \otimes \mathbf{v}) = \frac{[\{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_1\},\ \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_2\}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{1}{\epsilon_{123}}\{[v_1\mathbf{u}, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, v_2\mathbf{u}, \mathbf{g}_3] + [\mathbf{g}_1,\ \mathbf{g}_2, v_3\mathbf{u}]\}$$

$$= \frac{1}{\epsilon_{123}}\{(v_1\mathbf{u}) \cdot (\epsilon_{23i}\mathbf{g}^i) + (\epsilon_{31i}\mathbf{g}^i) \cdot (v_2\mathbf{u}) + (\epsilon_{12i}\mathbf{g}^i) \cdot (v_3\mathbf{u})\}$$

$$= \frac{1}{\epsilon_{123}}\{(v_1\mathbf{u}) \cdot (\epsilon_{231}\mathbf{g}^1) + (\epsilon_{312}\mathbf{g}^2) \cdot (v_2\mathbf{u}) + (\epsilon_{123}\mathbf{g}^3) \cdot (v_3\mathbf{u})\} = v_i u^i.$$

The second principal invariant $I_2(\mathbf{u} \otimes \mathbf{v}) =$

$$= \frac{[\{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_1\}, (\mathbf{u} \otimes \mathbf{v})\mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{(\mathbf{u} \otimes \mathbf{v})\,\mathbf{g}_2\}, (\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3] + [(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_1,\ \mathbf{g}_2, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{g}_3\}]}{[\mathbf{g}_1,\ \mathbf{g}_2, \mathbf{g}_3]}$$

$= \frac{1}{\epsilon_{123}}\{[v_1\mathbf{u}, v_2\mathbf{u}, \mathbf{g}_3] + [\mathbf{g}_1, v_2\mathbf{u}, v_3\mathbf{u}] + [v_1\mathbf{u},\ \mathbf{g}_2, v_3\mathbf{u}]\} = 0$. Vanishes – collinearity.

The third invariant $I_3(\mathbf{u} \otimes \mathbf{v}) = \frac{1}{\epsilon_{123}}[v_1\mathbf{u}, v_2\mathbf{u}, v_3\mathbf{u}]$ which also vanishes on account of collinearity

45. Define the cofactor of a tensor as, $\operatorname{cof}\mathbf{T} \equiv \mathbf{T}^{\mathrm{c}} \equiv \mathbf{T}^{-\mathbf{T}} \det \mathbf{T}$. Show that, for any pair of independent vectors $\mathbf{u}$ and $\mathbf{v}$ the cofactor satisfies, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$

First note that if $\mathbf{T}$ is invertible, the independence of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$. Consequently, we can define the non-vanishing

$$\mathbf{n} \equiv \mathbf{Tu} \times \mathbf{Tv} \neq \mathbf{0}.$$

It follows that $\mathbf{n}$ must be on the perpendicular line to both $\mathbf{Tu}$ and $\mathbf{Tv}$. Therefore,

$$\mathbf{n} \cdot \mathbf{Tu} = \mathbf{n} \cdot \mathbf{Tv} = \mathbf{0}.$$

We can also take a transpose and write,

$$\mathbf{u} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{v} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{0}$$

Showing that the vector $\mathbf{T}^{\mathrm{T}}\mathbf{n}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It follows that $\exists\ \alpha \in \mathfrak{R}$ such that

$$\mathbf{T}^{\mathrm{T}}\mathbf{n} = \alpha(\mathbf{u} \times \mathbf{v})$$

Therefore, $\mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v})$.

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ so that $\mathbf{u},\ \mathbf{v}$ and $\mathbf{w}$ are linearly independent, then we can take a scalar product of the above equation and obtain,

$$\mathbf{w} \cdot \mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$$

The LHS is also $\mathbf{Tw} \cdot (\mathbf{Tu} \times \mathbf{Tv}) = \mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw}$. In the equation, $\mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw} = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$, it is clear that

$$\alpha = \det \mathbf{T}$$

We have that, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v})$. And therefore, we have that,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v}).$$

46. Using direct notation and without going into components, Find the cofactor of a vector cross $\boldsymbol{\omega}\times$

Given independent vectors $\mathbf{u}$ and $\mathbf{v}$, consider the product,

$$\begin{aligned}\big((\boldsymbol{\omega}\times)\,\mathbf{u}\big)\times\big((\boldsymbol{\omega}\times)\,\mathbf{v}\big) &= (\boldsymbol{\omega}\times\ \mathbf{u})\times(\boldsymbol{\omega}\times\ \mathbf{v})\\ &= [(\boldsymbol{\omega}\times\ \mathbf{u})\cdot\mathbf{v}]\boldsymbol{\omega}-[(\boldsymbol{\omega}\times\ \mathbf{u})\cdot\boldsymbol{\omega}]\mathbf{v}\\ &= [\boldsymbol{\omega}\cdot\ (\mathbf{u}\times\mathbf{v})]\boldsymbol{\omega}\\ &= (\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\mathbf{u}\times\ \mathbf{v})\end{aligned}$$

Showing that the cofactor of $\boldsymbol{\omega}\times$ is the dyad $\boldsymbol{\omega}\otimes\boldsymbol{\omega}$.

47. Show that $(\alpha\mathbf{S})^{\mathrm{c}}=\alpha^2\mathbf{S}^{\mathrm{c}}$

$$\begin{aligned}(\alpha\mathbf{S})^{\mathrm{c}} &= (\det(\alpha\mathbf{S}))(\alpha\mathbf{S})^{-\mathrm{T}}\\ &= (\alpha^3\det(\mathbf{S}))\alpha^{-1}\mathbf{S}^{-\mathrm{T}}\\ &= (\alpha^2\det(\mathbf{S}))\mathbf{S}^{-\mathrm{T}}=\alpha^2\mathbf{S}^{\mathrm{c}}\end{aligned}$$

48. Show that $(\mathbf{S}^{-1})^{\mathrm{c}}=(\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$

$$\begin{aligned}(\mathbf{S}^{-1})^{\mathrm{c}} &= \det(\mathbf{S}^{-1})\,(\mathbf{S}^{-1})^{-\mathrm{T}}\\ &= (\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\end{aligned}$$

49. For an invertible tensor $\mathbf{S}$, Show that $\text{cof}(\text{cof}\,\mathbf{S}) = (\det \boldsymbol{S})\boldsymbol{S}$

$$\mathbf{S}^{\text{c}} = \det(\mathbf{S})\,\mathbf{S}^{-\text{T}}$$

So that,

$$\begin{aligned}\mathbf{S}^{\text{cc}} \equiv \text{cof}(\text{cof}\,\mathbf{S}) &= (\det \mathbf{S}^{\text{c}})(\mathbf{S}^{\text{c}})^{-\text{T}} \\ &= (\det \mathbf{S})^2[(\mathbf{S}^{\text{c}})^{-1}]^{\text{T}} \\ &= (\det \mathbf{S})^2[(\det \mathbf{S})^{-1}\mathbf{S}^{\text{T}}]^{\text{T}} \\ &= (\det \mathbf{S})^2(\det \mathbf{S})^{-1}\mathbf{S} \\ &= (\det \mathbf{S})\mathbf{S}\end{aligned}$$

as required.

50. Using direct notation and without going into components, show that the determinant of a vector cross is zero.

Given basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$, the third invariant of $\boldsymbol{\omega}\,\times$,

$$\begin{aligned} I_3(\boldsymbol{\omega}\,\times) &= \det(\boldsymbol{\omega}\,\times) \\ &= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, \boldsymbol{\omega}\times\mathbf{g}_2, \boldsymbol{\omega}\times\mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}\end{aligned}$$

$$= \frac{[\boldsymbol{\omega} \times \mathbf{g}_1, (\boldsymbol{\omega} \times)^c (\mathbf{g}_2 \times \mathbf{g}_3)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{[\boldsymbol{\omega} \times \mathbf{g}_1, (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{g}_2 \times \mathbf{g}_3)]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

upon noting that the cofactor, $(\boldsymbol{\omega} \times)^c = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.

And since $(\boldsymbol{\omega} \otimes \boldsymbol{\omega})$ is symmetric, the numerator above is,

$$(\boldsymbol{\omega} \times \mathbf{g}_1) \cdot (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{g}_2 \times \mathbf{g}_3) = (\mathbf{g}_2 \times \mathbf{g}_3) \cdot (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{g}_1)$$
$$= (\mathbf{g}_2 \times \mathbf{g}_3) \cdot [\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{g}_1)]\boldsymbol{\omega} = 0$$

so that $I_3(\boldsymbol{\omega} \times) = \det(\boldsymbol{\omega} \times) = 0$.

51. Show that the trace of the cofactor, $\mathrm{tr}(\boldsymbol{\omega} \times)^c = |\boldsymbol{\omega}|^2$

First note that $(\boldsymbol{\omega} \times)^c = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$. Therefore,

$$\mathrm{tr}(\boldsymbol{\omega} \times)^c = \mathrm{tr}(\boldsymbol{\omega} \otimes \boldsymbol{\omega})$$
$$= \boldsymbol{\omega} \cdot \boldsymbol{\omega} = |\boldsymbol{\omega}|^2$$

52. Show that for two invertible tensors **T** and **S**, $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$

The inverse of the product **TS** contracted with **TS** yields the unit vector

$$(\mathbf{TS})^{-1}\mathbf{TS} = \mathbf{1}$$

Observe that $\mathbf{S}^{-1}\mathbf{T}^{-1}\mathbf{TS} = \mathbf{S}^{-1}\mathbf{1S} = \mathbf{1}$.

It follows immediately that $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\,\mathbf{T}^{-1}$.

53. Use the direct notation to show that the cofactor of the product of two tensors is the product of the cofactors, that is, for vectors **S** and **T**, $(\mathbf{ST})^{c} = \mathbf{S}^{c}\mathbf{T}^{c}$

Consider vectors **u** and **v**.

$$\begin{aligned}\mathbf{T}^{c}(\mathbf{u}\times\mathbf{v}) &= \mathbf{Tu}\times\mathbf{Tv}\\ \mathbf{S}^{c}[\mathbf{T}^{c}(\mathbf{u}\times\mathbf{v})] &= \mathbf{S}^{c}\,[\mathbf{Tu}\times\mathbf{Tv}]\\ &= \mathbf{STu}\times\mathbf{STv} = (\mathbf{ST})^{c}(\mathbf{u}\times\mathbf{v})\end{aligned}$$

showing that $(\mathbf{ST})^{c} = \mathbf{S}^{c}\mathbf{T}^{c}$

54. Given that **I** is the identity tensor, use the result $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{c}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{c}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{I}) = \det\mathbf{S} + \det\mathbf{S}\,\mathrm{tr}(\mathbf{S}^{-1}) + \mathrm{tr}(\mathbf{S}) + 1$.

$\mathbf{T}\to\mathbf{I}$ in the given identity and noting that the unit tensor is self cofactor $\Rightarrow$

$$\begin{aligned}\det(\mathbf{S}+\mathbf{I}) &= \det\mathbf{S} + \mathrm{tr}(\mathbf{1}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{c}\mathbf{I}) + \det\mathbf{I}\\ &= \det\mathbf{S} + \mathrm{tr}(\mathbf{S}^{c}) + \mathrm{tr}(\mathbf{S}) + 1\\ &= \det\mathbf{S} + \det\mathbf{S}\,\mathrm{tr}(\mathbf{S}^{-1}) + \mathrm{tr}\,\mathbf{S} + 1\end{aligned}$$

55. For an arbitrary tensor $\mathbf{u}$, the vector cross is given as, $\mathbf{u}\,\times$. Use the result $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{u}\,\times) = \det\mathbf{S}\,(1 + (\mathbf{u}\,\times)\mathbf{S}^{-1}) + (\mathbf{u}\otimes\mathbf{u}):\mathbf{S}$.

$\mathbf{T}\to\mathbf{u}\,\times$, then $\mathbf{T}^{\mathrm{c}} = \mathbf{u}\otimes\mathbf{u}$ and $\mathbf{T}^{\mathrm{T}} = -\mathbf{u}\,\times$ and $\det(\mathbf{u}\,\times) = 0$.

$$\begin{aligned}\det(\mathbf{S}+\mathbf{u}\,\times) &= \det\mathbf{S} + \mathrm{tr}\left((\mathbf{u}\otimes\mathbf{u})\mathbf{S}^{\mathrm{T}}\right) - \mathrm{tr}\left(\mathbf{S}^{\mathrm{c}}(\mathbf{u}\,\times)\right)\\ &= \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{u}):\mathbf{S}^{\mathrm{T}} + (\mathbf{u}\,\times):\mathbf{S}^{\mathrm{c}}\end{aligned}$$

56. Given that $\mathbf{T}^{\mathrm{c}}$ is the cofactor of the tensor $\mathbf{T}$, show that $I_1(\mathbf{T}^{\mathrm{c}}) = I_2(\mathbf{T})$ that is, that the trace of the cofactor is the second principal invariant of the original tensor:

$$\begin{aligned}\mathrm{tr}(\mathbf{T}^{\mathrm{c}}) &= \frac{1}{2}\delta_{ijk}^{lmn}T_m^jT_n^k\mathbf{g}_l\cdot\mathbf{g}^i = I_1(\mathbf{T}^{\mathrm{c}})\\ &= \frac{1}{2}\delta_{ijk}^{lmn}T_m^jT_n^k\delta_l^i = \frac{1}{2}\delta_{ijk}^{imn}T_m^jT_n^k\\ &= \frac{1}{2}\left(\delta_j^m\delta_k^n - \delta_k^m\delta_j^n\right)T_m^jT_n^k\\ &= \frac{1}{2}\left(T_j^jT_k^k - T_k^jT_j^k\right) = I_2(\mathbf{T})\end{aligned}$$

57. Use the fact that the cofactor of any tensor can be written as $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^{\mathrm{T}}$ to show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$ that $(\mathbf{u}\otimes\mathbf{v})^{\mathrm{c}} = \mathbf{0}$.

$$
\begin{aligned}
(\mathbf{u}\otimes\mathbf{v})^2 &= (\mathbf{u}\otimes\mathbf{v})(\mathbf{u}\otimes\mathbf{v}) \\
&= (\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\otimes\mathbf{v}) \\
I_2(\mathbf{u}\otimes\mathbf{v}) &= \frac{1}{2}\left(\mathrm{tr}^2\,\mathbf{u}\otimes\mathbf{v} - \mathrm{tr}(\mathbf{u}\otimes\mathbf{v})^2\right) \\
&= \frac{1}{2}\left((\mathbf{u}\cdot\mathbf{v})^2 - (\mathbf{u}\cdot\mathbf{v})\ \mathrm{tr}(\mathbf{u}\otimes\mathbf{v})\right) = 0 \\
(\mathbf{u}\otimes\mathbf{v})^{\mathrm{c}} &= [(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\otimes\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\otimes\mathbf{v}) + 0]^{\mathrm{T}} \\
&= 0
\end{aligned}
$$

58. For arbitrary tensors $\mathbf{u}$ and $\mathbf{v}$ the dyadic, $\mathbf{u}\otimes\mathbf{v}$. Use the result $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{u}\otimes\mathbf{v}) = \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{v})\!:\mathbf{S}$.

$\mathbf{T}\to\mathbf{u}\otimes\mathbf{v}$, then $\mathbf{T}^{\mathrm{c}} = \mathbf{0}$ and $\mathbf{T}^{\mathrm{T}} = \mathbf{v}\otimes\mathbf{u}$ and $\det(\mathbf{u}\otimes\mathbf{v}) = 0$.

$$
\begin{aligned}
\det(\mathbf{S}+\mathbf{u}\otimes\mathbf{v}) &= \det\mathbf{S} + \mathbf{0} + \mathrm{tr}\left((\mathbf{v}\otimes\mathbf{u})\mathbf{S}^{\mathrm{c}}\right) + 0 \\
&= \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{v})\!:\mathbf{S}^{\mathrm{c}}
\end{aligned}
$$

59. For the tensors $\mathbf{A}$ and $\mathbf{B}$, use direct methods to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$

Select linearly independent tensors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent. Now,

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

60. Given that the cofactor $\mathbf{A}^{\mathrm{c}} \equiv \operatorname{cof} \mathbf{A} = \mathbf{A}^{-\mathrm{T}} \det \mathbf{A}$ satisfies $\mathbf{Aa} \times \mathbf{Ab} = \mathbf{A}^{\mathrm{c}}(\mathbf{a} \times \mathbf{b})$. Show by direct methods that transposing does not alter the determinant of a tensor.

$$\det \mathbf{A} = \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{\mathbf{Aa} \cdot \mathbf{Ab} \times \mathbf{Ac}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{\mathbf{Aa} \cdot \mathbf{A}^{\mathrm{c}}(\mathbf{b} \times \mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{\mathrm{cT}} \mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{-1} \det \mathbf{A}^{\mathrm{T}}\, \mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \det \mathbf{A}^{\mathrm{T}} \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{-1} \mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A}^{\mathrm{T}}$$

upon noting that $\mathbf{A}^{\mathrm{T}} \mathbf{Aa} = \mathbf{Ia} = \mathbf{a}$.

61. For a scalar α show that $\det \alpha\mathbf{A} = \alpha^3 \det \mathbf{A}$

Given that $\det \mathbf{A} = \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, then

$$\det \alpha\mathbf{A} = \frac{[\alpha\mathbf{Aa}, \alpha\mathbf{Ab}, \alpha\mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \alpha^3 \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \alpha^3 \det \mathbf{A}$$

62. Define the inner product of tensors $\mathbf{T}$ and $\mathbf{S}$ as $\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^T\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^T)$ show that $I_1(\mathbf{T}) = \mathbf{T}:\mathbf{I}$

$$\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^T\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^T)$$

Let $\mathbf{S} = \mathbf{I}$;

$$\mathbf{T}:\mathbf{I} = \mathrm{tr}(\mathbf{T}^T\mathbf{I}) = \mathrm{tr}(\mathbf{T}\mathbf{I})$$
$$= \mathrm{tr}(\mathbf{T}) = I_1(\mathbf{T})$$

63. Given arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the tensor $\mathbf{Q}$ is said to be orthogonal if $(\mathbf{Q}\boldsymbol{a}) \cdot (\mathbf{Q}\boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$ show that the inverse of $\mathbf{Q}$ is its transpose. and that $\mathbf{Q}$ is the cofactor of itself.

By definition of the transpose, we have that,

$$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$

Clearly, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = 1$ which makes the transpose the same as the inverse tensor.

A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse is equal to its transpose.

64. An orthogonal tensor $\mathbf{Q}$ is said to be "proper orthogonal" if its determinant $|\mathbf{Q}| = +1$. Show that a proper orthogonal tensor is the cofactor of itself. Show also that its first two invariants are equal.

$$\mathrm{cof}\,\mathbf{Q} = \det\mathbf{Q}\ \mathbf{Q}^{-\mathrm{T}} = +1\,(\mathbf{Q}^{\mathrm{T}})^{-1} = 1\,(\mathbf{Q}^{-1})^{-1} = \mathbf{Q}$$

$$I_2(\mathbf{Q}) = I_1(\mathbf{Q}^{\mathrm{c}})$$

The second principal invariant for any vector is equal to the first principal invariant of its co-factor. But we find here that $\mathbf{Q} = \mathbf{Q}^{\mathrm{c}}$. It follows that the first two invariants of a proper orthogonal tensor are equal. The third invariant, $I_3(\mathbf{Q}) = \det(\mathbf{Q})$=1. All essential information on an orthogonal tensor is known once we know its trace!

65. For any invertible tensor $\mathbf{S}$ and a scalar α show that show that the cofactor of the product of α and $\mathbf{S}$ equals $\alpha^2 \times$ the cofactor of $\mathbf{S}$, that is, $(\alpha\mathbf{S})^{\mathrm{c}} = \alpha^2\mathbf{S}^{\mathrm{c}}$

$$\begin{aligned}(\alpha\mathbf{S})^{\mathrm{c}} &= (\det(\alpha\mathbf{S}))(\alpha\mathbf{S})^{-\mathrm{T}} \\ &= (\alpha^3\det(\mathbf{S}))\alpha^{-1}\boldsymbol{S}^{-\mathrm{T}} \\ &= (\alpha^2\det(\mathbf{S}))\mathbf{S}^{-\mathrm{T}} \\ &= \alpha^2\mathbf{S}^{\mathrm{c}}\end{aligned}$$

66. For an invertible tensor show that the cofactor of the cofactor is the product of the original tensor and its determinant $\mathbf{S}^{cc} = (\det \mathbf{S})\mathbf{S}$

$$\mathbf{S}^{c} = \det(\mathbf{S})\,\mathbf{S}^{-T}$$

So that,

$$\begin{aligned}\mathbf{S}^{cc} &= (\det \mathbf{S}^{c})(\mathbf{S}^{c})^{-T} \\ &= (\det \mathbf{S})^{2}[(\mathbf{S}^{c})^{-1}]^{T} = (\det \mathbf{S})^{2}[(\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}]^{T} \\ &= (\det \mathbf{S})^{2}(\det \mathbf{S})^{-1}\mathbf{S} \\ &= (\det \mathbf{S})\mathbf{S}\end{aligned}$$

as required.

67. Given that $\boldsymbol{\Omega}$ is a skew tensor with the corresponding axial vector $\boldsymbol{\omega}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\boldsymbol{\Omega}\mathbf{u} \times \boldsymbol{\Omega}\mathbf{v} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$ and, hence conclude that $\boldsymbol{\Omega}^{c} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.

$$\begin{aligned}\boldsymbol{\Omega}\mathbf{u} \times \boldsymbol{\Omega}\mathbf{v} &= (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) = (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) \\ &= [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\omega}]\mathbf{v} = [\boldsymbol{\omega} \cdot (\mathbf{u} \times \mathbf{v})]\boldsymbol{\omega} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

But by definition, the cofactor must satisfy,

$$\mathbf{\Omega u} \times \mathbf{\Omega v} = \mathbf{\Omega}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

which compared with the previous equation yields the desired result that

$$\mathbf{\Omega}^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}).$$

68. Show, using direct notation, that the cofactor of a tensor can be written as $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^{\mathrm{T}}$ even if $\mathbf{S}$ is not invertible. I_1, I_2 are the first two invariants of $\mathbf{S}$.

For any three linearly independent vectors, the trace of a tensor $\mathbf{T}$

$$\mathrm{tr}\,\mathbf{T} \equiv I_1(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

Replacing $\mathbf{g}_1$ by $\mathbf{Tg}_1$ in the above equation, we have,

$$\mathrm{tr}\,\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] = [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]$$

Or, upon rearrangement,

$$[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3] = \mathrm{tr}\,\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

But, the second Invariant,

$$I_2(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{\mathrm{tr}\,\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{\operatorname{tr}\mathbf{T}\,[\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + \mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3)}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{[(\operatorname{tr}\mathbf{T})\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

so that,

$$[(I_2(\mathbf{T})\mathbf{1})\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = [(\operatorname{tr}\mathbf{T})\mathbf{T}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

From which we can write that

$$I_2(\mathbf{T})\mathbf{1} = (\operatorname{tr}\mathbf{T})\mathbf{T} - \mathbf{T}^2 + \mathbf{T}^{\mathrm{cT}}$$

or,

$$\mathbf{T}^{\mathrm{c}} = (\mathbf{T}^2 - I_1(\mathbf{T})\mathbf{T} + I_2(\mathbf{T})\mathbf{1})^{\mathrm{T}}$$

69. By appealing to the Cayley-Hamilton theorem, show that, that the inverse of an invertible tensor $\mathbf{S}$ can be written as $\mathbf{S}^{-1} = \frac{1}{\det\mathbf{S}}(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})$

The characteristic equation for $\mathbf{S}$ can be written as,

$$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3\mathbf{I} = 0$$

Multiplying by the inverse, $\mathbf{S}^{-1}$, we have,

$$\mathbf{S}^2 - I_1\mathbf{S} + I_2 - I_3\mathbf{S}^{-1} = 0$$

from which the result,

$$\mathbf{S}^{-1} = \frac{1}{\det \mathbf{S}}(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})$$

immediately follows.

70. By direct notation and the relationship, $\mathbf{T}^c = (\mathbf{T}^2 - I_1(\mathbf{T})\mathbf{T} + I_2(\mathbf{T})\mathbf{1})^T$ show that the second invariant of a tensor is half the difference between of the square of its trace and the trace of its square.

Take the trace of the given equation,

$$\operatorname{tr} \mathbf{T}^c = \operatorname{tr} \mathbf{T}^2 - I_1(\mathbf{T})I_1(\mathbf{T}) + 3I_2(\mathbf{T})$$

But recall that $\operatorname{tr} \mathbf{T}^c = I_2(\mathbf{T})$. It therefore follows that,

$$\begin{aligned} 2I_2(\mathbf{T}) &= I_1^2(\mathbf{T}) - \operatorname{tr} \mathbf{T}^2 \\ &= \operatorname{tr}^2 \mathbf{T} - \operatorname{tr} \mathbf{T}^2 \end{aligned}$$

So that,

$$I_2(\mathbf{T}) = \frac{1}{2}(\operatorname{tr}^2 \mathbf{T} - \operatorname{tr} \mathbf{T}^2)$$

71. For an arbitrary vector, show that the cofactor of its vector cross its tensor product with itself. That is $(\mathbf{u}\,\times)^c = \mathbf{u} \otimes \mathbf{u}$

First recall the result that for any tensor $\mathbf{S}$, the cofactor $\mathbf{S}^c = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^T$

$$
\begin{aligned}
(\mathbf{u}\times)^2 &= \left(\epsilon^{i\alpha j}u_\alpha \mathbf{g}_i \otimes \mathbf{g}_j\right)\left(\epsilon_{l\beta m}u^\beta \mathbf{g}^l \otimes \mathbf{g}^m\right)\\
&= \epsilon^{i\alpha j}\epsilon_{l\beta m}u_\alpha u^\beta(\mathbf{g}_i \otimes \mathbf{g}^m)\delta_j^l\\
&= \epsilon^{i\alpha j}\epsilon_{j\beta m}u_\alpha u^\beta(\mathbf{g}_i \otimes \mathbf{g}^m)\\
&= \epsilon^{i\alpha j}\epsilon_{\beta m j}u_\alpha u^\beta(\mathbf{g}_i \otimes \mathbf{g}^m)\\
&= \left(\delta_\beta^i\delta_m^\alpha - \delta_m^i\delta_\beta^\alpha\right)u_\alpha u^\beta(\mathbf{g}_i \otimes \mathbf{g}^m)\\
&= \left(u_m u^i - \delta_m^i u_\alpha u^\alpha\right)\mathbf{g}_i \otimes \mathbf{g}^m\\
&= \ \mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{1}\\
\mathrm{tr}[(\mathbf{u}\times)^2] &= \ \mathbf{u}\cdot\mathbf{u} - 3\,\mathbf{u}\cdot\mathbf{u} = -\,2\,\mathbf{u}\cdot\mathbf{u}\\
\mathrm{tr}[(\mathbf{u}\times)] &= 0
\end{aligned}
$$

But from previous result,

$$
\begin{aligned}
(\mathbf{u}\times)^{\mathrm{c}} &= \left((\mathbf{u}\times)^2 - (\mathbf{u}\times)\mathrm{tr}(\mathbf{u}\times) + \frac{1}{2}[\mathrm{tr}^2(\mathbf{u}\times) - \mathrm{tr}((\mathbf{u}\times)^2)]\mathbf{1}\right)^{\mathrm{T}}\\
&= \left(\mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{1} - 0 + \frac{1}{2}[0 + 2\,\mathbf{u}\cdot\mathbf{u}]\mathbf{1}\right)^{\mathrm{T}}\\
&= (\mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{1} - 0 + [(\,\mathbf{u}\cdot\mathbf{u})]\mathbf{1})^{\mathrm{T}}\\
&= \mathbf{u}\otimes\mathbf{u}
\end{aligned}
$$

72. Given a Euclidean Vector Space $\mathscr{E}$ a tensor $\mathbf{Q}$ is said to be rotation if in addition to satisfying $(\mathbf{Q}\boldsymbol{a})\cdot(\mathbf{Q}\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b}\ \forall \boldsymbol{a}, \boldsymbol{b} \in \mathscr{E}$ its determinant $(\det \mathbf{Q}) = +1$. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Q}\mathbf{u})\times(\mathbf{Q}\mathbf{v})$ if $\mathbf{Q}$ is a rotation. That is, that the cofactor of $\mathbf{Q}$ is $\mathbf{Q}$ itself

We can write that

$$\mathbf{T}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Q}\mathbf{u})\times(\mathbf{Q}\mathbf{v})$$

where

$$\mathbf{T} = \mathrm{cof}(\mathbf{Q}) = \det(\mathbf{Q})\,\mathbf{Q}^{-\mathrm{T}}$$

Now that $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = 1$, and because it is orthogonal, its inverse is its transpose:

$$\mathbf{Q}^{-\mathrm{T}} = (\mathbf{Q}^{-1})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q}$$

This implies that $\mathbf{T} = \mathbf{Q}$ and consequently,

$$\mathbf{Q}(\mathbf{u}\times\mathbf{v}) = (\mathbf{Q}\mathbf{u})\times(\mathbf{Q}\mathbf{v})$$

73. For a proper orthogonal tensor $\mathbf{Q}$, show that the eigenvalue equation always yields an eigenvalue of $+1$. This means that there is always a solution for the equation, $\mathbf{Qu} = \mathbf{u}$.

For any invertible tensor, note that the cofactor is defined as,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

For a proper orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = 1$, and its inverse is its transpose. It therefore follows that,

$$\mathbf{Q}^{\mathbf{c}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

It is easily shown that $\mathrm{tr}\, \mathbf{Q}^{\mathrm{c}} = I_2(\mathbf{Q})$. It follows from the above that, $I_2(\mathbf{Q}) = I_{\mathbf{1}}(\mathbf{Q})$. For the general eigenvalue equation, $\mathbf{Qu} = \lambda\mathbf{u}$, the characteristic equation is,

$$\mathbf{d}\mathrm{et}\,(\mathbf{Q} - \boldsymbol{\lambda}\mathbf{1}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

where $Q_1, Q_2 (= Q_1)$ and $Q_3 (= 1)$ are the first second and third invariants of the orthogonal tensor respectively. In this particular case, the characteristic equation becomes,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

Which is obviously satisfied by $\lambda = 1$. Hence there is always a solution to $\mathbf{Qu} = \mathbf{u}$.

74. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that $\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\times)$

We first observe that,

$$\mathbf{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)[\mathbf{e} \times (\mathbf{e} \otimes \mathbf{e})]$$

The last term vanishes immediately on account of the fact that $\mathbf{e} \otimes \mathbf{e}$ is a symmetric tensor. (The contraction of a symmetric and an antisymmetric tensor always vanishes). Consequently, we have,

$$\mathbf{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} = \mathbf{e} \otimes \mathbf{e}$$

which again means that $\mathbf{Q}(\theta)$ has no effect on $\mathbf{e} \otimes \mathbf{e}$ so that,

$$\begin{aligned}\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) &= \cos(\theta)\mathbf{1} + \big(1 - \cos(\theta)\big)\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times) - \mathbf{e} \otimes \mathbf{e} \\ &= \cos(\theta)\,(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\times)\end{aligned}$$

as required.

75. For an arbitrary unit vector $\mathbf{e}$, show that the skew tensor, $\mathbf{W} = (\mathbf{e}\times)$ is such that $\mathbf{W}^2 \equiv (\mathbf{e}\times)(\mathbf{e}\times) = (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}$

$$\begin{aligned}(\mathbf{e}\times)(\mathbf{e}\times) &= \big(\epsilon^{ijk} e_j \mathbf{g}_i \otimes \mathbf{g}_k\big)\big(\epsilon_{\alpha\beta\gamma} e^{\beta} \mathbf{g}^{\alpha} \otimes \mathbf{g}^{\gamma}\big) \\ &= \delta^{ijk}_{\alpha\beta\gamma} e_j e^{\beta} \mathbf{g}_i \otimes \mathbf{g}^{\gamma} \delta^{\alpha}_{k}\end{aligned}$$

$$
\begin{aligned}
&= \delta^{ijk}_{k\beta\gamma} e_j e^\beta \mathbf{g}_i \otimes \mathbf{g}^\gamma \\
&= \left(\delta^i_\beta \delta^j_\gamma - \delta^j_\beta \delta^i_\gamma\right) e_j e^\beta \mathbf{g}_i \otimes \mathbf{g}^\gamma \\
&= e_\gamma e^\beta \mathbf{g}_\beta \otimes \mathbf{g}^\gamma - e_\beta e^\beta \mathbf{g}_i \otimes \mathbf{g}^i \\
&= e_\gamma e^\beta \mathbf{g}_\beta \otimes \mathbf{g}^\gamma - (\mathbf{e} \cdot \mathbf{e}) \mathbf{g}_i \otimes \mathbf{g}^i \\
&= (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}
\end{aligned}
$$

upon noting that the dot product of the unit vector with itself is unity.

76. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times) \equiv \mathbf{W}$ is the vector cross of $\mathbf{e}$. Show that for, $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)$ [Note that $\mathbf{e} \otimes \mathbf{e} = \mathbf{W}^2 + \mathbf{I}$]

Using the noted result,

$$
\begin{aligned}
\mathbf{Q}(\theta) &= \cos\theta\,\mathbf{I} + (1 - \cos\theta)\mathbf{e} \otimes \mathbf{e} + \sin\theta\,(\mathbf{e}\times) \\
&= \cos\theta\,\mathbf{I} + (1 - \cos\theta)(\mathbf{W}^2 + \mathbf{I}) + \mathbf{W}\sin\theta \\
&= \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1 - \cos\theta)
\end{aligned}
$$

77. Use the fact that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ where $\mathbf{W} \equiv (\mathbf{e}\,\times)$ - the vector cross of the unit tensor, rotates every vector about the axis of $\mathbf{e}$ by the angle θ to find the tensor that rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_3\}$.

Clearly, the rotation axis here is the unit vector $\mathbf{e}_3$ and the angle of rotation is $\frac{\pi}{2}$. Consequently, since $\mathbf{e}_3 = \{0,0,1\}$,

$$\mathbf{W} \equiv (\mathbf{e}_3\,\times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \mathbf{W}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{Q}\left(\frac{\pi}{2}\right) &= \mathbf{I} + \mathbf{W}\sin\frac{\pi}{2} + \mathbf{W}^2\left(1-\cos\frac{\pi}{2}\right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This same tensor can be found directly by recognizing that the tensor, $\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ so that the tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

78. Given that $\mathbf{e}_1 = \{1,0,0\}, \mathbf{e}_2 = \{0,1,0\}, \mathbf{e}_3 = \{0,0,1\}, \mathbf{e}_4 = \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \mathbf{e}_5 = \left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \mathbf{e}_6 = \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}$, Find the tensor that transforms from $\{\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_3\}$ to $\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$.

Tensor, $\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. The tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_4 \otimes \mathbf{e}_2 + \mathbf{e}_5 \otimes \mathbf{e}_1 - \mathbf{e}_6 \otimes \mathbf{e}_3$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

79. Write the characteristic equation of the Tensor whose components are $\begin{pmatrix} 6 & 5 & 4 \\ 5 & 6 & 4 \\ 4 & 4 & 3 \end{pmatrix}$. Show that one of its eigenvalues equals 1 and there is a corresponding eigenvector, $\{1 \quad -1 \quad 0\}$

```
M = {{6, 5, 4}, {5, 6, 4}, {4, 4, 3}}
Eigenvalues[M]
Eigenvectors[M]
```

{{6, 5, 4}, {5, 6, 4}, {4, 4, 3}}

$\{7 + 4\sqrt{3}, 1, 7 - 4\sqrt{3}\}$

$\left\{\left\{-\frac{-9-5\sqrt{3}}{2\left(3+2\sqrt{3}\right)}, -\frac{-9-5\sqrt{3}}{2\left(3+2\sqrt{3}\right)}, 1\right\}, \{-1, 1, 0\}, \left\{-\frac{9-5\sqrt{3}}{2\left(-3+2\sqrt{3}\right)}, -\frac{9-5\sqrt{3}}{2\left(-3+2\sqrt{3}\right)}, 1\right\}\right\}$

80. Show that $\mathbf{Q} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & 1/2 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$ is an orthogonal tensor. Is it proper orthogonal?

Compute the tensor $\mathbf{QFQ}^{\mathrm{T}}$ where $\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find the eigenvectors and eigenvalues of the tensor **F.**

```
Q = {{3 / 4, Sqrt[3] / 4, 1 / 2}, {Sqrt[3] / 4, 1 / 4, -Sqrt[3] / 2},
  {-1 / 2, Sqrt[3] / 2, 0}}
A = {{1, 0, 0}, {0, 2, 0}, {0, 0, 3}}
F = Transpose[Q]
```

$$\left\{\left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right\}, \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}\right\}$$

$$\{\{1, 0, 0\}, \{0, 2, 0\}, \{0, 0, 3\}\}$$

$$\left\{\left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \left\{\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right\}\right\}$$

```
Q.A.F
```

$$\left\{\left\{\frac{27}{16}, -\frac{7\sqrt{3}}{16}, \frac{3}{8}\right\}, \left\{-\frac{7\sqrt{3}}{16}, \frac{41}{16}, \frac{\sqrt{3}}{8}\right\}, \left\{\frac{3}{8}, \frac{\sqrt{3}}{8}, \frac{7}{4}\right\}\right\}$$

```
Eigenvalues [Q.A.F]
```

$$\{3, 2, 1\}$$

```
G = Q.A.F
```

$$\left\{\left\{\frac{27}{16}, -\frac{7\sqrt{3}}{16}, \frac{3}{8}\right\}, \left\{-\frac{7\sqrt{3}}{16}, \frac{41}{16}, \frac{\sqrt{3}}{8}\right\}, \left\{\frac{3}{8}, \frac{\sqrt{3}}{8}, \frac{7}{4}\right\}\right\}$$

```
Eigenvectors[G]
```

$$\left\{\left\{-\frac{1}{\sqrt{3}}, 1, 0\right\}, \left\{\frac{1}{2}, \frac{1}{2\sqrt{3}}, 1\right\}, \left\{-\frac{3}{2}, -\frac{\sqrt{3}}{2}, 1\right\}\right\}$$

81. Find the rotation tensor around an axis parallel to the unit vector, $\mathbf{e} = \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$ through an angle $\frac{\pi}{3}$.

The skew tensor $(\mathbf{e}\,\times) = \mathbf{W} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$.

$$(\mathbf{e}\,\times)^2 = \mathbf{W}^2 = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{5}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

$$\mathbf{Q}\left(\frac{\pi}{6}\right) = \mathbf{I} + \mathbf{W}\sin\frac{\pi}{6} + \mathbf{W}^2\left(1 - \cos\frac{\pi}{6}\right)$$

$$= \begin{pmatrix} 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & -\frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \\ \frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) \\ -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) & -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & 1+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0.888354 & -0.430577 & 0.159465 \\ 0.385919 & 0.888354 & 0.248782 \\ -0.248782 & -0.159465 & 0.955341 \end{pmatrix}$$

The inverse of this tensor is its transpose and its determinant is unity. Clearly, it is the rotation tensor we seek.

82. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\,\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the rotation tensor around the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis through an angle θ.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\,\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

Along the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis, $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2}$. Consequently, $\mathbf{w} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$. $\mathbf{W}\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = (\mathbf{w}\times) = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$, $\mathbf{W}^2\left(\frac{\pi}{4}, \frac{\pi}{2}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$

And the rotation tensor for this axis is,

$$\begin{aligned}\mathbf{Q}(\theta) &= \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta) \\ &= \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{2}(1-\cos\theta) & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{2}(1+\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{pmatrix}.\end{aligned}$$

83. Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the general rotation through an angle θ.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

$$\mathbf{Q}(\alpha,\beta,\theta) = \mathbf{I} + \mathbf{W}(\alpha,\beta)\sin\theta + \mathbf{W}^2(\alpha,\beta)(1-\cos\theta) =$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 1:

$$\{(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\beta))+1,$$
$$\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))-\cos(\beta)\sin(\theta),$$
$$\sin(\alpha)\sin(\beta)\sin(\theta)+\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 2:

$$\{\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))+\cos(\beta)\sin(\theta),$$
$$(1-\cos(\theta))\left(-\cos^2(\alpha)\sin^2(\beta)-\cos^2(\beta)\right)+1,$$
$$\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\cos(\alpha)\sin(\beta)\sin(\theta)\}$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 3

$$\{\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\sin(\alpha)\sin(\beta)\sin(\theta),$$
$$\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))+\cos(\alpha)\sin(\beta)\sin(\theta),$$
$$(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\alpha)\sin^2(\beta))+1\}$$

84. Let the spectral form of the tensor $\mathbf{S} = \lambda_1 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal set. For a positive integer n, find the spectral form of $\mathbf{S}^n$ and that of $\mathbf{S}^{-1}$.

$$\begin{aligned}\mathbf{S}^2 &= (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3)(\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &= (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1}) + (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_2 \mathbf{e_2} \otimes \mathbf{e_2}) \\ &\qquad + (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3) + \cdots + (\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3)(\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &= \lambda_1^2 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2^2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3^2 \mathbf{e}_3 \otimes \mathbf{e}_3\end{aligned}$$

repeated multiplication leads to,

$$\mathbf{S}^n = \lambda_1^n \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2^n \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3^n \mathbf{e}_3 \otimes \mathbf{e}_3$$

85. Given that the skew tensor $(\mathbf{e}\,\times) \equiv \mathbf{W}$, and that $\mathbf{Q}(\theta) \equiv \mathbf{I} + \mathbf{W} \sin\theta + \mathbf{W}^2(1 - \cos\theta)$ is the rotation along the axis $\mathbf{e}$ through the angle θ, Find out if the set $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent.

First note that $\mathbf{W}$ is antisymmetric but $\mathbf{W}^2 = (\mathbf{e} \otimes \mathbf{e}) - \mathbf{I}$ is the linear combination of two symmetric tensors, and therefore symmetric. Assume that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ to be linearly dependent. It means we can find α, β and γ not all equal to zero such that

$$\alpha \mathbf{I} + \beta \mathbf{W} + \gamma \mathbf{W}^2 = 0$$

Since α, β and γ are not all equal to zero, we assume in particular that $\beta \neq 0$.

Consequently, we can write,

$$\mathbf{W} = -\frac{\alpha}{\beta}\mathbf{I} - \frac{\gamma}{\beta}\mathbf{W}^2$$

In which we have expressed the anti-symmetric tensor $\mathbf{W}$ as a linear combination of two symmetric tensors! A contradiction! We can conclude that the set $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent.

86. Given that every rotation tensor $\mathbf{Q}$ can be expressed in terms of the skew tensor $\mathbf{W}(\equiv \mathbf{e}\,\times)$ as a function of the rotation angle θ: $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ and that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent set of tensors, show that, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^{\mathrm{T}}\}$ is also a linearly independent set.

Assume that the tensor set, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^{\mathrm{T}}\}$ is linearly dependent. It means we can find α, β and γ not all equal to zero such that

$$\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^{\mathrm{T}} = 0$$

Since $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$, we substitute and obtain,

$$\begin{aligned}
&\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^{\mathrm{T}} = \\
&= \alpha\mathbf{I} + \beta\big(\mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) + \gamma\big(\mathbf{I} - \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) \\
&\qquad = (\alpha + \beta + \gamma)\mathbf{I} + (\beta - \gamma)\,\mathbf{W}\sin\theta + (\beta + \gamma)\mathbf{W}^2(1-\cos\theta) \\
&\qquad = a\mathbf{I} + b\mathbf{W} + c\mathbf{W}^2 = 0
\end{aligned}$$

if we write $(\alpha + \beta + \gamma) = a, (\beta - \gamma)\sin\theta = b$ and $(\beta + \gamma)(1 - \cos\theta) = c$ thereby contradicting the well-known fact that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is a linearly independent set.

87. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Given that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$, and that, for any scalar α, $\mathbf{Q}(\theta)(\alpha\mathbf{e}) = \alpha\mathbf{e}$, What is the physical meaning of $\mathbf{Q}(\theta)$?

$\mathbf{Q}(\theta)$ is a rotation about the vector $\mathbf{e}$ counterclockwise through an angle θ. It therefore does not alter the magnitude or direction of any vector in the direction of $\mathbf{e}$; for any other vector, it has no effect on the magnitude but affects direction.

88. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Show that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$. What is the physical meaning of $\mathbf{Q}(\theta)$?

Let the scalar $x \equiv \mathbf{e} \cdot \mathbf{u}$ be the projection of $\mathbf{u}$ onto the unit vector $\mathbf{e}$. The square of the magnitude of $\mathbf{v}$ is $|\mathbf{v}|^2$

$$= \mathbf{v} \cdot \mathbf{v} = \left(\cos\theta(\mathbf{1u}) + (1 - \cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\right) \cdot \left(\cos\theta(\mathbf{1u}) + (1 - \cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\right) \cdot$$

$$
\begin{aligned}
&= \left(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\right)^2 \\
&= (\mathbf{u}\cos\theta)\cdot\left(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\right) \\
&+x\mathbf{e}\cdot\left(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\right)(1-\cos\theta) \\
&+(\mathbf{e}\times\mathbf{u})\cdot\left(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e}\times\mathbf{u})\right)\sin\theta \\
&= \mathbf{u}^2\cos^2\theta + 2(\cos\theta - \cos^2\theta)x^2 + 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2x^2 \\
&\quad + 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{e}\times\mathbf{u})^2 \\
&= \mathbf{u}^2\cos^2\theta + 2(\cos\theta - \cos^2\theta)x^2 + 2(\mathbf{e}\times\mathbf{u}\cdot\mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2x^2 \\
&\quad + 2x(\mathbf{e}\times\mathbf{u}\cdot\mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{u}^2 - x^2) \\
&= \mathbf{u}^2(\cos^2\theta + \sin^2\theta) + x^2[2(\cos\theta - \cos^2\theta) + (1-\cos\theta)^2 - \sin^2\theta] \\
&= \mathbf{u}^2
\end{aligned}
$$

As the term in square brackets vanish when expanded.

89. If for an arbitrary unit vector **e**, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1-\cos(\theta))\mathbf{e}\otimes\mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of **e.** Show that for arbitrary $0 < \alpha, \beta \le 2\pi$, that $\mathbf{Q}(\alpha+\beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$.

It is convenient to write $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ in terms of their i, j components; we assume that the unit vector $\mathbf{e} = (x_1, x_2, x_3)$:

$$[\mathbf{Q}(\alpha)]_{ij} = \cos\alpha\,\delta_{ij} + (1-\cos\alpha)x_i x_j - \sin\alpha\,\epsilon_{ijk}x_k$$

Consequently, we can write for the product $\mathbf{Q}(\alpha)\mathbf{Q}(\beta)$,

$$
\begin{aligned}
&[\mathbf{Q}(\alpha)\mathbf{Q}(\beta)]_{ij} = [\mathbf{Q}(\alpha)]_{ik}[\mathbf{Q}(\beta)]_{kj} = \\
&= [\cos\alpha\,\delta_{ik} + (1-\cos\alpha)x_i x_k - \sin\alpha\,\epsilon_{ikl}x_l]\big[\cos\beta\,\delta_{kj} + (1-\cos\beta)x_k x_j \\
&\qquad - \sin\beta\,\epsilon_{kjn}x_n\big] \\
&= \cos\alpha\cos\beta\,\delta_{ik}\delta_{kj} + \cos\alpha\,(1-\cos\beta)\delta_{ik}x_k x_j - \cos\alpha\sin\beta\,\delta_{ik}\epsilon_{kjn}x_n \\
&\qquad + (1-\cos\alpha)\cos\beta\,x_i x_k\delta_{kj} + (1-\cos\alpha)(1-\cos\beta)x_i x_k^2 x_j \\
&\qquad - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ikl}x_l\delta_{kj} \\
&\qquad - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l \\
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j \\
&\qquad + (1-\cos\alpha)(1-\cos\beta)x_i x_j - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} \\
&\qquad - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l \\
&= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j \\
&\qquad + (1-\cos\alpha)(1-\cos\beta)x_i x_j - \boxed{(1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn}} \\
&\qquad - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \boxed{\sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j} \\
&\qquad + \sin\alpha\sin\beta\,\big(\delta_{lj}\delta_{in} - \delta_{ln}\delta_{ji}\big)x_n x_l \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\delta_{ij} + [1 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]x_i x_j \\
&\qquad - [(\cos\alpha\sin\beta - \sin\alpha\cos\beta)]\epsilon_{ijn}x_n \\
&= [\mathbf{Q}(\alpha+\beta)]_{ij}
\end{aligned}
$$

With the boxed terms vanishing on account of antisymmetric contraction with symmetry.

90. If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\,\times)$ where $(\mathbf{e}\,\times)$ is the vector cross of $\mathbf{e}$. Show that . $\mathbf{Q}(\theta)$ is a periodic tensor function with period 2π. [Hint: $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$

Since $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$ we can write that $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi)$. But a direct substitution shows that, $\mathbf{Q}(0) = \mathbf{Q}(2\pi) = \mathbf{1}$. We therefore have that,

$\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi) = \mathbf{Q}(\alpha)$

which completes the proof. The above results show that $\mathbf{Q}(\alpha)$ is a rotation along the unit vector $\mathbf{e}$ through an angle α.

91. Given that $\mathbf{Q}$ is an orthogonal tensor, show that the principal invariants of a tensor S satisfy $I_k(\mathbf{QSQ}^{\mathrm{T}}) = I_k(\mathbf{S}), k = 1,2,$ or 3, that is, Rotations and orthogonal transformations do not change the Invariants.

$$\begin{aligned} I_1(\mathbf{QSQ}^{\mathrm{T}}) &= \mathrm{tr}(\mathbf{QSQ}^{\mathrm{T}}) \\ &= \mathrm{tr}(\mathbf{Q}^{\mathrm{T}}\mathbf{QS}) = \mathrm{tr}(\mathbf{S}) \\ &= I_1(\mathbf{S}) \end{aligned}$$

$$
\begin{aligned}
I_2(\mathbf{QSQ}^{\mathrm{T}}) &= \frac{1}{2}[\mathrm{tr}^2(\mathbf{QSQ}^{\mathrm{T}}) - \mathrm{tr}(\mathbf{QSQ}^{\mathrm{T}}\mathbf{QSQ}^{\mathrm{T}})] \\
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{QS^2Q}^{\mathrm{T}})] \\
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{Q}^{\mathrm{T}}\mathbf{QS^2})] \\
&= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S^2})] = I_2(\mathbf{S}) \\
I_3(\mathbf{QSQ}^{\mathrm{T}}) &= \det(\mathbf{QSQ}^{\mathrm{T}}) \\
&= \det(\mathbf{Q}^{\mathrm{T}}\mathbf{QS}) = \det(\mathbf{S}) \\
&= I_3(\mathbf{S})
\end{aligned}
$$

Hence $I_k(\mathbf{QSQ}^{\mathrm{T}}) = I_k(\boldsymbol{S}),\ \ k = 1{,}2, \text{ or } 3$

92. Define $\mathbf{Lin}^+$ as the set of all tensors with a positive determinant. Show that $\mathbf{Lin}^+$ is invariant under G, the proper orthogonal group of all rotations, in the sense that for any tensor $\mathbf{A} \in \mathrm{Lin}^+\ \mathbf{Q} \in \mathbf{G} \Rightarrow \mathbf{QAQ}^{\mathbf{T}} \in \mathrm{Lin}^+$

Since we are given that $\mathbf{A} \in \mathrm{Lin}^+$, the determinant of $\boldsymbol{A}$ is positive. Consider $\det(\mathbf{QAQ}^{\mathrm{T}})$**.** We observe the fact that the determinant of a product of tensors is the product of their determinants (proved above). We see clearly that, $\det(\mathbf{QAQ}^{\mathrm{T}}) = \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det(\mathbf{Q}^{\mathrm{T}})$**.** Since $\mathbf{Q}$ is a rotation**,** $\det(\mathbf{Q}) = \det(\mathbf{Q}^{\mathrm{T}}) =$

1. Consequently, we see that,

$\det(\mathbf{QAQ}^{\mathrm{T}}) = \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det(\mathbf{Q}^{\mathrm{T}}) = \det(\mathbf{QAQ}^{\mathrm{T}}) = 1 \times \det(\mathbf{A}) \times 1 = \det(\mathbf{A})$

Hence the determinant of $\mathbf{QAQ}^{\mathrm{T}}$ is also positive and therefore $\mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Lin}^{+}$.

93. Define $\mathbf{Sym}$ as the set of all symmetric tensors. Show that $\mathbf{Sym}$ is invariant under G where is the proper orthogonal group of all rotations, in the sense that for any tensor $\mathbf{A} \in \mathbf{Sym}$ every $\mathbf{Q} \in G \Rightarrow \mathbf{QAQ}^{\mathrm{T}} \in \mathbf{Sym}$.

Since we are given that $\mathrm{A} \in \mathrm{Sym}$, we inspect the tensor $\mathbf{QAQ}^{\mathrm{T}}$. Its transpose is, $(\mathbf{QAQ}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}}\mathbf{AQ}^{\mathrm{T}} = \mathbf{QAQ}^{\mathrm{T}}$. So that $\mathbf{QAQ}^{\mathrm{T}}$ is symmetric and therefore $\mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Sym}$. so that the transformation is invariant.

94. Define Psym as the set of all symmetric, positive definite tensors. Show that Psym is invariant under G where G is the proper orthogonal group of all rotations, in the sense that for any tensor $\mathrm{A} \in \mathrm{Psym}$ $\mathbf{Q} \in \mathrm{G} \Rightarrow \mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Psym}$. (G285)

Since we are given that $\mathbf{A} \in \mathrm{Psym}$, it means its characteristic equation has roots that are all positive. This equation can be written as

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

The eigenvalues are the roots of the above equation. We now try to find the characteristic equation of the tensor $\mathbf{QAQ}^{\mathrm{T}}$. Following the above equation, if α is an eigenvalue of $\mathbf{QAQ}^\mathrm{T}$, then,

$$\begin{aligned}\left|\mathbf{QAQ}^\mathrm{T} - \alpha\mathbf{I}\right| &= \left|\mathbf{QAQ}^\mathrm{T} - \alpha\mathbf{QIQ}^\mathrm{T}\right| \\ &= \left|\mathbf{Q}(\mathbf{A} - \alpha\mathbf{I})\mathbf{Q}^\mathrm{T}\right| \\ &= \det(\mathbf{Q}) \times \det(\mathbf{A} - \alpha\mathbf{I}) \times \det(\mathbf{Q}^\mathrm{T}) \\ &= \det(\mathbf{A} - \alpha\mathbf{I}) = 0.\end{aligned}$$

Clearly, $\mathbf{QAQ}^\mathrm{T}$ has the same characteristic equations as $\mathbf{A}$ and hence they have the same eigenvalues. Since $\mathbf{A} \in \mathrm{Psym}$ we have reached the same conclusion that $\mathbf{QAQ}^\mathrm{T} \in \mathrm{Psym}$.

95. A tensor function $\mathbf{\Phi}(\mathbf{A})$ in the domain A $(ie, \mathbf{A} \in A)$ is said to be invariant under G if for every $\mathbf{Q} \in \mathrm{G}$, $\mathbf{\Phi}(\mathbf{QAQ}^\mathrm{T}) = \mathbf{Q\Phi}(\mathbf{A})\mathbf{Q}^\mathrm{T}$. Show that if $\mathbf{\Phi}_1(\mathbf{A})$ and $\mathbf{\Phi}_2(\mathbf{A})$ are both invariant under G, then the product function $\mathbf{\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})$ is also invariant under G.

$\mathbf{\Phi}_1(\mathbf{A})$ and $\mathbf{\Phi}_2(\mathbf{A})$ are both invariant under G, therefore, $\mathbf{\Phi}_1(\mathbf{QAQ}^\mathrm{T}) = \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{Q}^\mathrm{T}$ and $\mathbf{\Phi}_2(\mathbf{QAQ}^\mathrm{T}) = \mathbf{Q\Phi}_2(\mathbf{A})\mathbf{Q}^\mathrm{T}$. Clearly,

$$\begin{aligned}\mathbf{\Phi}_1(\mathbf{QAQ}^\mathrm{T})\mathbf{\Phi}_2(\mathbf{QAQ}^\mathrm{T}) &= \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{Q}^\mathrm{T}\,\mathbf{Q\Phi}_2(\mathbf{A})\mathbf{Q}^\mathrm{T} \\ &= \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})\mathbf{Q}^\mathrm{T}\end{aligned}$$

Which obviously shows that $\mathbf{\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})$ satisfies the conditions for invariance under G.

96. Suppose that $\mathbf{U}$ and $\mathbf{C}$ are symmetric, positive-definite tensors with $\mathbf{U}^2 = \mathbf{C}$, write the invariants of $\mathbf{C}$ in terms of $\mathbf{U}$

$$I_1(\mathbf{C}) = \operatorname{tr}(\mathbf{U}^2) = I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})$$

By the Cayley-Hamilton theorem,

$$\mathbf{U}^3 - I_1\mathbf{U}^2 + I_2\mathbf{U} - I_3\mathbf{1} = \mathbf{0}$$

which contracted with $\mathbf{U}$ gives,

$$\mathbf{U}^4 - I_1\mathbf{U}^3 + I_2\mathbf{U}^2 - I_3\mathbf{U} = \mathbf{0}$$

so that,

$$\mathbf{U}^4 = I_1\mathbf{U}^3 - I_2\mathbf{U}^2 + I_3\mathbf{U}$$

and

$$\begin{aligned}\operatorname{tr}(\mathbf{U}^4) &= I_1\operatorname{tr}(\mathbf{U}^3) - I_2\operatorname{tr}(\mathbf{U}^2) + I_3\operatorname{tr}(\mathbf{U})\\ &= I_1(\mathbf{U})\left(I_1^3(\mathbf{U}) - 3I_1(\mathbf{U})I_2(\mathbf{U}) + 3I_3(\mathbf{U})\right) - I_2(\mathbf{U})\left(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\right)\\ &\quad + I_1(\mathbf{U})I_3(\mathbf{U})\\ &= I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U}) + 4I_1(\mathbf{U})I_3(\mathbf{U}) + 2I_2^2(U)\end{aligned}$$

But,

$$I_2(\mathbf{C}) = \frac{1}{2}[I_1^2(\mathbf{C}) - \operatorname{tr}(\mathbf{C}^2)] = \frac{1}{2}[I_1^2(\mathbf{U}^2) - \operatorname{tr}(\mathbf{U}^4)] = \frac{1}{2}[\operatorname{tr}^2(\mathbf{U}^2) - \operatorname{tr}(\mathbf{U}^4)]$$

$$= \frac{1}{2}\left[\left(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\right)^2 - \mathrm{tr}(\mathbf{U}^4)\right]$$

$$= \frac{1}{2}\left[\boxed{I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U})} + 4I_2^2(\mathbf{U}) - \left(\boxed{I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U})} + 4I_1(\mathbf{U})I_3(\mathbf{U}) + 2I_2^2(\mathbf{U})\right)\right]$$

The boxed items cancel out so that,

$$I_2(\mathbf{C}) = I_2^2(\mathbf{U}) - 2I_1(\mathbf{U})I_3(\mathbf{U})$$

as required.

$$I_3(\mathbf{C}) = \det(\mathbf{C}) = \det(\mathbf{U}^2) = (\det(\mathbf{U}))^2 = I_3^2(\mathbf{U})$$