Homework 2.1

1. For any tensor $\mathbf{S}$, show that, $(\mathbf{S}\mathbf{g}^{\alpha}) \otimes \mathbf{g}_{\alpha} = \mathbf{S}$
2. Gurtin 2.6.1
3. Show that that if the tensor $\mathbf{T}$ is invertible, for any vector $\mathbf{k}$, $\mathbf{T}\mathbf{k} = \mathbf{o}$ automatically means that $\mathbf{k} = \mathbf{o}$.
4. Show that if the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are independent and $\mathbf{T}$ is invertible, then the vectors $\mathbf{T}\mathbf{u}, \mathbf{T}\mathbf{v}$ and $\mathbf{T}\mathbf{w}$ are also independent.
5. Show that $\mathbf{w} \times (\mathbf{w} \otimes \mathbf{w}) = \mathbf{0}$ and that $(\mathbf{w}\times)(\mathbf{w}\times) = \mathbf{w} \otimes \mathbf{w} - \|\mathbf{w}\|^2\mathbf{1}$
6. Gurtin 2.8.5
7. Gurtin 2.9.1
8. Gurtin 2.9.2
9. Gurtin 2.9.4

Due March 21, 2016

Homework 2.2

11. Gurtin 2.11.1 d&e
12. Gurtin 2.11.3
13. Gurtin 2.11.4
14. Gurtin 2.11.5
15. Let $\mathbf{Q}$ be a rotation. For any pair of independent vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Q}\mathbf{u}) \times (\mathbf{Q}\mathbf{v})$
16. For a proper orthogonal tensor $\mathbf{Q}$, show that the eigenvalue equation always yields an eigenvalue of +1.
17. For an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{1} + (\mathbf{1} - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the skew tensor whose ij component is $\epsilon_{jik}e_k$, show that $\mathbf{Q}(\theta)(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) = \cos(\boldsymbol{\theta})(\mathbf{1} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\times)$.
18. For an arbitrary unit vector $\mathbf{e}$ and the tensor, $\mathbf{Q}(\theta)$ defined as above, Show for an arbitrary vector $\mathbf{u}$ that $\mathbf{v} = \mathbf{Q}(\theta)\mathbf{u}$ has the same magnitude as $\mathbf{u}$.

Due March 28, 2016

Homework 2.3

1. Gurtin 2.13.1
2. If the reciprocal relationship, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$ is satisfied, what relationship is there between the tensor bases (1) $\mathbf{g}_i \otimes \mathbf{g}_j$ and $\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$, and (2) $\mathbf{g}^i \otimes \mathbf{g}_j$ and $\mathbf{g}_\alpha \otimes \mathbf{g}^\beta$?
3. Gurtin 2.14.1
4. Gurtin 2.14.2
5. Gurtin 2.14.3
6. Gurtin 2.14.4
7. Gurtin 2.14.5
8. Gurtin 2.15 1-3a, 3b, 3c
9. Gurtin 2.16 1-8

Due April 4, 2016

Quiz

For a given a tensor $\mathbf{T}$ and its transpose $\mathbf{T}^{\mathrm{T}}$, Write out expressions for the

1. Symmetric Part
2. Skew Part
3. Spherical Part
4. Deviatoric Part.

What is the magnitude of $\mathbf{T}$?

Tensor Algebra

Tensors as Linear Mappings

March 15 to March 22, 2016

No	Topics	From Slide	Date
0	Home Work & Due dates & Quiz	1	
1	Definitions, Special Tensors	7	15 March
2	Scalar Functions or Invariants	17	
3	Inner Product, Euclidean Tensors	26	
4	The Tensor Product	29	
5	Tensor Basis & Component Representation	40	
6	The Vector Cross, Axial Vectors	60	
7	The Cofactor	68	22 March
8	Orthogonal Tensors	88	
9	Eigenvalue Problem, Spectral Decomposition & Cayley Hamilton	100	Weekend

Second Order Tensor

A second Order Tensor $\boldsymbol{T}$ is a linear mapping from a vector space to itself. Given $\boldsymbol{u} \in \mathscr{V}$ the mapping,

$$\boldsymbol{T}:\ \mathscr{V} \to \mathscr{V}$$

states that $\exists\ \boldsymbol{w} \in \mathscr{V}$ such that,

$$\boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{w}.$$

Every other definition of a second order tensor can be derived from this simple definition. The tensor character of an object can be established by observing its action on a vector.

Linearity

* The mapping is linear. This means that if we have two runs of the process, we first input $\boldsymbol{u}$ and later input $\mathbf{v}$. The outcomes $\boldsymbol{T}(\boldsymbol{u})$ and $\boldsymbol{T}(\mathbf{v})$, added would have been the same as if we had added the inputs $\boldsymbol{u}$ and $\mathbf{v}$ first and supplied the sum of the vectors as input. More compactly, this means,

$$\boldsymbol{T}(\boldsymbol{u}+\mathbf{v})=\boldsymbol{T}(\boldsymbol{u})+\boldsymbol{T}(\mathbf{v})$$

Linearity

Linearity further means that, for any scalar α and tensor $\boldsymbol{T}$

$$\boldsymbol{T}(\alpha\boldsymbol{u}) = \alpha\boldsymbol{T}(\boldsymbol{u})$$

The two properties can be added so that, given $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$, then

$$\boldsymbol{T}(\alpha\boldsymbol{u} + \beta\mathbf{v}) = \alpha\boldsymbol{T}(\boldsymbol{u}) + \beta\boldsymbol{T}(\mathbf{v})$$

Since we can think of a tensor as a process that takes an input and produces an output, two tensors are equal only if they produce the same outputs when supplied with the same input. The sum of two tensors is the tensor that will give an output which will be the sum of the outputs of the two tensors when each is given that input.

Vector Space

In general, $\alpha, \beta \in \mathscr{R}$, $\boldsymbol{u}, \mathbf{v} \in \mathscr{V}$ and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\alpha \boldsymbol{S}\boldsymbol{u} + \beta \boldsymbol{T}\boldsymbol{u} = (\alpha \boldsymbol{S} + \beta \boldsymbol{T})\boldsymbol{u}$$

With the definition above, the set of tensors constitute a vector space with its rules of addition and multiplication by a scalar. It will become obvious later that it also constitutes a Euclidean vector space with its own rule of the inner product.

Special Tensors

Notation.

It is customary to write the tensor mapping without the parentheses. Hence, we can write,

$$\boldsymbol{T}\boldsymbol{u} \equiv \boldsymbol{T}(\boldsymbol{u})$$

For the mapping by the tensor $\boldsymbol{T}$ on the vector variable and dispense with the parentheses unless when needed.

Zero Tensor or Annihilator

The annihilator $\mathbf{O}$ is defined as the tensor that maps all vectors to the zero vector, $\mathbf{o}$:

$$\mathbf{O}\mathbf{u} = \mathbf{o}, \qquad \forall \mathbf{u} \in \mathscr{V}$$

The Identity

The identity tensor $\mathbf{I}$ is the tensor that leaves every vector unaltered. $\forall \boldsymbol{u} \in \mathscr{V}$,

$$\mathbf{Iu} = \mathbf{u}$$

Furthermore, $\forall \alpha \in \mathscr{R}$, the tensor, $\alpha\mathbf{I}$ is called a spherical tensor.

The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\boldsymbol{T} \in \mathscr{T}$ and $\boldsymbol{u} \in \mathscr{V}$, the mapping $\boldsymbol{w} \equiv \boldsymbol{Tu}$ produces a vector.

The Inverse

Consider a linear mapping that, operating on $\boldsymbol{w}$, produces our original argument, $\boldsymbol{u}$, if we can find it:

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{u}$$

As a linear mapping, operating on a vector, clearly, $\boldsymbol{Y}$ is a tensor. It is called the inverse of $\boldsymbol{T}$ because,

$$\boldsymbol{Y}\boldsymbol{w} = \boldsymbol{Y}\boldsymbol{T}\boldsymbol{u} = \boldsymbol{u}$$

So that the composition $\boldsymbol{Y}\boldsymbol{T} = \mathbf{I}$, the identity mapping. For this reason, we write,

$$\boldsymbol{Y} = \boldsymbol{T}^{-1}$$

Inverse

It is easy to show that if $\boldsymbol{YT} = \mathbf{I}$, then $\boldsymbol{TY} = \boldsymbol{YT} = \mathbf{I}$.

* ***HW: Show this.***

The set of invertible sets is closed under composition. It is also closed under inversion. It forms a group with the identity tensor as the group's neutral element

Answer

Given that $\mathbf{YT} = \mathbf{I}$ we want to show that $\mathbf{TY} = \mathbf{YT} = \mathbf{I}$.

Consider $\mathbf{TYTu}$ where $\mathbf{u}$ is a vector. Since $\mathbf{YT} = \mathbf{I}$, it follows that $\mathbf{TYTu} = \mathbf{TIu} = \mathbf{Tu} \equiv \mathbf{v}$ where $\mathbf{v}$ is a vector. Clearly,

$$\mathbf{TYTu} = \mathbf{TYv} = \mathbf{v}$$

which immediately shows that $\mathbf{TY} = \mathbf{I}$ as required to be shown.

Transposition of Tensors

Given $\mathbf{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\mathbf{A}^{\mathrm{T}}$ satisfying

$$\mathbf{w} \cdot \left(\mathbf{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\mathbf{A}\mathbf{w})$$

Is called the transpose of $\mathbf{A}$.

A tensor indistinguishable from its transpose is said to be symmetric.

Invariants

There are certain mappings from the space of tensors to the real space. Such mappings are called Invariants of the Tensor. Three of these, called Principal invariants play key roles in the application of tensors to continuum mechanics. We shall define them shortly.

The definition given here is free of any association with a coordinate system. It is a good practice to derive any other definitions from these fundamental ones:

The Trace

If we write

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are arbitrary vectors.

For any second order tensor $\boldsymbol{T}$, and linearly independent $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the linear mapping $I_1 \colon \mathscr{T} \to \mathscr{R}$

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{T}\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{T}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \boldsymbol{T}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Is independent of the choice of the basis vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is called the First Principal Invariant of $\boldsymbol{T}$ or Trace of $\boldsymbol{T} \equiv \mathrm{tr}(\boldsymbol{T}) \equiv I_1(\boldsymbol{T})$

Invariance of the Trace

* $I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{[\boldsymbol{Ta},\boldsymbol{b},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{Tb},\boldsymbol{c}]+[\boldsymbol{a},\boldsymbol{b},\boldsymbol{Tc}]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$

Let us refer each vector to a covariant basis so that, $\boldsymbol{a} = a^i \mathbf{g}_i$, $\boldsymbol{b} = b^j \mathbf{g}_j$, and $\boldsymbol{c} = c^k \mathbf{g}_k$. Hence,

$$I_1(\boldsymbol{T}) \equiv \mathrm{tr}(\boldsymbol{T}) = \frac{\left[\boldsymbol{T}\left(a^i \mathbf{g}_i\right), b^j \mathbf{g}_j, c^k \mathbf{g}_k\right] + \left[a^i \mathbf{g}_i, \boldsymbol{T}\left(b^j \mathbf{g}_j\right), c^k \mathbf{g}_k\right] + \left[a^i \mathbf{g}_i, b^j \mathbf{g}_j, \boldsymbol{T}\left(c^k \mathbf{g}_k\right)\right]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

$$= \frac{a^i b^j c^k \left(\left[\boldsymbol{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right] + \left[\mathbf{g}_i, \boldsymbol{T}\mathbf{g}_j, \mathbf{g}_k\right] + \left[\mathbf{g}_i, \mathbf{g}_j, \boldsymbol{T}\mathbf{g}_k\right]\right)}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

But $\left[\mathbf{T}\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right] + \left[\mathbf{g}_i, \mathbf{T}\mathbf{g}_j, \mathbf{g}_k\right] + \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{T}\mathbf{g}_k\right] = T_\alpha^\alpha \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]$. We have that

$$I_1(\boldsymbol{T}) = \frac{a^i b^j c^k T_\alpha^\alpha \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right]}{\epsilon_{ijk} a^i b^j c^k}$$

$$= \frac{\epsilon_{ijk} a^i b^j c^k}{\epsilon_{ijk} a^i b^j c^k} T_\alpha^\alpha = T_\alpha^\alpha$$

Which, in either case, is obviously independent of the choice of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

The Trace

The trace is a linear mapping. It is easily shown that $\alpha, \beta \in \mathscr{R}$, and $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

$$\operatorname{tr}(\alpha\boldsymbol{S} + \beta\boldsymbol{T}) = \alpha\operatorname{tr}(\boldsymbol{S}) + \beta\operatorname{tr}(\boldsymbol{T})$$

HW. Show this by appealing to the linearity of the vector space.

While the trace of a tensor is linear, the other two principal invariants are nonlinear. We now proceed to define them

Square of the trace

The second principal invariant $I_2(\boldsymbol{S})$ is related to the trace. In fact, you may come across books that define it so. However, the most common definition is that

$$I_2(\boldsymbol{S}) = \frac{1}{2}\left[I_1^2(\boldsymbol{S}) - I_1(\boldsymbol{S}^2)\right]$$

Independently of the trace, we can also define the second principal invariant as,

Second Principal Invariant

The Second Principal Invariant, $I_2(\boldsymbol{T})$, using the same notation as above is

$$\frac{[(\boldsymbol{Ta}),(\boldsymbol{Tb}),\boldsymbol{c}]+[\boldsymbol{a},(\boldsymbol{Tb}),(\boldsymbol{Tc})]+[(\boldsymbol{Ta}),\boldsymbol{b},(\boldsymbol{Tc})]}{[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]}$$

$$=\frac{1}{2}\left[\operatorname{tr}^2(\boldsymbol{T})-\operatorname{tr}\left(\boldsymbol{T}^2\right)\right]$$

that is half the square of trace minus the trace of the square of $\boldsymbol{T}$ which is the second principal invariant.

* This quantity remains unchanged for any arbitrary selection of basis vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.

The Determinant

The third mapping from tensors to the real space underlying the tensor is the determinant of the tensor. While you may be familiar with that operation and can easily extract a determinant from a matrix, it is important to understand the definition for a tensor that is independent of the component expression. The latter remains relevant even when we have not expressed the tensor in terms of its components in a particular coordinate system.

The Determinant

As before, For any second order tensor $\boldsymbol{T}$, and any linearly independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

* The determinant of the tensor $\boldsymbol{T}$,

$$\det(\boldsymbol{T}) = \frac{[(\boldsymbol{Ta}), (\boldsymbol{Tb}), (\boldsymbol{Tc})]}{[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}]}$$

(In the special case when the basis vectors are orthonormal, the denominator is unity)

Other Principal Invariants

* It is good to note that there are other principal invariants that can be defined. The ones we defined here are the ones you are most likely to find in other texts.
* An invariant is a scalar derived from a tensor that remains unchanged in any coordinate system. Mathematically, it is a mapping from the tensor space to the real space. Or simply **a scalar valued function of the tensor**.

Deviatoric Tensors

* When the trace of a tensor is zero, the tensor is said to be traceless. A traceless tensor is also called a deviatoric tensor.
* Given any tensor $\mathbf{S}$, A deviatoric tensor may be created from $\mathbf{S}$ by the following process:

$$\mathbf{S}_0 \equiv \text{dev}\,\mathbf{S} \equiv \mathbf{S} - \frac{1}{3}(\text{tr}\,\mathbf{S})\mathbf{I} = \mathbf{S} - s\mathbf{I}$$

So that $s = \frac{1}{3}(\text{tr}\,\mathbf{S})$; $s\mathbf{I}$ is called the spherical part, and $\mathbf{S}_0$ as defined here is called the deviatoric part of $\mathbf{S}$.

Every tensor thus admits the decomposition,

$$\mathbf{S} = \mathbf{S}_0 + s\mathbf{I}$$

Inner Product of Tensors

The trace provides a simple way to define the inner product of two second-order tensors. Given $\boldsymbol{S}, \boldsymbol{T} \in \mathscr{T}$

The trace,

$$\mathrm{tr}\left(\boldsymbol{S}^T\boldsymbol{T}\right) = \mathrm{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Is a scalar, independent of the coordinate system chosen to represent the tensors. This is defined as the inner or scalar product of the tensors $\boldsymbol{S}$ and $\boldsymbol{T}$. That is,

$$\boldsymbol{S}:\boldsymbol{T} \equiv \mathrm{tr}\left(\boldsymbol{S}^T\boldsymbol{T}\right) = \mathrm{tr}(\boldsymbol{S}\boldsymbol{T}^T)$$

Attributes of a Euclidean Space

The trace automatically induces the concept of the norm of a vector (This is not the determinant! Note!!) The square root of the scalar product of a tensor with itself is the norm, magnitude or length of the tensor:

$$\|\boldsymbol{T}\| = \sqrt{\mathrm{tr}(\boldsymbol{T}^T\boldsymbol{T})} = \sqrt{\boldsymbol{T}:\boldsymbol{T}}$$

Distance and angles

Furthermore, the distance between two tensors as well as the angle they contain are defined. The scalar distance $d(\boldsymbol{S}, \boldsymbol{T})$between tensors $\boldsymbol{S}$ and $\boldsymbol{T}$:

$$d(\boldsymbol{S}, \boldsymbol{T}) = \|\boldsymbol{S} - \boldsymbol{T}\| = \|\boldsymbol{T} - \boldsymbol{S}\|$$

And the angle $\theta(\boldsymbol{S}, \boldsymbol{T})$,

$$\theta = \cos^{-1} \frac{\boldsymbol{S} : \boldsymbol{T}}{\|\boldsymbol{S}\| \|\boldsymbol{T}\|}$$

The Tensor Product

A product mapping from two vector spaces to $\mathscr{T}$ is defined as the tensor product. It has the following properties:

$$"\otimes": \mathscr{V} \times \mathscr{V} \to \mathscr{T}$$

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}$$

It is an ordered pair of vectors. It acts on any other vector by creating a new vector in the direction of its first vector as shown above. This product of two vectors is called a tensor product or a simple dyad.

Dyad Properties

It is very easily shown that the transposition of dyad is simply a reversal of its order. (Shown below).

The tensor product is linear in its two factors.

Based on the obvious fact that for any tensor $\boldsymbol{T}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathscr{V}$, $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{T}\boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = [(\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}]\boldsymbol{w}$

It is clear that

$$\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}$$

Show this neatly by operating either side on a vector

Furthermore, the contraction,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes \left(\boldsymbol{T}^T \boldsymbol{v}\right)$$

A fact that can be established by operating each side on the same vector.

Composition with Tensors

Operate on the vector $\boldsymbol{z}$ and let $\boldsymbol{Tz} = \boldsymbol{w}$. On the LHS,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{Tz} = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

On the RHS, we have:

$$\left(\boldsymbol{u} \otimes \left(\boldsymbol{T}^T \boldsymbol{v}\right)\right)\boldsymbol{z} = \boldsymbol{u}\left(\left(\boldsymbol{T}^T \boldsymbol{v}\right) \cdot \boldsymbol{z}\right) = \boldsymbol{u}\left(\boldsymbol{z} \cdot \left(\boldsymbol{T}^T \boldsymbol{v}\right)\right)$$

Since the contents of both sides of the dot are vectors and dot product of vectors is commutative. Clearly,

$$\boldsymbol{u}\left(\boldsymbol{z} \cdot \left(\boldsymbol{T}^T \boldsymbol{v}\right)\right) = \boldsymbol{u}\left(\boldsymbol{v} \cdot (\boldsymbol{Tz})\right)$$

follows from the definition of transposition. Hence,

$$\left(\boldsymbol{u} \otimes \left(\boldsymbol{T}^T \boldsymbol{v}\right)\right)\boldsymbol{z} = \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}$$

Transpose of a Dyad

Recall that for $\boldsymbol{w}, \mathbf{v} \in \mathscr{V}$, The tensor $\boldsymbol{A}^{\mathrm{T}}$ satisfying

$$\boldsymbol{w} \cdot \left(\boldsymbol{A}^{\mathrm{T}}\mathbf{v}\right) = \mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w})$$

Is called the transpose of $\boldsymbol{A}$. Now let $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{b}$ a dyad.

$$\begin{aligned}
\mathbf{v} \cdot (\boldsymbol{A}\boldsymbol{w}) &= \\
&= \mathbf{v} \cdot [(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{w}] = \mathbf{v} \cdot [\boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{w})] \\
&= (\mathbf{v} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{b})(\mathbf{v} \cdot \boldsymbol{a}) \\
&= \boldsymbol{w} \cdot (\boldsymbol{b} \otimes \boldsymbol{a})\mathbf{v}
\end{aligned}$$

So that $(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} = \boldsymbol{b} \otimes \boldsymbol{a}$

Showing that the transpose of a dyad is simply a reversal of its factors.

If $\mathbf{n}$ is the unit normal to a given plane, show that the tensor $\mathbf{T} \equiv \mathbf{1} - \mathbf{n} \otimes \mathbf{n}$ is such that $\mathbf{Tu}$ is the projection of the vector $\mathbf{u}$ to the plane in question.

Consider the fact that

$$\mathbf{Tu} = \mathbf{1u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} = \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n}$$

The above vector equation shows that $\mathbf{Tu}$ is what remains after we have subtracted the projection $(\mathbf{n} \cdot \mathbf{u})\mathbf{n}$ onto the normal. Obviously, this is the projection to the plane itself. $\mathbf{T}$ as we shall see later is called a tensor projector.

Substitution Operation

Consider a contravariant vector component a^k let us take a product of this with the Kronecker Delta:

$$\delta^i_j a^k$$

which gives us a third-order object. Let us now perform a contraction across (by taking the superscript index from A^k and the subscript from δ^i_j) to arrive at,

* $d^i = \delta^i_j a^j$
* Observe that the only free index remaining is the superscript i as the other indices have been contracted (it is consequently a summation index) out in the implied summation. Let us now expand the RHS above, we find,

Substitution

$$d^i = \delta^i_j a^j = \delta^i_1 a^1 + \delta^i_2 a^2 + \delta^i_3 a^3$$

Note the following cases:

* if $i = 1$, we have $d^1 = a^1$, if $i = 2$, we have $d^2 = a^2$ if $i = 3$, we have $d^3 = a^3$. This leads us to conclude therefore that the contraction, $\delta^i_j a^j = a^i$. Indicating that that the Kronecker Delta, in a contraction, merely substitutes its own other symbol for the symbol on the vector a^j it was contracted with. This fact, that the Kronecker Delta does this in general earned it the alias of "**Substitution Operator**".

Dyad on Dyad Composition

For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x} \in \mathscr{V}$, We can show that the dyad composition,

$$(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{u} \otimes \boldsymbol{x})(\boldsymbol{v} \cdot \boldsymbol{w})$$

Again, the proof is to show that both sides produce the same result when they act on the same vector. Let $\boldsymbol{y} \in \mathscr{V}$, then the LHS on $\boldsymbol{y}$ yields:

$$\begin{aligned}(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{w} \otimes \boldsymbol{x})\boldsymbol{y} &= (\boldsymbol{u} \otimes \boldsymbol{v})[\boldsymbol{w}(\boldsymbol{x} \cdot \boldsymbol{y})] \\ &= \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w})(\boldsymbol{x} \cdot \boldsymbol{y})\end{aligned}$$

Which is obviously the result from the RHS also.

This therefore makes it straightforward to contract dyads by breaking and joining as seen above.

Trace of a Dyad

Show that the trace of the tensor product $\mathbf{u} \otimes \mathbf{v}$ is $\mathbf{u} \cdot \mathbf{v}$.

Given any three independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, (No loss of generality in letting the three independent vectors be the curvilinear basis vectors $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$). Using the above definition of trace, we can write that,

Trace of a Dyad

$$\mathrm{tr}(\mathbf{u}\otimes\mathbf{v}) = \frac{[\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{g}_1\},\mathbf{g}_2,\mathbf{g}_3] + [\mathbf{g}_1,\{(\mathbf{u}\otimes\mathbf{v})\,\mathbf{g}_2\},\mathbf{g}_3] + [\mathbf{g}_1,\mathbf{g}_2,\{(\mathbf{u}\otimes\mathbf{v})\mathbf{g}_3\}]}{[\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3]}$$

$$= \frac{1}{\epsilon_{123}}\{[v_1\mathbf{u},\mathbf{g}_2,\mathbf{g}_3] + [\mathbf{g}_1,v_2\mathbf{u},\mathbf{g}_3] + [\mathbf{g}_1,\mathbf{g}_2,v_3\mathbf{u}]\}$$

$$= \frac{1}{\epsilon_{123}}\left\{(v_1\mathbf{u})\cdot\left(\epsilon_{23i}\mathbf{g}^i\right) + \left(\epsilon_{31i}\mathbf{g}^i\right)\cdot(v_2\mathbf{u}) + \left(\epsilon_{12i}\mathbf{g}^i\right)\cdot(v_3\mathbf{u})\right\}$$

$$= \frac{1}{\epsilon_{123}}\left\{(v_1\mathbf{u})\cdot(\epsilon_{231}\mathbf{g}^1) + \left(\epsilon_{312}\mathbf{g}^2\right)\cdot(v_2\mathbf{u}) + \left(\epsilon_{123}\mathbf{g}^3\right)\cdot(v_3\mathbf{u})\right\} = v_i u^i$$

Other Invariants of a Dyad

* It is easy to show that for a tensor product

$$\boldsymbol{D} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathscr{V}$$

$$\boldsymbol{I}_2(\boldsymbol{D}) = \boldsymbol{I}_3(\boldsymbol{D}) = 0$$

HW. Show that this is so.

We proved earlier that $\boldsymbol{I}_1(\boldsymbol{D}) = \boldsymbol{u} \cdot \boldsymbol{v}$

Furthermore, if $\boldsymbol{T} \in \mathscr{T}$, then,

$$\operatorname{tr}(\boldsymbol{Tu} \otimes \boldsymbol{v}) = \operatorname{tr}(\boldsymbol{w} \otimes \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{v} = \boldsymbol{Tu} \cdot \boldsymbol{v}$$

Tensor Bases & Component Representation

Given $\boldsymbol{T} \in \mathscr{T}$, for any basis vectors $\mathbf{g}_i \in \mathscr{V}, i = 1,2,3$

$$\boldsymbol{T}_j \equiv \boldsymbol{T}\mathbf{g}_j \in \mathscr{V}, j = 1,2,3$$

by the law of tensor mapping. We proceed to find the components of $\boldsymbol{T}_j$ on this same basis. Its covariant components, just like in any other vector are the scalars,

$$(\boldsymbol{T}_\alpha)_j = \mathbf{g}_\alpha \cdot \boldsymbol{T}_j$$

Specifically, these components are $\left((\boldsymbol{T}_1)_j, (\boldsymbol{T}_2)_j, (\boldsymbol{T}_3)_j\right)$

Tensor Components

We can dispense with the parentheses and write that

$$T_{\alpha j} \equiv (T_\alpha)_j = \boldsymbol{T}_j \cdot \mathbf{g}_\alpha$$

So that the vector

$$\boldsymbol{T}\mathbf{g}_j = \boldsymbol{T}_j = T_{\alpha j}\mathbf{g}^\alpha$$

The components T_{ij} can be found by taking the dot product of the above equation with $\mathbf{g}_i$:

$$\mathbf{g}_i \cdot (\boldsymbol{T}\mathbf{g}_j) = T_{\alpha j}(\mathbf{g}_i \cdot \mathbf{g}^\alpha) = T_{ij}$$

$$T_{ij} = \mathbf{g}_i \cdot (\boldsymbol{T}\mathbf{g}_j)$$

$$= \mathrm{tr}(\boldsymbol{T}\mathbf{g}_j \otimes \mathbf{g}_i) = \boldsymbol{T}: (\mathbf{g}_i \otimes \mathbf{g}_j)$$

Tensor Components

The component T_{ij} is simply the result of the inner product of the tensor $\boldsymbol{T}$ on the tensor product $\mathbf{g}_i \otimes \mathbf{g}_j$. These are the components of $\boldsymbol{T}$ on the product dual of this particular product base.

This is a general result and applies to all product bases:

It is straightforward to prove the results on the following table:

Tensor Components

Components of T	Derivation	Full Representation
T_{ij}	$\boldsymbol{T}:(\mathbf{g}_i \otimes \mathbf{g}_j)$	$\boldsymbol{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$
T^{ij}	$\boldsymbol{T}:(\mathbf{g}^i \otimes \mathbf{g}^j)$	$\boldsymbol{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$
$T_i^{.j}$	$\boldsymbol{T}:(\mathbf{g}_i \otimes \mathbf{g}^j)$	$\boldsymbol{T} = T_i^{.j}\mathbf{g}^i \otimes \mathbf{g}_j$
$T_{.i}^{j}$	$\boldsymbol{T}:(\mathbf{g}^j \otimes \mathbf{g}_i)$	$\boldsymbol{T} = T_{.i}^{j}\mathbf{g}_j \otimes \mathbf{g}^i$

* $\mathbf{T}:\left(\mathbf{g}_i \otimes \mathbf{g}_j\right) = \operatorname{tr}\left(\mathbf{T}\left(\mathbf{g}_j \otimes \mathbf{g}_i\right)\right)$
$$= \operatorname{tr}\left(\mathbf{T}\mathbf{g}_j \otimes \mathbf{g}_i\right) = \mathbf{T}\mathbf{g}_j \cdot \mathbf{g}_i$$
$$= \mathbf{g}_i \cdot \mathbf{T}\mathbf{g}_j$$
* $\mathbf{I}:\left(\mathbf{g}_i \otimes \mathbf{g}_j\right) = \operatorname{tr}\left(\mathbf{I}\left(\mathbf{g}_j \otimes \mathbf{g}_i\right)\right)$
$$= \operatorname{tr}\left(\mathbf{I}\mathbf{g}_j \otimes \mathbf{g}_i\right) = \mathbf{I}\mathbf{g}_j \cdot \mathbf{g}_i$$
$$= \mathbf{g}_i \cdot \mathbf{I}\mathbf{g}_j = g_{ij}$$

IdentityTensor Components

It is easily verified from the definition of the identity tensor and the inner product that: (HW Verify this)

Components of 1	Derivation	Full Representation
$(\mathbf{I})_{ij} = g_{ij}$	$\mathbf{I}:(\mathbf{g}_i \otimes \mathbf{g}_j)$	$\mathbf{I} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$
$(\mathbf{I})^{ij} = g^{ij}$	$\mathbf{I}:\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)$	$\mathbf{I} = g^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$
$(\mathbf{I})_i^{\cdot j} = \delta_i^{\cdot j}$	$\mathbf{I}:(\mathbf{g}_i \otimes \mathbf{g}^j)$	$\mathbf{I} = \delta_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \mathbf{g}^i \otimes \mathbf{g}_i$
$(\mathbf{I})_{.i}^{j} = \delta_{.i}^{j}$	$\mathbf{I}:(\mathbf{g}^j \otimes \mathbf{g}_i)$	$\mathbf{I} = \delta_{.i}^{j}\mathbf{g}_j \otimes \mathbf{g}^i = \mathbf{g}_j \otimes \mathbf{g}^j$

Showing that the Kronecker deltas are the components of the identity tensor in certain (not all) coordinate bases.

Kronecker and Metric Tensors

- The above table shows the interesting relationship between the metric components and Kronecker deltas.
- Obviously, they are the same tensors under different bases vectors.

Component Representation

It is easy to show that the above tables of component representations are valid. For any $\mathbf{v} \in \mathscr{V}$, and $\boldsymbol{T} \in \mathscr{T}$,

$$\left(\boldsymbol{T} - T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right)\mathbf{v} = \boldsymbol{T}\mathbf{v} - T_{ij}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)\mathbf{v}$$

Expanding the vector in contravariant components, we have,

* $$\begin{aligned}\boldsymbol{T}\mathbf{v} - T_{ij}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)\mathbf{v} &= \boldsymbol{T}v^\alpha\mathbf{g}_\alpha - T_{ij}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right)v^\alpha\mathbf{g}_\alpha \\ &= \boldsymbol{T}v^\alpha\mathbf{g}_\alpha - T_{ij}v^\alpha\mathbf{g}^i\left(\mathbf{g}^j \cdot \mathbf{g}_\alpha\right) \\ &= \boldsymbol{T}v^\alpha\mathbf{g}_\alpha - T_{ij}v^\alpha\mathbf{g}^i\delta_\alpha^j \\ &= \boldsymbol{T}_\alpha v^\alpha - T_{ij}v^j\mathbf{g}^i = \boldsymbol{T}_\alpha v^\alpha - \boldsymbol{T}_j v^j \\ &= \boldsymbol{o}\end{aligned}$$

$$\therefore \boldsymbol{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$$

Symmetry

The transpose of $\boldsymbol{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$ is $\boldsymbol{T}^T = T_{ij}\mathbf{g}^j \otimes \mathbf{g}^i$.

If $\boldsymbol{T}$ is symmetric, then,

$$T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = T_{ij}\mathbf{g}^j \otimes \mathbf{g}^i = T_{ji}\mathbf{g}^i \otimes \mathbf{g}^j$$

Clearly, in this case,

$$T_{ij} = T_{ji}$$

It is straightforward to establish the same for contravariant components. This result is impossible to establish for mixed tensor components:

Symmetry

For mixed tensor components,

$$\boldsymbol{T} = T_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j$$

The transpose,

$$\boldsymbol{T}^{\mathrm{T}} = T_i^{\cdot j}\mathbf{g}_j \otimes \mathbf{g}^i = T_j^{\cdot i}\mathbf{g}_i \otimes \mathbf{g}^j$$

While symmetry implies that,

$$\boldsymbol{T} = T_i^{\cdot j}\mathbf{g}^i \otimes \mathbf{g}_j = \boldsymbol{T}^{\mathrm{T}} = T_j^{\cdot i}\mathbf{g}_i \otimes \mathbf{g}^j$$

We are not able to exploit the dummy variables to bring the two sides to a common product basis. Hence the symmetry is not expressible in terms of their components.

AntiSymmetry

* A tensor is antisymmetric if its transpose is its negative. In product bases that are either covariant or contravariant, antisymmetry, like symmetry can be expressed in terms of the components:

The transpose of $\boldsymbol{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$ is $\boldsymbol{T}^T = T_{ij}\mathbf{g}^j \otimes \mathbf{g}^i$.

If $\boldsymbol{T}$ is antisymmetric, then,

$$T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = -T_{ij}\mathbf{g}^j \otimes \mathbf{g}^i = -T_{ji}\mathbf{g}^i \otimes \mathbf{g}^j$$

Clearly, in this case,

$$T_{ij} = -T_{ji}$$

It is straightforward to establish the same for contravariant components. Antisymmetric tensors are also said to be skew-symmetric.

Symmetric & Skew Parts of Tensors

For any tensor $\mathbf{T}$, define the symmetric and skew parts $\text{sym}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{\mathrm{T}}\right)$, and $\text{skw}\,\mathbf{T} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^{\mathrm{T}}\right)$. It is easy to show the following:

$$\mathbf{T} = \text{sym}\,\mathbf{T} + \text{skw}\,\mathbf{T}$$

$$\text{skw}(\text{sym}\,\mathbf{T}) = \text{sym}(\text{skw}\,\mathbf{T}\,) = 0$$

We can also write that,

$$\text{sym}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} + T_{ji}\right)\mathbf{g}^i \otimes \mathbf{g}^j$$

and

$$\text{skw}\,\mathbf{T} = \frac{1}{2}\left(T_{ij} - T_{ji}\right)\mathbf{g}^i \otimes \mathbf{g}^j$$

Composition

Composition of tensors in component form follows the rule of the composition of dyads.

$$\boldsymbol{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j,$$

$$\boldsymbol{S} = S^{ij}\ \mathbf{g}_i \otimes \mathbf{g}_j$$

$$\begin{aligned}\boldsymbol{TS} &= \left(T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j\right)\left(S^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\\ &= T^{ij}S^{\alpha\beta}\left(\mathbf{g}_i \otimes \mathbf{g}_j\right)\left(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\\ &= T^{ij}S^{\alpha\beta}\mathbf{g}_i \otimes \mathbf{g}_\beta g_{j\alpha}\\ &= T^{i.}_{.j}S^{j\beta}\mathbf{g}_i \otimes \mathbf{g}_\beta\\ &= T^{i.}_{.\alpha}S^{\alpha j}\mathbf{g}_i \otimes \mathbf{g}_j\end{aligned}$$

Addition

* Addition of two tensors of the same order is the addition of their components provided they are refereed to the same product basis.

Component Addition

Components	$\boldsymbol{T}+\boldsymbol{S}$
$T_{ij}+S_{ij}$	$\left(T_{ij}+S_{ij}\right)\mathbf{g}^i\otimes\mathbf{g}^j$
$T^{ij}+S^{ij}$	$\left(T^{ij}+S^{ij}\right)\mathbf{g}_i\otimes\mathbf{g}_j$
$T_i^{.j}+S_i^{.j}$	$\left(T_i^{.j}+S_i^{.j}\right)\mathbf{g}^i\otimes\mathbf{g}_j$
$T_{.i}^{j}+S_{.i}^{j}$	$\left(T_{.i}^{j}+S_{.i}^{j}\right)\mathbf{g}_j\otimes\mathbf{g}^i$

Component Representation of Invariants

- Invoking the definition of the three principal invariants, we now find expressions for these in terms of the components of tensors in various product bases.
- First note that for $\boldsymbol{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$, the triple product, $[\{\boldsymbol{T}\mathbf{g}_1\}, \mathbf{g}_2, \mathbf{g}_3] = \left[\left\{\left(T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right)\mathbf{g}_1\right\}, \mathbf{g}_2, \mathbf{g}_3\right]$

$$= \left[\left\{T_{ij}\mathbf{g}^i \delta_1^j\right\}, \mathbf{g}_2, \mathbf{g}_3\right] = \left[T_{i1}\mathbf{g}^i \cdot (\epsilon_{231}\mathbf{g}^1)\right] = T_{i1}g^{i1}\epsilon_{231}$$

- Recall that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$

The Trace

The Trace of the Tensor $\mathbf{T} = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$

$$\mathrm{tr}(\mathbf{T}) = \frac{[\{\mathbf{T}\mathbf{g}_1\}, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{\mathbf{T}\mathbf{g}_2\}, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \{\mathbf{T}\mathbf{g}_3\}]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{T_{i1}g^{i1}\epsilon_{231} + T_{i2}g^{i2}\epsilon_{312} + T_{i3}g^{i3}\epsilon_{123}}{\epsilon_{123}}$$

$$= T_{i1}g^{i1} + T_{i2}g^{i2} + T_{i3}g^{i3} = T_{ij}g^{ij} = T_i^{.i}$$

Write the second tensor invariant in terms of components

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken as given. Consequently,

$$I_2(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

The first of the numerator terms can be simplified as,

$$\begin{aligned}[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] &= \left[\left(T_{i1}\mathbf{g}^i\right), \left(T_{j2}\mathbf{g}^j\right), \mathbf{g}_3\right] \\ &= \left[\left(T_{i1}g^{\alpha i}\mathbf{g}_\alpha\right), \left(T_{j2}g^{\beta j}\mathbf{g}_\beta\right), \mathbf{g}_3\right]\end{aligned}$$

The other terms are similarly simplified. Clearly,

$$\begin{aligned}I_2(\mathbf{T}) &= \frac{T_1^\alpha T_2^\beta [\mathbf{g}_\alpha, \mathbf{g}_\beta, \mathbf{g}_3] + T_2^\beta T_3^\gamma [\mathbf{g}_1, \mathbf{g}_\beta, \mathbf{g}_\gamma] + T_1^\alpha T_3^\gamma [\mathbf{g}_\alpha, \mathbf{g}_2, \mathbf{g}_\gamma]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\ &= \frac{T_1^\alpha T_2^\beta \epsilon_{\alpha\beta 3} + T_2^\beta T_3^\gamma \epsilon_{1\beta\gamma} + T_1^\alpha T_3^\gamma \epsilon_{\alpha 2\gamma}}{\epsilon_{123}} \\ &= \frac{\left[\left(T_1^1 T_2^2 - T_1^2 T_2^1\right) + \left(T_2^2 T_3^3 - T_2^3 T_3^2\right) + \left(T_3^3 T_1^1 - T_3^1 T_1^3\right)\right]\epsilon_{123}}{\epsilon_{123}} \\ &= \frac{1}{2}\left(T_\alpha^\alpha T_\beta^\beta - T_\beta^\alpha T_\alpha^\beta\right)\end{aligned}$$

Determinant

Express the third tensor invariant in terms of its components.

As previously observed, any three linearly independent vectors can be treated as the basis of a coordinate system, $\mathbf{g}_i, i = 1,2,3$. The existence of the dual of these vectors can be taken for granted. Consequently, for any tensor $\mathbf{T}$,

$$I_3(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} = \frac{\left[\left(T_{i1}\mathbf{g}^i\right), \left(T_{j2}\mathbf{g}^j\right), \left(T_{k3}\mathbf{g}^k\right)\right]}{\epsilon_{123}}$$

$$= \frac{\left[\left(T_{i1}g^{\alpha i}\mathbf{g}_\alpha\right), \left(T_{j2}g^{\beta j}\mathbf{g}_\beta\right), \left(T_{k3}g^{\gamma k}\mathbf{g}_\gamma\right)\right]}{\epsilon_{123}}$$

$$= \frac{T_1^\alpha T_2^\beta T_3^\gamma \epsilon_{\alpha\beta\gamma}}{\epsilon_{123}} = e_{\alpha\beta\gamma} T_1^\alpha T_2^\beta T_3^\gamma = \det \mathbf{T}$$

The Vector Cross

Given a vector $\boldsymbol{u} = u^i \mathbf{g}_i$, the tensor

$$(\boldsymbol{u}\times) \equiv \epsilon_{i\alpha j} u^\alpha \mathbf{g}^i \otimes \mathbf{g}^j$$

is called a vector cross. The following relation is easily established between a the vector cross and its associated vector:

$$\forall \mathbf{v} \in \mathscr{V}, (\boldsymbol{u}\times)\mathbf{v} = \boldsymbol{u} \times \mathbf{v}$$

The vector cross is *traceless* and *antisymmetric*. *(HW. Show this). The converse is the axial vector.*

Traceless tensors are also called deviatoric or deviator tensors.

Axial Vector

- For any antisymmetric tensor $\mathbf{\Omega}$, $\exists \boldsymbol{\omega} \in \mathscr{V}$, such that

$$\mathbf{\Omega} = (\boldsymbol{\omega} \times)$$

$\boldsymbol{\omega}$ which can always be found, is called the axial vector to the skew tensor.

It can be proved that

$$\boldsymbol{\omega} = -\frac{1}{2}\epsilon^{ijk}\Omega_{jk}\mathbf{g}_i = -\frac{1}{2}\epsilon_{ijk}\Omega^{jk}\mathbf{g}^i$$

(HW: Prove it by contracting both sides of $\Omega_{ij} = \epsilon_{i\alpha j}\omega^{\alpha}$ with $\epsilon^{ij\beta}$ while noting that $\epsilon^{ij\beta}\epsilon_{i\alpha j} = \delta^{ij\beta}_{i\alpha j} = -2\delta^{\beta}_{\alpha}$)

Examples

Gurtin 2.8.5 Show that for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the inner product $(\mathbf{u}\times):(\mathbf{v}\times) = 2\mathbf{u}\cdot\mathbf{v}$. Hence show that $\|\mathbf{u}\times\| = \sqrt{2}\|\mathbf{u}\|$

$(\mathbf{u}\times) = \epsilon^{ijk}u_j\mathbf{g}_i\otimes\mathbf{g}_k$, $(\mathbf{v}\times) = \epsilon_{lmn}v^m\mathbf{g}^l\otimes\mathbf{g}^n$. Hence,

$$
\begin{aligned}
(\mathbf{u}\times):(\mathbf{v}\times) &= \epsilon^{ijk}\epsilon_{lmn}u_jv^m(\mathbf{g}_i\otimes\mathbf{g}_k):\left(\mathbf{g}^l\otimes\mathbf{g}^n\right)\\
&= \epsilon^{ijk}\epsilon_{lmn}u_jv^m\left(\mathbf{g}_i\cdot\mathbf{g}^l\right)(\mathbf{g}_k\cdot\mathbf{g}^n)\\
&= \epsilon^{ijk}\epsilon_{lmn}u_jv^m\delta_i^l\delta_k^n = \epsilon^{ijk}\epsilon_{imk}u_jv^m\\
&= 2\delta_m^j u_jv^m = 2u_jv^j = 2\,\mathbf{u}\cdot\,\mathbf{v}
\end{aligned}
$$

The rest of the result follows by setting $\mathbf{u}=\mathbf{v}$

HW. Redo this proof using the contravariant alternating tensor components, ϵ^{ijk} and ϵ^{lmn}.

For vectors u, v and w, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times$.

The tensor $(\mathbf{u}\times) = -\epsilon_{lmn}u^n\mathbf{g}^l\otimes\mathbf{g}^m$

Similarly, $(\mathbf{v}\times) = -\epsilon^{\alpha\beta\gamma}v_\gamma\mathbf{g}_\alpha\otimes\mathbf{g}_\beta$ and $(\mathbf{w}\times) = -\epsilon^{ijk}w_k\mathbf{g}_i\otimes\mathbf{g}_j$. Clearly,

$$
\begin{aligned}
&(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)\\
&= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}^l\otimes\mathbf{g}^m)(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= -\epsilon^{\alpha\beta\gamma}\epsilon_{lmn}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\delta^l_\beta\delta^m_i\\
&= -\epsilon^{\alpha l\gamma}\epsilon_{lin}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&\qquad\qquad = -\epsilon^{l\alpha\gamma}\epsilon_{lni}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&= -(\delta^\alpha_n\delta^\gamma_i - \delta^\alpha_i\delta^\gamma_n)\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&= -\epsilon^{ijk}u^\alpha v_i w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j) + \epsilon^{ijk}u^\gamma v_\gamma w_k(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= [\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]
\end{aligned}
$$

Index Raising & Lowering

$$g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j \qquad \text{and} \qquad g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j$$

These two quantities turn out to be fundamentally important to any space that which either of these two basis vectors can span. They are called the covariant and contravariant metric tensors. They are the quantities that metrize the space in the sense that any measurement of length, angles areas etc are dependent on them.

Index Raising & Lowering

Now we start with the fact that the contravariant and covariant components of a vector $\boldsymbol{a}$, $a^j = \boldsymbol{a} \cdot \mathbf{g}^j$, $a_j = \boldsymbol{a} \cdot \mathbf{g}_j$ respectively. We can express the vector $\boldsymbol{a}$ with respect to the reciprocal basis as

$$\boldsymbol{a} = a_i \mathbf{g}^i$$

Consequently,

$$a^j = \boldsymbol{a} \cdot \mathbf{g}^j = a_i \mathbf{g}^i \cdot \mathbf{g}^j = g^{ij} a_i$$

The effect of g^{ij} contracting g^{ij} with a_i is to raise and substitute its index.

Index Raising & Lowering

With similar arguments, it is easily demonstrated that,

$$a_i = g_{ij}a^j$$

So that g_{ij}, in a contraction, lowers and substitutes the index. This rule is a general one. These two components are able to raise or lower indices in tensors of higher orders as well. They are called index raising and index lowering operators.

Associated Tensors

Tensor components such as a_i and a^j related through the index-raising and index lowering metric tensors as we have on the previous slide, are called associated vectors. In higher order quantities, they are associated tensors.

* Note that associated tensors, so called, are mere tensor components of the same tensor in different bases.

Cofactor Definition

We will define the cofactor of a tensor as,

$$\text{cofac}\,\mathbf{T} \equiv \mathbf{T}^{\mathrm{c}} \equiv \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$$

and proceed to show that, for any pair of independent vectors $\mathbf{u}$ and $\mathbf{v}$ the cofactor satisfies,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

We will further find an invariant component representation for the cofactor tensor. Lastly, in this section, we will find an important relationship between the trace of the cofactor and second invariant of the tensor itself: $\mathrm{tr}(\mathbf{T}^{\mathrm{c}}) = I_2(\mathbf{T})$

Transformed Basis

First note that if $\mathbf{T}$ is invertible, the independence of the vectors $\mathbf{u}$ and $\mathbf{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$. Consequently we can define the non-vanishing

$$\mathbf{n} \equiv \mathbf{Tu} \times \mathbf{Tv} \neq 0.$$

It follows that $\mathbf{n}$ must be on the perpendicular line to both $\mathbf{Tu}$ and $\mathbf{Tv}$. Therefore,

$$\mathbf{n} \cdot \mathbf{Tu} = \mathbf{n} \cdot \mathbf{Tv} = 0.$$

We can also take a transpose and write,

$$\mathbf{u} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{v} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{n} = 0$$

Showing that the vector $\mathbf{T}^{\mathrm{T}}\mathbf{n}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It follows that $\exists\ \alpha \in \mathscr{R}$ such that

$$\mathbf{T}^{\mathrm{T}}\mathbf{n} = \alpha(\mathbf{u} \times \mathbf{v})$$

Cofactor Transformation

Therefore, $\mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v})$.

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ so that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent, then we can take a scalar product of the above equation and obtain,

$$\mathbf{w} \cdot \mathbf{T}^{\mathrm{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$$

The LHS is also $\mathbf{Tw} \cdot (\mathbf{Tu} \times \mathbf{Tv}) = \mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw}$. In the equation, $\mathbf{Tu} \times \mathbf{Tv} \cdot \mathbf{Tw} = \alpha(\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})$, it is clear that

$$\alpha = \det \mathbf{T}$$

We therefore have that, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}\,(\mathbf{u} \times \mathbf{v})$.

Cofactor Tensor

We therefore have that,

$$\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T} \, (\mathbf{u} \times \mathbf{v}).$$

This quantity, $\mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$ is the cofactor of $\mathbf{T}$. If we write,

$$\operatorname{cofac} \mathbf{T} \equiv \mathbf{T}^{\mathrm{c}} \equiv \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$$

we can see that the cofactor satisfies, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$

We now express the cofactor in its general components.

$$\begin{aligned}
\mathbf{T}^{\mathrm{c}} &= (\mathbf{T}^{\mathrm{c}})_i^{\alpha} \mathbf{g}_\alpha \otimes \mathbf{g}^i = (\mathbf{g}^\alpha \cdot \mathbf{T}^{\mathrm{c}} \mathbf{g}_i) \mathbf{g}_\alpha \otimes \mathbf{g}^i \\
&= \frac{1}{2} \epsilon_{ijk} \left[\mathbf{g}^\alpha \cdot \mathbf{T}^{\mathrm{c}} \left(\mathbf{g}^j \times \mathbf{g}^k \right) \right] \mathbf{g}_\alpha \otimes \mathbf{g}^i \\
&= \frac{1}{2} \epsilon_{ijk} \left[\mathbf{g}^\alpha \cdot \left(\mathbf{T} \mathbf{g}^j \right) \times \left(\mathbf{T} \mathbf{g}^k \right) \right] \mathbf{g}_\alpha \otimes \mathbf{g}^i.
\end{aligned}$$

Cofactor Components

The scalar in brackets,

$$\mathbf{g}^{\alpha}\cdot\left(\mathbf{T}\mathbf{g}^{j}\right)\times\left(\mathbf{T}\mathbf{g}^{k}\right)=\mathbf{g}^{\alpha}\cdot\epsilon^{lmn}\left(\mathbf{g}_m\cdot\mathbf{T}\mathbf{g}^{j}\right)\left(\mathbf{g}_n\cdot\mathbf{T}\mathbf{g}^{k}\right)\mathbf{g}_l$$
$$=\delta_l^{\alpha}\epsilon^{lmn}\left(\mathbf{g}_m\cdot\mathbf{T}\mathbf{g}^{j}\right)\left(\mathbf{g}_n\cdot\mathbf{T}\mathbf{g}^{k}\right)$$
$$=\delta_l^{\alpha}\epsilon^{lmn}T_m^j T_n^k=\epsilon^{\alpha mn}T_m^j T_n^k$$

Inserting this above, we therefore have, in invariant component form,

$$\mathbf{T}^{\mathrm{c}}=\frac{1}{2}\epsilon_{ijk}\left[\mathbf{g}^{\alpha}\cdot\left(\mathbf{T}\mathbf{g}^{j}\right)\times\left(\mathbf{T}\mathbf{g}^{k}\right)\right]\mathbf{g}_{\alpha}\otimes\mathbf{g}^{i}$$
$$=\frac{1}{2}\epsilon_{ijk}\epsilon^{\alpha mn}T_m^j T_n^k\mathbf{g}_{\alpha}\otimes\mathbf{g}^{i}$$
$$=\frac{1}{2}\delta_{ijk}^{lmn}T_m^j T_n^k\mathbf{g}_l\otimes\mathbf{g}^{i}$$

Trace of the Cofactor

For any invertible tensor, show that the trace of the cofactor is the second principal invariant of the original tensor: $I_1(\mathbf{T}^{\mathrm{c}}) = I_2(\mathbf{T})$

$$\begin{aligned}
\mathrm{tr}(\mathbf{T}^{\mathrm{c}}) &= \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \mathbf{g}_l \cdot \mathbf{g}^i = I_1(\mathbf{T}^{\mathrm{c}}) \\
&= \frac{1}{2}\delta_{ijk}^{lmn} T_m^j T_n^k \delta_l^i = \frac{1}{2}\delta_{ijk}^{imn} T_m^j T_n^k \\
&= \frac{1}{2}\left(\delta_j^m \delta_k^n - \delta_k^m \delta_j^n\right) T_m^j T_n^k = \frac{1}{2}\left(T_j^j T_k^k - T_k^j T_j^k\right) \\
&= I_2(\mathbf{T})
\end{aligned}$$

Determinants

Show that the determinant of a product is the product of the determinants

$$\boldsymbol{C} = \boldsymbol{AB} \;\Rightarrow C_j^i = A_m^i B_j^m$$

so that the determinant of $\mathbf{C}$ in component form is,

$$\begin{aligned}
\det \mathbf{AB} = \det \mathbf{C} &= e^{ijk} C_i^1 C_j^2 C_k^3 \\
&= e^{ijk} A_l^1 B_i^l A_m^2 B_j^m A_n^3 B_k^n \\
&= \left(A_l^1 A_m^2 A_n^3\right) e^{ijk} B_i^l B_j^m B_k^n \\
&= \left(A_l^1 A_m^2 A_n^3 e^{ijk}\right)\left(B_i^1 B_j^2 B_k^3 e^{lmn}\right) \\
&= \det \mathbf{A} \times \det \mathbf{B}
\end{aligned}$$

Determinants

Here we establish the equality assumed above that, $e_{\alpha\beta\gamma} \det \mathbf{T}$

$$= e_{ijk} T_\alpha^i T_\beta^j T_\gamma^k = e_{ijk} T_1^i T_2^j T_3^k e_{\alpha\beta\gamma}$$

We do this by first establishing the fact that the LHS is completely antisymmetric in α, β and γ. We note that the indices i, j and k are dummy and therefore,

$$e_{ijk} T_\alpha^i T_\beta^j T_\gamma^k = e_{kji} T_\alpha^k T_\beta^j T_\gamma^i = e_{kji} T_\gamma^i T_\alpha^k T_\beta^j = -e_{ijk} T_\gamma^i T_\beta^j T_\alpha^k$$

Showing that a simple swap of α and γ changes the sign. This is similarly true for the other pairs in the lower symbols. Thus we establish anti-symmetry in α, β and γ.

Noting that both sides take the same values when α, β and γ are equal to 1, 2 and 3 respectively. The arrangement of the indices makes this value positive or negative in the same antisymmetric way. This completes the proof. Similarly we can write,

$$e^{ijk} T_i^\alpha T_j^\beta T_k^\gamma = e^{ijk} T_i^1 T_j^2 T_k^3 e^{\alpha\beta\gamma} = e^{\alpha\beta\gamma} \det \mathbf{T}$$

$$\det \alpha \boldsymbol{C} = \epsilon^{ijk}\left(\alpha C_i^1\right)\left(\alpha C_j^2\right)\left(\alpha C_k^3\right) = \alpha^3 \det \boldsymbol{C}$$

For any invertible tensor we show that $\det\left(\boldsymbol{S}^{\mathrm{C}}\right) = (\det \boldsymbol{S})^2$

The inverse of tensor $\boldsymbol{S}$,

$$\boldsymbol{S}^{-1} = (\det \boldsymbol{S})^{-1}\left(\boldsymbol{S}^{\mathrm{C}}\right)^{\mathrm{T}}$$

let the scalar $\alpha = \det \boldsymbol{S}$. We can see clearly that,

$$\boldsymbol{S}^{\mathrm{C}} = \alpha \boldsymbol{S}^{-T}$$

Taking the determinant of this equation, we have,

$$\det\left(\boldsymbol{S}^{\mathrm{C}}\right) = \alpha^3 \det\left(\boldsymbol{S}^{-T}\right) = \alpha^3 \det\left(\boldsymbol{S}^{-1}\right)$$

as the transpose operation has no effect on the value of a determinant. Noting that the determinant of an inverse is the inverse of the determinant, we have,

$$\det\left(\boldsymbol{S}^{\mathrm{C}}\right) = \alpha^3 \det\left(\boldsymbol{S}^{-1}\right) = \frac{\alpha^3}{\alpha} = (\det \boldsymbol{S})^2$$

Cofactor

Show that $(\alpha\boldsymbol{S})^{\mathrm{C}} = \alpha^2\boldsymbol{S}^{\mathrm{C}}$

Ans

$$(\alpha\boldsymbol{S})^{\mathrm{C}} = (\det(\alpha\boldsymbol{S}))(\alpha\boldsymbol{S})^{-\mathrm{T}} = \left(\alpha^3 \det(\boldsymbol{S})\right)\alpha^{-1}\boldsymbol{S}^{-\mathrm{T}}$$
$$= \left(\alpha^2 \det(\boldsymbol{S})\right)\boldsymbol{S}^{-\mathrm{T}} = \alpha^2\boldsymbol{S}^{\mathrm{C}}$$

Show that $(\boldsymbol{S}^{-1})^{\mathrm{C}} = (\det \boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$

Ans.

$$(\boldsymbol{S}^{-1})^{\mathrm{C}} = \det(\boldsymbol{S}^{-1})\,(\boldsymbol{S}^{-1})^{-T} = (\det \boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$$

(d) Show that $\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1}=(\det\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$

Ans.

$$\boldsymbol{S}^{\mathrm{C}}=\det(\boldsymbol{S})\,\boldsymbol{S}^{-T}$$

Consequently,

$$\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1}=(\det\boldsymbol{S})^{-1}\left(\boldsymbol{S}^{-T}\right)^{-1}=(\det\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$$

(e) Show that $\left(\boldsymbol{S}^{\mathrm{C}}\right)^{\mathrm{C}}=(\det\boldsymbol{S})\boldsymbol{S}$

Ans.

$$\boldsymbol{S}^{\mathrm{C}}=\det(\boldsymbol{S})\,\boldsymbol{S}^{-T}$$

So that,

$$\left(\boldsymbol{S}^{\mathrm{C}}\right)^{\mathrm{C}}=(\det\boldsymbol{S}^{\mathrm{C}})\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-T}=(\det\boldsymbol{S})^{2}\left[\left(\boldsymbol{S}^{\mathrm{C}}\right)^{-1}\right]^{T}$$

$$=(\det\boldsymbol{S})^{2}\left[(\det\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\right]^{T}=(\det\boldsymbol{S})^{2}(\det\boldsymbol{S})^{-1}\boldsymbol{S}=(\det\boldsymbol{S})\boldsymbol{S}$$

as required.

3. **Show that for any invertible tensor $\boldsymbol{S}$ and any vector $\boldsymbol{u}$,**

$$[(\boldsymbol{S}\boldsymbol{u})\times] = \boldsymbol{S}^{\mathrm{c}}(\boldsymbol{u}\times)\boldsymbol{S}^{-\mathbf{1}}$$

where $\boldsymbol{S}^{\mathrm{C}}$ and $\boldsymbol{S}^{-\mathbf{1}}$ are the cofactor and inverse of $\boldsymbol{S}$ respectively.

By definition,

$$\boldsymbol{S}^{c} = (\det \boldsymbol{S})\boldsymbol{S}^{-\mathrm{T}}$$

We are to prove that,

$$[(\boldsymbol{S}\boldsymbol{u})\times] = \boldsymbol{S}^{c}(\boldsymbol{u}\times)\boldsymbol{S}^{-\mathbf{1}} = (\det \boldsymbol{S})\boldsymbol{S}^{-\mathrm{T}}(\boldsymbol{u}\times)\boldsymbol{S}^{-\mathbf{1}}$$

or that,

$$\boldsymbol{S}^{\mathrm{T}}[(\boldsymbol{S}\boldsymbol{u})\times] = (\boldsymbol{u}\times)(\det \boldsymbol{S})\boldsymbol{S}^{-\mathbf{1}} = (\boldsymbol{u}\times)(\boldsymbol{S}^{c})^{\mathbf{T}}$$

On the RHS, the contravariant ij component of $\boldsymbol{u}\times$ is

$$(\boldsymbol{u}\times)^{ij} = \epsilon^{i\alpha j}u_{\alpha}$$

which is exactly the same as writing, $(\boldsymbol{u}\times) = \epsilon^{i\alpha l}\, u_{\alpha}\mathbf{g}_i \otimes \mathbf{g}_l$ in the invariant form.

We now turn to the LHS;

$$[(\boldsymbol{Su})\times] = \epsilon^{l\alpha k}(\boldsymbol{Su})_\alpha \mathbf{g}_l \otimes \mathbf{g}_k = \epsilon^{l\alpha k} S_\alpha^j u_j \mathbf{g}_l \otimes \mathbf{g}_k$$

Now, $\boldsymbol{S} = S^{i\cdot}_{\cdot\beta}\mathbf{g}_i \otimes \mathbf{g}^\beta$ so that its transpose, $\boldsymbol{S}^{\mathrm{T}} = S_\beta^i \mathbf{g}^\beta \otimes \mathbf{g}_i = S_i^\beta \mathbf{g}^i \otimes \mathbf{g}_\beta$ so that

$$\begin{aligned}
\boldsymbol{S}^{\mathrm{T}}[(\boldsymbol{Su})\times] &= \epsilon^{l\alpha k} S_i^\beta S_\alpha^j u_j \mathbf{g}^i \otimes \mathbf{g}_\beta \cdot \mathbf{g}_l \otimes \mathbf{g}_k \\
&= \epsilon^{l\alpha k} S_{il} S_\alpha^j u_j \mathbf{g}^i \otimes \mathbf{g}_k \\
&= \epsilon^{l\alpha k} S_l^i S_\alpha^j u_j \mathbf{g}_i \otimes \mathbf{g}_k \\
&= \epsilon^{\alpha\beta k} u_j S_\alpha^i S_\beta^j \mathbf{g}_i \otimes \mathbf{g}_k = (\boldsymbol{u}\times)(\boldsymbol{S}^c)^{\mathrm{T}}.
\end{aligned}$$

* Show that $\left[\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)\times\right] = \boldsymbol{S}(\boldsymbol{u}\times)\boldsymbol{S}^{\mathrm{T}}$

The LHS in component invariant form can be written as:

$$\left[\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)\times\right] = \epsilon^{ijk}\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)_j \mathbf{g}_i \otimes \mathbf{g}_k$$

where $\left(\boldsymbol{S}^{\mathrm{C}}\right)_j^{\beta} = \frac{1}{2}\epsilon_{jab}\epsilon^{\beta cd}S_c^a S_d^b$ so that

$$\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)_j = \left(\boldsymbol{S}^{\mathrm{C}}\right)_j^{\beta} u_\beta = \frac{1}{2}\epsilon_{jab}\epsilon^{\beta cd}u_\beta S_c^a S_d^b$$

Consequently,

$$\left[\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)\times\right] = \frac{1}{2}\epsilon^{ijk}\epsilon_{jab}\epsilon^{\beta cd}u_\beta S_c^a S_d^b \mathbf{g}_i \otimes \mathbf{g}_k$$
$$= \frac{1}{2}\epsilon^{\beta cd}\left(\delta_a^k\delta_b^i - \delta_b^k\delta_a^i\right)u_\beta S_c^a S_d^b \mathbf{g}_i \otimes \mathbf{g}_k$$
$$= \frac{1}{2}\epsilon^{\beta cd}u_\beta\left(S_c^k S_d^i - S_c^i S_d^k\right)\mathbf{g}_i \otimes \mathbf{g}_k$$
$$= \epsilon^{\beta cd}u_\beta S_c^k S_d^i \mathbf{g}_i \otimes \mathbf{g}_k$$

On the RHS, $(\boldsymbol{u}\times)\boldsymbol{S}^{\mathrm{T}} = \epsilon^{\alpha\beta\gamma}u_\beta S_\gamma^k \mathbf{g}_\alpha \otimes \mathbf{g}_k$. We can therefore write,

$$\boldsymbol{S}(\boldsymbol{u}\times)\boldsymbol{S}^{\mathrm{T}} = \epsilon^{\alpha\beta\gamma}u_\beta S_\alpha^i S_\gamma^k \mathbf{g}_i \otimes \mathbf{g}_k =$$

Which on a closer look is exactly the same as the LHS so that,

$\left[\left(\boldsymbol{S}^{\mathrm{C}}\boldsymbol{u}\right)\times\right] = \boldsymbol{S}(\boldsymbol{u}\times)\boldsymbol{S}^{\mathrm{T}}$

as required.

* 4. Let $\mathbf{\Omega}$ be skew with axial vector $\boldsymbol{\omega}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\mathbf{\Omega u} \times \mathbf{\Omega v} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$ and, hence conclude that $\mathbf{\Omega}^{\mathrm{C}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.
*

$$\begin{aligned}\mathbf{\Omega u} \times \mathbf{\Omega v} &= (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) = (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v}) \\ &= [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\omega}]\mathbf{v} = [\boldsymbol{\omega} \cdot (\mathbf{u} \times \mathbf{v})]\boldsymbol{\omega} \\ &= (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

But by definition, the cofactor must satisfy,

$$\mathbf{\Omega u} \times \mathbf{\Omega v} = \mathbf{\Omega}^{c}(\mathbf{u} \times \mathbf{v})$$

which compared with the previous equation yields the desired result that

$$\mathbf{\Omega}^{\mathrm{C}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}).$$

5. Show that the cofactor of a tensor can be written as

$$\boldsymbol{S}^{\mathrm{C}} = \left(\boldsymbol{S}^2 - I_1\boldsymbol{S} + I_2\mathbf{1}\right)^{\mathrm{T}}$$

even if $\boldsymbol{S}$ is not invertible. I_1, I_2 are the first two invariants of $\boldsymbol{S}$.

Ans.

The above equation can be written more explicitly as,

$$\boldsymbol{S}^{\mathrm{C}} = \left(\boldsymbol{S}^2 - \mathrm{tr}(\boldsymbol{S})\boldsymbol{S} + \left[\frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}\left(\boldsymbol{S}^2\right)\right]\right]\mathbf{1}\right)^{\mathrm{T}}$$

In the invariant component form, this is easily seen to be,

$$\boldsymbol{S}^{\mathrm{C}} = \left(S^i_\eta S^\eta_j - S^\alpha_\alpha S^i_j + \frac{1}{2}\left(S^\alpha_\alpha S^\beta_\beta - S^\alpha_\beta S^\beta_\alpha\right)\delta^i_j\right)\mathbf{g}^j \otimes \mathbf{g}_i$$

But we know that the cofactor can be obtained directly from the equation,

$$(\boldsymbol{S}^{\mathrm{c}}) = \frac{1}{2}\epsilon^{i\beta\gamma}\epsilon_{j\lambda\eta}S_\beta^\lambda S_\gamma^\eta \mathbf{g}_i \otimes \mathbf{g}^j = \frac{1}{2}\begin{bmatrix} \delta_j^i & \delta_\lambda^i & \delta_\eta^i \\ \delta_j^\beta & \delta_\lambda^\beta & \delta_\eta^\beta \\ \delta_j^\gamma & \delta_\lambda^\gamma & \delta_\eta^\gamma \end{bmatrix} S_\beta^\lambda S_\gamma^\eta \mathbf{g}_i \otimes \mathbf{g}^j$$

$$\frac{1}{2}\left(\delta_j^i \begin{bmatrix} \delta_\lambda^\beta & \delta_\eta^\beta \\ \delta_\lambda^\gamma & \delta_\eta^\gamma \end{bmatrix} - \delta_\lambda^i \begin{bmatrix} \delta_j^\beta & \delta_\eta^\beta \\ \delta_j^\gamma & \delta_\eta^\gamma \end{bmatrix} + \delta_\eta^i \begin{bmatrix} \delta_j^\beta & \delta_\lambda^\beta \\ \delta_j^\gamma & \delta_\lambda^\gamma \end{bmatrix}\right) S_\beta^\lambda S_\gamma^\eta \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta_j^i \left(\delta_\lambda^\beta \delta_\eta^\gamma - \delta_\eta^\beta \delta_\lambda^\gamma\right) - \delta_\lambda^i \left(\delta_j^\beta \delta_\eta^\gamma - \delta_\eta^\beta \delta_j^\gamma\right) + \delta_\eta^i \left(\delta_j^\beta \delta_\lambda^\gamma - \delta_\lambda^\beta \delta_j^\gamma\right)\right] S_\beta^\lambda S_\gamma^\eta \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta_j^i \left(S_\alpha^\alpha S_\beta^\beta - S_\eta^\lambda S_\lambda^\eta\right) - 2S_j^i S_\alpha^\alpha + 2S_\eta^i S_j^\eta \right] \mathbf{g}_i \otimes \mathbf{g}^j$$

Using the above, Show that the cofactor of a vector cross$(\mathbf{u}\times)$ is $\mathbf{u}\otimes\mathbf{u}$

$$\begin{aligned}
(\mathbf{u}\times)^2 &= \left(\epsilon^{i\alpha j}u_\alpha\mathbf{g}_i\otimes\mathbf{g}_j\right)\left(\epsilon_{l\beta m}u^\beta\mathbf{g}^l\otimes\mathbf{g}^m\right)\\
&= \epsilon^{i\alpha j}\epsilon_{l\beta m}u_\alpha u^\beta(\mathbf{g}_i\otimes\mathbf{g}^m)\delta_j^l = \epsilon^{i\alpha j}\epsilon_{j\beta m}u_\alpha u^\beta(\mathbf{g}_i\otimes\mathbf{g}^m) = \epsilon^{i\alpha j}\epsilon_{\beta mj}u_\alpha u^\beta(\mathbf{g}_i\otimes\mathbf{g}^m)\\
&= \left(\delta_\beta^i\delta_m^\alpha-\delta_m^i\delta_\beta^\alpha\right)u_\alpha u^\beta(\mathbf{g}_i\otimes\mathbf{g}^m) = \left(u_m u^i-\delta_m^i u_\alpha u^\alpha\right)\mathbf{g}_i\otimes\mathbf{g}^m = \mathbf{u}\otimes\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{1}
\end{aligned}$$

$$\mathrm{tr}[(\mathbf{u}\times)^2] = \mathbf{u}\cdot\mathbf{u}-3\,\mathbf{u}\cdot\mathbf{u} = -\,2\,\mathbf{u}\cdot\mathbf{u}$$

$$\mathrm{tr}[(\mathbf{u}\times)] = 0$$

But from previous result,

$$\begin{aligned}
(\mathbf{u}\times)^{\mathrm{C}} &= \left((\mathbf{u}\times)^2-(\mathbf{u}\times)\mathrm{tr}(\mathbf{u}\times)+\left[\frac{1}{2}\left[\mathrm{tr}^2(\mathbf{u}\times)-\mathrm{tr}\left((\mathbf{u}\times)^2\right)\right]\right]\mathbf{1}\right)^{\mathrm{T}}\\
&= \left(\mathbf{u}\otimes\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{1}-0+\left[\frac{1}{2}[0+2\,\mathbf{u}\cdot\mathbf{u}]\right]\mathbf{1}\right)^{\mathrm{T}}\\
&= (\mathbf{u}\otimes\mathbf{u}-(\mathbf{u}\cdot\mathbf{u})\mathbf{1}-0+[(\mathbf{u}\cdot\mathbf{u})]\mathbf{1})^{\mathrm{T}}\\
&= \mathbf{u}\otimes\mathbf{u}
\end{aligned}$$

Show that $(\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{C}}=\mathbf{O}$

In component form,

$$\boldsymbol{u}\otimes\boldsymbol{u}=u^i u_j \mathbf{g}_i\otimes\mathbf{g}^j$$

So that

$$(\boldsymbol{u}\otimes\boldsymbol{u})^2=\left(u^i u_j \mathbf{g}_i\otimes\mathbf{g}^j\right)\left(u^l u_m \mathbf{g}_l\otimes\mathbf{g}^m\right)=u^i u_j u^l u_m \mathbf{g}_i\otimes\mathbf{g}^m\delta_m^j$$
$$=u^i u_j u^j u_m \mathbf{g}_i\otimes\mathbf{g}^m=(\boldsymbol{u}\otimes\boldsymbol{u})(\boldsymbol{u}\cdot\boldsymbol{u})$$

Clearly,

$\mathrm{tr}[(\boldsymbol{u}\otimes\boldsymbol{u})]=\boldsymbol{u}\cdot\boldsymbol{u}$, $\mathrm{tr}^2[(\boldsymbol{u}\otimes\boldsymbol{u})]=(\boldsymbol{u}\cdot\boldsymbol{u})^2$

and $\mathrm{tr}[(\boldsymbol{u}\otimes\boldsymbol{u})^2]=(\boldsymbol{u}\cdot\boldsymbol{u})^2 (\boldsymbol{u}\otimes\boldsymbol{u})^{\mathrm{C}}=\Big((\boldsymbol{u}\otimes\boldsymbol{u})^2-(\boldsymbol{u}\otimes\boldsymbol{u})\mathrm{tr}(\boldsymbol{u}\otimes\boldsymbol{u})+$

Orthogonal Tensors

Given a Euclidean Vector Space $\mathcal{E}$, a tensor $\boldsymbol{Q}$ is said to be orthogonal if, $\forall \boldsymbol{a}, \boldsymbol{b} \in \mathcal{E}$,

$$(\boldsymbol{Qa}) \cdot (\boldsymbol{Qb}) = \boldsymbol{a} \cdot \boldsymbol{b}$$

Specifically, we can allow $\boldsymbol{a} = \boldsymbol{b}$, so that

$$(\boldsymbol{Qa}) \cdot (\boldsymbol{Qa}) = \boldsymbol{a} \cdot \boldsymbol{a}$$

Or

$$||\boldsymbol{Qa}|| = ||\boldsymbol{a}||$$

In which case the mapping leaves the magnitude unaltered.

Orthogonal Tensors

Let $\boldsymbol{q} = \boldsymbol{Qa}$

$$(\boldsymbol{Qa}) \cdot (\boldsymbol{Qb}) = \boldsymbol{q} \cdot \boldsymbol{Qb} = \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

By definition of the transpose, we have that,

$$\boldsymbol{q} \cdot \boldsymbol{Qb} = \boldsymbol{b} \cdot \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{q} = \boldsymbol{b} \cdot \boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{Qa} = \boldsymbol{b} \cdot \boldsymbol{a}$$

Clearly, $\boldsymbol{Q}^{\boldsymbol{T}}\boldsymbol{Q} = \boldsymbol{1}$

A condition necessary and sufficient for a tensor $\boldsymbol{Q}$ *to be orthogonal is that* $\boldsymbol{Q}$ *be invertible and its inverse equal to its transpose.*

Orthogonal

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det\left(\boldsymbol{Q}^T\boldsymbol{Q}\right) = \left(\det \boldsymbol{Q}^T\right)(\det \boldsymbol{Q}) = (\det \boldsymbol{Q})^2 = 1$$

Which clearly shows that

$$(\det \boldsymbol{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "*proper orthogonal*".

Rotation & Reflection

A rotation is a proper orthogonal tensor while a reflection is not.

Rotation

* **Let $\boldsymbol{Q}$ be a rotation. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Qu}) \times (\mathbf{Qv})$**

This question is the same as showing that the cofactor of $\boldsymbol{Q}$ is $\boldsymbol{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\boldsymbol{T}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

where

$$\mathbf{T} = \operatorname{cof}(\boldsymbol{Q}) = \det(\boldsymbol{Q})\,\boldsymbol{Q}^{-\mathrm{T}}$$

Now that $\boldsymbol{Q}$ is a rotation, $\det(\boldsymbol{Q}) = 1$, and

$$\boldsymbol{Q}^{-\mathrm{T}} = (\boldsymbol{Q}^{-1})^{T} = (\boldsymbol{Q}^{\mathrm{T}})^{T} = \boldsymbol{Q}$$

This implies that $\boldsymbol{T} = \boldsymbol{Q}$ and consequently,

$$\boldsymbol{Q}(\mathbf{u} \times \mathbf{v}) = (\boldsymbol{Q}\mathbf{u}) \times (\boldsymbol{Q}\mathbf{v})$$

For a proper orthogonal tensor Q, show that the eigenvalue equation always yields an eigenvalue of +1. This means that there is always a solution for the equation,

$$\boldsymbol{Qu} = \boldsymbol{u}$$

For any invertible tensor,

$$\mathbf{S}^{\mathrm{c}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

For a proper orthogonal tensor $\mathbf{Q}$, $\det \mathbf{Q} = 1$. It therefore follows that,

$$\mathbf{Q}^{\mathrm{C}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

It is easily shown that $\mathrm{tr}\mathbf{Q}^{\mathrm{C}} = I_2(\mathbf{Q})$ (HW Show this Romano 26)

Characteristic equation for $\boldsymbol{Q}$ is,

$$\det(\mathbf{Q} - \lambda\mathbf{1}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

Or,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

Which is obviously satisfied by $\lambda = 1$.

If for an arbitrary unit vector e, the tensor, $\boldsymbol{Q}(\theta) = \cos(\theta)\boldsymbol{I} + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the skew tensor whose ij component is $\epsilon_{jik}e_k$, show that $\boldsymbol{Q}(\theta)(\boldsymbol{I} - \mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\boldsymbol{I} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\times)$.

$$\boldsymbol{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)[\mathbf{e} \times (\mathbf{e} \otimes \mathbf{e})]$$

The last term vanishes immediately on account of the fact that $\mathbf{e} \otimes \mathbf{e}$ is a symmetric tensor. We therefore have,

$$\boldsymbol{Q}(\theta)(\mathbf{e} \otimes \mathbf{e}) = \cos(\theta)(\mathbf{e} \otimes \mathbf{e}) + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} = \mathbf{e} \otimes \mathbf{e}$$

which again mean that $\boldsymbol{Q}(\theta)$ so that

$$\begin{aligned}\boldsymbol{Q}(\theta)(\boldsymbol{I} - \mathbf{e} \otimes \mathbf{e}) &= \cos(\theta)\boldsymbol{I} + \big(1 - \cos(\boldsymbol{\theta})\big)\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times) - \mathbf{e} \otimes \mathbf{e} \\ &= \cos(\theta)(\boldsymbol{I} - \mathbf{e} \otimes \mathbf{e}) + \sin(\theta)(\mathbf{e}\times)\end{aligned}$$

as required.

If for an arbitrary unit vector e, the tensor, $\boldsymbol{Q}(\theta) = \cos(\theta)\boldsymbol{I} + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the skew tensor whose ij component is $\epsilon_{jik} e_k$. Show for an arbitrary vector u that $\mathbf{v} = \boldsymbol{Q}(\theta)\mathbf{u}$ has the same magnitude as u.

Given an arbitrary vector $\mathbf{u}$, compute the vector $\mathbf{v} = \boldsymbol{Q}(\theta)\mathbf{u}$. Clearly,

$$\mathbf{v} = \cos(\theta)\mathbf{u} + (1 - \cos(\boldsymbol{\theta}))(\mathbf{u}\cdot\mathbf{e})\mathbf{e} + \sin(\theta)(\mathbf{e}\times\mathbf{u})$$

The square of the magnitude of $\mathbf{v}$ is

$\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^{\mathbf{2}}$

$2\square + 2\cos(\boldsymbol{\theta})\,(1 - \cos(\boldsymbol{\theta}))(\mathbf{u}\cdot\mathbf{e})^{\mathbf{2}} = \cos^2\theta\ \mathbf{u}\cdot\mathbf{u} + (1 - \cos(\boldsymbol{\theta}))(\mathbf{u}\cdot\mathbf{e})^{\mathbf{2}}(1$

If for an arbitrary unit vector e, the tensor, $\boldsymbol{Q}(\theta) = \cos(\theta)\boldsymbol{I} + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the skew tensor whose ij component is $\epsilon_{jik}e_k$. Show for an arbitrary $0 \le \alpha, \beta \le 2\pi$, that $\boldsymbol{Q}(\alpha+\beta) = \boldsymbol{Q}(\alpha)\boldsymbol{Q}(\beta)$.

It is convenient to write $\boldsymbol{Q}(\alpha)$ and $\boldsymbol{Q}(\beta)$ in terms of their components: The ij component of

$$[\boldsymbol{Q}(\alpha)]_{ij} = (\cos\alpha)\delta_{ij} + (1-\cos\alpha)e_i e_j - (\sin\alpha)\,\epsilon_{ijk}e_k$$

Consequently, we can write,

$$\begin{aligned}
[\boldsymbol{Q}(\alpha)\boldsymbol{Q}(\beta)]_{ij} &= [\boldsymbol{Q}(\alpha)]_{\boldsymbol{ik}}[\boldsymbol{Q}(\beta)]_{kj} = \\
&= [(\cos\alpha)\delta_{ik} + (1-\cos\alpha)e_i e_k - (\sin\alpha)\,\epsilon_{ikl}e_l]\big[(\cos\beta)\delta_{kj} + (1-\cos\beta)e_k e_j - (\sin\beta)\,\epsilon_{kjm}e_m\big] \\
&= (\cos\alpha\cos\beta)\,\delta_{ik}\delta_{kj} + \cos\alpha(1-\cos\beta)\delta_{ik}e_k e_j - \cos\alpha\sin\beta\,\epsilon_{kjm}e_m\delta_{ik} + \cos\beta(1-\cos\alpha)\delta_{kj}e_i e_k + (1-\cos\alpha)(1-\cos\beta)e_i e_k e_k e_j \\
&\quad - (1-\cos\alpha)e_i e_k\,(\sin\beta)\,\epsilon_{kjm}e_m - (\sin\alpha\cos\beta)\,\epsilon_{ikl}e_l\delta_{kj} - (\sin\alpha)(1-\cos\beta)e_k e_j\,\epsilon_{ikl}e_l + (\sin\alpha\sin\beta)\,\epsilon_{ikl}\epsilon_{kjm}e_l e_m \\
&= (\cos\alpha\cos\beta)\,\delta_{ij} + \cos\alpha(1-\cos\beta)e_i e_j - \cos\alpha\sin\beta\,\epsilon_{ijm}e_m + \cos\beta(1-\cos\alpha)e_i e_j + (1-\cos\alpha)(1-\cos\beta)e_i e_j - (\sin\alpha\cos\beta)\,\epsilon_{ijl}e_l \\
&\quad + (\sin\alpha\sin\beta)\,\big(\delta_{lj}\delta_{im} - \delta_{lm}\delta_{ji}\big)e_l e_m \\
&= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\,\delta_{ij} + [1 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]e_i e_j - [\cos\alpha\sin\beta - \sin\alpha\cos\beta]\epsilon_{ijm}e_m \\
&= [\boldsymbol{Q}(\alpha+\beta)]_{ij}
\end{aligned}$$

Use the results of 52 and 55 above to show that the tensor $\boldsymbol{Q}(\theta) = \cos(\theta)\boldsymbol{I} + (1 - \cos(\boldsymbol{\theta}))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ is periodic with a period of 2π.

From 55 we can write that $\boldsymbol{Q}(\alpha + 2\pi) = \boldsymbol{Q}(\alpha)\boldsymbol{Q}(2\pi)$. But from 52, $\boldsymbol{Q}(0) = \boldsymbol{Q}(2\pi) = \boldsymbol{I}$. We therefore have that,

$$\boldsymbol{Q}(\alpha + 2\pi) = \boldsymbol{Q}(\alpha)\boldsymbol{Q}(2\pi) = \boldsymbol{Q}(\alpha)$$

which completes the proof. The above results show that $\boldsymbol{Q}(\alpha)$ is a rotation along the unit vector $\mathbf{e}$ through an angle α.

Define Lin^+as the set of all tensors with a positive determinant. Show that $\mathrm{Lin}^+$is invariant under G where is the proper orthogonal group of all rotations, in the sense that for any tensor $\mathbf{A} \in \mathrm{Lin}^+$ $\mathbf{Q} \in \mathrm{G} \Rightarrow \mathbf{QAQ}^\mathrm{T} \in \mathrm{Lin}^+$.(G285)

Since we are given that $\mathbf{A} \in \mathrm{Lin}^+$, the determinant of $\mathbf{A}$ is positive. Consider $\det\left(\mathbf{QAQ}^\mathrm{T}\right)$. We observe the fact that the determinant of a product of tensors is the product of their determinants (proved above). We see clearly that,

$\det\left(\mathbf{QAQ}^\mathrm{T}\right) = \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det\left(\mathbf{Q}^\mathrm{T}\right)$. Since $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = \det\left(\mathbf{Q}^\mathrm{T}\right) = 1$. Consequently we see that,

$$\begin{aligned}\det\left(\mathbf{QAQ}^\mathrm{T}\right) &= \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det\left(\mathbf{Q}^\mathrm{T}\right)\\ &= \det\left(\mathbf{QAQ}^\mathrm{T}\right)\\ &= 1 \times \det(\mathbf{A}) \times 1\\ &= \det(\mathbf{A})\end{aligned}$$

Hence the determinant of $\mathbf{QAQ}^\mathrm{T}$ is also positive and therefore $\mathbf{QAQ}^\mathrm{T} \in \mathrm{Lin}^+$.

Define Sym **as the set of all symmetric tensors. Show that** Sym **is invariant under G where is the proper orthogonal group of all rotations, in the sense that for any tensor** $\mathrm{A} \in \mathrm{Sym}$ **every** $\mathbf{Q} \in G \Rightarrow \mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Sym}$**. (G285)**

Since we are given that $\mathrm{A} \in \mathrm{Sym}$, we inspect the tensor $\mathbf{QAQ}^{\mathrm{T}}$. Its transpose is, $\left(\mathbf{QAQ}^{\mathrm{T}}\right)^{\mathrm{T}} = \left(\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{AQ}^{\mathrm{T}} = \mathbf{QAQ}^{\mathrm{T}}$. So that $\mathbf{QAQ}^{\mathrm{T}}$ is symmetric and therefore $\mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Sym}$. so that the transformation is invariant.

Central to the usefulness of tensors in Continuum Mechanics is the **Eigenvalue Problem** and its consequences.

- These issues lead to the mathematical representation of such physical properties as Principal stresses, Principal strains, Principal stretches, Principal planes, Natural frequencies, Normal modes, Characteristic values, resonance, equivalent stresses, theories of yielding, failure analyses, Von Mises stresses, etc.
- As we can see, these seeming unrelated issues are all centered around the eigenvalue problem of tensors. Symmetry groups, and many other constructs that simplify analyses cannot be understood outside a thorough understanding of the eigenvalue problem.
- At this stage of our study of Tensor Algebra, we shall go through a simplified study of the eigenvalue problem. This study will reward any diligent effort. The converse is also true. A superficial understanding of the Eigenvalue problem will cost you dearly.

The Eigenvalue Problem

Recall that a tensor $\boldsymbol{T}$ is a linear transformation for $\boldsymbol{u} \in \mathscr{V}$

$$\boldsymbol{T}\colon\ \mathscr{V} \to \mathscr{V}$$

states that $\exists\ \boldsymbol{w} \in \mathscr{V}$ such that,

$$\boldsymbol{T}\boldsymbol{u} \equiv \boldsymbol{T}(\boldsymbol{u}) = \boldsymbol{w}$$

Generally, $\boldsymbol{u}$ and its image, $\boldsymbol{w}$ are independent vectors for an arbitrary tensor $\boldsymbol{T}$. The eigenvalue problem considers the special case when there is a linear dependence between $\boldsymbol{u}$ and $\boldsymbol{w}$.

Eigenvalue Problem

Here the image $\boldsymbol{w} = \lambda \boldsymbol{u}$ where $\lambda \in \mathscr{R}$

$$\boldsymbol{T}\boldsymbol{u} = \lambda \boldsymbol{u}$$

The vector $\boldsymbol{u}$, if it can be found, that satisfies the above equation, is called an eigenvector while the scalar λ is its corresponding eigenvalue.

The eigenvalue problem examines the existence of the eigenvalue and the corresponding eigenvector as well as their consequences.

In order to obtain such solutions, it is useful to write out this equation in its component form:

$$T^i_j u^j \mathbf{g}_i = \lambda u^i \mathbf{g}_i$$

so that,

$$\left(T^i_j - \lambda \delta^i_j\right) u^j \mathbf{g}_i = \mathbf{o}$$

the zero vector. Each component must vanish identically so that we can write

$$\left(T^i_j - \lambda \delta^i_j\right) u^j = 0$$

From the fundamental law of algebra, the above equations can only be possible for arbitrary values of u^j if the determinant,

$$\left|T_j^i - \lambda\delta_j^i\right|$$

Vanishes identically. Which, when written out in full, yields,

$$\begin{vmatrix} T_1^1 - \lambda & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 - \lambda & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 - \lambda \end{vmatrix} = 0$$

Expanding, we have,

$$
\begin{aligned}
&-T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 \\
&+ T_1^1T_2^2T_3^3 + T_2^1T_1^2\lambda - T_1^1T_2^2\lambda + T_3^1T_1^3\lambda + T_3^2T_2^3\lambda \\
&- T_1^1T_3^3\lambda - T_2^2T_3^3\lambda + T_1^1\lambda^2 + T_2^2\lambda^2 + T_3^3\lambda^2 - \lambda^3 = 0
\end{aligned}
$$

$$
\begin{aligned}
&= -T_3^1T_2^2T_1^3 + T_2^1T_3^2T_1^3 + T_3^1T_1^2T_2^3 - T_1^1T_3^2T_2^3 - T_2^1T_1^2T_3^3 \\
&+ T_1^1T_2^2T_3^3 \\
&+ \left(T_2^1T_1^2 - T_1^1T_2^2 + T_3^1T_1^3 + T_3^2T_2^3 - T_1^1T_3^3 - T_2^2T_3^3\right)\lambda \\
&+ \left(T_1^1 + T_2^2 + T_3^3\right)\lambda^2 - \lambda^3 = 0
\end{aligned}
$$

or

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Principal Invariants Again

* This is the characteristic equation for the tensor $\boldsymbol{T}$. From here we are able, in the best cases, to find the three eigenvalues. Each of these can be used in to obtain the corresponding eigenvector.
* The above coefficients are the same invariants we have seen earlier!

Positive Definite Tensors

A tensor $\boldsymbol{T}$ is Positive Definite if for all $\boldsymbol{u} \in \mathscr{V}$,

$$\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{u} > 0$$

It is easy to show that the eigenvalues of a symmetric, positive definite tensor are all greater than zero. (HW: Show this, and its converse that if the eigenvalues are greater than zero, the tensor is symmetric and positive definite. Hint, use the spectral decomposition.)

Cayley- Hamilton Theorem

- We now state without proof (See Dill for proof) the important ***Caley-Hamilton*** theorem: Every tensor satisfies its own characteristic equation. That is, the characteristic equation not only applies to the eigenvalues but must be satisfied by the tensor $\mathbf{T}$ itself. This means,

$$\mathbf{T}^3 - I_1\mathbf{T}^2 + I_2\mathbf{T} - I_3\mathbf{1} = \boldsymbol{O}$$

is also valid.

- This fact is used in continuum mechanics to obtain the ***spectral decomposition*** of important material and spatial tensors.

Spectral Decomposition

* It is easy to show that when the tensor is symmetric, its three eigenvalues are all real. When they are distinct, corresponding eigenvectors are orthogonal. It is therefore possible to create a basis for the tensor with an orthonormal system based on the normalized eigenvectors. This leads to what is called a *spectral decomposition* of a symmetric tensor in terms of a coordinate system formed by its eigenvectors:

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i$$

Where $\mathbf{n}_i$ is the normalized eigenvector corresponding to the eigenvalue λ_i.

Multiplicity of Roots

* The above spectral decomposition is a special case where the eigenbasis forms an Orthonormal Basis. Clearly, all symmetric tensors are diagonalizable.
* Multiplicity of roots, when it occurs robs this representation of its uniqueness because two or more coefficients of the eigenbasis are now the same.
* The uniqueness is recoverable by the ingenious device of eigenprojection.

Eigenprojectors

Case 1: All Roots equal.

* The three orthonormal eigenvectors in an ONB obviously constitutes an Identity tensor **1.** The unique spectral representation therefore becomes

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i = \lambda \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{n}_i$$

since $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ in this case.

Eigenprojectors

Case 2: Two Roots equal: $\lambda_1$unique while $\lambda_2 = \lambda_3$

In this case,

$$\mathbf{T} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 (\mathbf{1} - \mathbf{n}_1 \otimes \mathbf{n}_1)$$

since $\lambda_2 = \lambda_3$ in this case.

The eigenspace of the tensor is made up of the projectors:

$$\boldsymbol{P}_1 = \mathbf{n}_1 \otimes \mathbf{n}_1$$

and

$$\boldsymbol{P}_2 = \mathbf{1} - \mathbf{n}_2 \otimes \mathbf{n}_2$$

Eigenprojectors

The eigen projectors in all cases are based on the normalized eigenvectors of the tensor. They constitute the eigenspace even in the case of repeated roots. They can be easily shown to be:

1. Idempotent: $\boldsymbol{P}_i\,\boldsymbol{P}_i = \boldsymbol{P}_i$ (no sums)
2. Orthogonal: $\boldsymbol{P}_i\,\boldsymbol{P}_j = \boldsymbol{O}$ (the anihilator)
3. Complete: $\sum_{i=1}^{n} \boldsymbol{P}_i = \mathbf{1}$ (the identity)

Tensor Functions

* For symmetric tensors (with real eigenvalues and consequently, a defined spectral form in all cases), the tensor equivalent of real functions can easily be defined:
* Trancendental as well as other functions of tensors are defined by the following maps:

$$\boldsymbol{F}: \mathcal{Sym} \to \mathcal{Sym}$$

Maps a symmetric tensor into a symmetric tensor. The latter is the spectral form such that,

$$\boldsymbol{F}(\boldsymbol{T}) \equiv \sum_{i=1}^{3} f(\lambda_i)\mathbf{n}_i \otimes \mathbf{n}_i$$

Tensor functions

- Where $f(\lambda_i)$ is the relevant real function of the ith eigenvalue of the tensor $\boldsymbol{T}$.
- Whenever the tensor is symmetric, for any map,

$$f:\mathscr{R}\rightarrow \mathscr{R}, \quad \exists\, \boldsymbol{F}: \mathscr{Sym} \rightarrow \mathscr{Sym}$$

As defined above. The tensor function is defined uniquely through its spectral representation.

Show that the principal invariants of a tensor $\boldsymbol{S}$ satisfy $I_k(\boldsymbol{QSQ}^{\mathrm{T}}) = I_k(\boldsymbol{S}),\ k = 1,2,\text{or } 3$ Rotations and orthogonal transformations do not change the Invariants

$$I_1(\boldsymbol{QSQ}^{\mathrm{T}}) = \mathrm{tr}(\boldsymbol{QSQ}^{\mathrm{T}}) = \mathrm{tr}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{QS}) = \mathrm{tr}(\boldsymbol{S}) = I_1(\boldsymbol{S})$$

$$\begin{aligned} I_2(\boldsymbol{QSQ}^{\mathrm{T}}) &= \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{QSQ}^{\mathrm{T}}) - \mathrm{tr}(\boldsymbol{QSQ}^{\mathrm{T}}\boldsymbol{QSQ}^{\mathrm{T}})\right] \\ &= \frac{1}{2}\left[\mathrm{I}_1^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{QS}^2\boldsymbol{Q}^{\mathrm{T}})\right] \\ &= \frac{1}{2}\left[\mathrm{I}_1^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{QS}^2)\right] \\ &= \frac{1}{2}\left[\mathrm{I}_1^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)\right] = I_2(\boldsymbol{S}) \end{aligned}$$

$$\begin{aligned} I_3(\boldsymbol{QSQ}^{\mathrm{T}}) &= \det(\boldsymbol{QSQ}^{\mathrm{T}}) \\ &= \det(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{QS}) \\ &= \det(\boldsymbol{S}) = I_3(\boldsymbol{S}) \end{aligned}$$

Hence $I_k(\boldsymbol{QSQ}^{\mathrm{T}}) = I_k(\boldsymbol{S}),\ k = 1,2,\text{or } 3$

Coordinate Transformation

Given the basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ and their dual, $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$, Show that for any other basis pair, $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3$ and their dual, $\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3$, the relationship,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = \delta_\alpha^\beta$$

holds.

By simply reversing the step, it is immediately obvious that,

$$(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) = [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta$$

Observe that the expression in the parentheses is the unit tensor – having no effect on a vector; it follows that,

$$\begin{aligned}(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha)(\boldsymbol{\gamma}_i \cdot \mathbf{g}^\beta) &= [(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}^i)\mathbf{g}_\alpha] \cdot \mathbf{g}^\beta \\ &= \mathbf{g}_\alpha \cdot \mathbf{g}^\beta = \delta_\alpha^\beta\end{aligned}$$

The Principal Invariants of a Tensor are equal in all coordinate systems.

* We showed earlier that, for any tensor $\mathbf{T}$, and a set $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ of basis vectors, the first principal Invariant, the trace,

$$\mathrm{tr}(\mathbf{T}) \equiv \frac{[\{\mathbf{T}\mathbf{g}_1\}, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \{\mathbf{T}\mathbf{g}_2\}, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \{\mathbf{T}\mathbf{g}_3\}]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$

$$= \frac{T_{i1} g^{i1} \epsilon_{231} + T_{i2} g^{i2} \epsilon_{312} + T_{i3} g^{i3} \epsilon_{123}}{\epsilon_{123}}$$

$$= T_{i1} g^{i1} + T_{i2} g^{i2} + T_{i3} g^{i3} = T_{ij} g^{ij} = T_i^{.i}$$

* We proceed to show that this quantity has the same value in with another independent set of basis vectors. Let $(\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3)$ and their dual, $(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{\gamma}^3)$ be such a set. It is easy to establish that the new set will be related to the old in;

The Principal Invariants of a Tensor are equal in all coordinate systems.

$$\boldsymbol{\gamma}_j = (\boldsymbol{\gamma}_j \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha, \text{ and } \boldsymbol{\gamma}^i = (\boldsymbol{\gamma}^i \cdot \mathbf{g}_\beta)\mathbf{g}^\beta$$

Let us write the $(i,j)^{th}$ components in the $\boldsymbol{\gamma}$ − bases as $\tilde{T}_i^{.j}$.

Clearly,

$$\begin{aligned}\tilde{T}_i^{.j} = \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^j &= (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)\mathbf{g}_\alpha \cdot \left[\mathbf{T}(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta)\mathbf{g}^\beta\right] \\ &= (\boldsymbol{\gamma}_i \cdot \mathbf{g}^\alpha)(\boldsymbol{\gamma}^j \cdot \mathbf{g}_\beta)\left[\mathbf{g}_\alpha \cdot \mathbf{T}\mathbf{g}^\beta\right] \\ \tilde{T}_i^{.i} = \boldsymbol{\gamma}_i \cdot \mathbf{T}\boldsymbol{\gamma}^i &= \left[(\boldsymbol{\gamma}^i \otimes \boldsymbol{\gamma}_i)\mathbf{g}^\alpha\right] \cdot \mathbf{g}_\beta \left[T_\alpha^{.\beta}\right] \\ &= \mathbf{g}^{\boldsymbol{\alpha}} \cdot \mathbf{g}_\beta T_\alpha^{.\beta} = \delta_\beta^\alpha T_\alpha^{.\beta} = T_{.\alpha}^{\alpha}\end{aligned}$$

So that the trace has the same value in any arbitrarily chosen coordinate system including curvilinear ones whether orthogonal or not.

The Principal Invariants of a Tensor are equal in all coordinate systems.

By an earlier example, we established that the second principal Invariant,

$$I_2(\mathbf{T}) = \frac{1}{2}\left(T_\alpha^\alpha T_\beta^\beta - T_\beta^\alpha T_\alpha^\beta\right) = \frac{1}{2}\left(I_1^2(\mathbf{T}) - I_1(\mathbf{T}^2)\right)$$

That is, half of the square of the trace minus the trace of the square. These therefore are unchanged by coordinate transformation.

Lastly, an application of the Cayley Hamilton theorem shows that,

$$\begin{aligned} I_1(\boldsymbol{S}^3) &= I_1(\boldsymbol{S})I_1(\boldsymbol{S}^2) - I_2(\boldsymbol{S})I_1(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \\ &= I_1(\boldsymbol{S})\left(I_1^2(\boldsymbol{S}) - 2I_2(\boldsymbol{S})\right) - I_1(\boldsymbol{S})I_2(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \\ &= I_1^3(\boldsymbol{S}) - 3I_1(\boldsymbol{S})I_2(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \end{aligned}$$

Or, $I_3(\boldsymbol{S}) = \frac{1}{3}\left(I_1(\boldsymbol{S}^3) - I_1^3(\boldsymbol{S}) + 3I_1(\boldsymbol{S})I_2(\boldsymbol{S})\right)$

which show that the third invariant is itself expressible in terms of traces only. It is therefore invariant in value as a result of coordinate transformation.

Show that, for any tensor $\boldsymbol{S}$, $\mathrm{tr}\left(\boldsymbol{S}^2\right) = I_1^2(\boldsymbol{S}) - 2I_2(\boldsymbol{S})$ and $\mathrm{tr}\left(\boldsymbol{S}^3\right) = I_1^3(\boldsymbol{S}) - 3I_1I_2(\boldsymbol{S}) + 3I_3(\boldsymbol{S})$

$$\begin{aligned} I_2(\boldsymbol{S}) &= \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}\left(\boldsymbol{S}^2\right)\right] \\ &= \frac{1}{2}\left[I_1^2(\boldsymbol{S}) - \mathrm{tr}\left(\boldsymbol{S}^2\right)\right] \end{aligned}$$

So that,

$$\mathrm{tr}\left(\boldsymbol{S}^2\right) = I_1^2(\boldsymbol{S}) - 2I_2(\boldsymbol{S})$$

By the Cayley-Hamilton theorem,

$$\boldsymbol{S}^3 - I_1\boldsymbol{S}^2 + I_2\boldsymbol{S} - I_3\mathbf{1} = \mathbf{0}$$

Taking a trace of the above equation, we can write that,

$$\mathrm{tr}\left(\boldsymbol{S}^3 - I_1\boldsymbol{S}^2 + I_2\boldsymbol{S} - I_3\mathbf{1}\right) = \mathrm{tr}(\boldsymbol{S}^3) - I_1\mathrm{tr}\left(\boldsymbol{S}^2\right) + I_2\mathrm{tr}(\boldsymbol{S}) - 3I_3 = 0$$

so that,

$$\begin{aligned} \mathrm{tr}\left(\boldsymbol{S}^3\right) &= I_1(\boldsymbol{S})\mathrm{tr}\left(\boldsymbol{S}^2\right) - I_2(\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \\ &= I_1(\boldsymbol{S})\left(I_1^2(\boldsymbol{S}) - 2I_2(\boldsymbol{S})\right) - I_1(\boldsymbol{S})I_2(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \\ &= I_1^3(\boldsymbol{S}) - 3I_1I_2(\boldsymbol{S}) + 3I_3(\boldsymbol{S}) \end{aligned}$$

As required.

Suppose that $\mathbf{U}$ and $\mathbf{C}$ are symmetric, positive-definite tensors with $\mathbf{U}^2 = \mathbf{C}$, *write the invariants of C in terms of U*

$$I_1(\mathbf{C}) = \operatorname{tr}\left(\mathbf{U}^2\right) = I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})$$

By the Cayley-Hamilton theorem,

$$\mathbf{U}^3 - I_1\mathbf{U}^2 + I_2\mathbf{U} - I_3\mathbf{1} = \mathbf{0}$$

which contracted with $\boldsymbol{U}$ gives,

$$\mathbf{U}^4 - I_1\mathbf{U}^3 + I_2\mathbf{U}^2 - I_3\mathbf{U} = \mathbf{0}$$

so that,

$$\mathbf{U}^4 = I_1\mathbf{U}^3 - I_2\mathbf{U}^2 + I_3\mathbf{U}$$

and

$$\begin{aligned}\operatorname{tr}(\mathbf{U}^4) &= I_1\operatorname{tr}\left(\mathbf{U}^3\right) - I_2\operatorname{tr}\left(\mathbf{U}^2\right) + I_3\operatorname{tr}(\mathbf{U}) \\ &= I_1(\mathbf{U})\left(I_1^3(\mathbf{U}) - 3I_1(\mathbf{U})I_2(\mathbf{U}) + 3I_3(\mathbf{U})\right) \\ &- I_2(\mathbf{U})\left(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\right) + I_1(\mathbf{U})I_3(\mathbf{U}) \\ &= I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U}) + 4I_1(\mathbf{U})I_3(\mathbf{U})\end{aligned}$$

But,

$$I_2(\mathbf{C}) = \frac{1}{2}\left[I_1^2(\mathbf{C}) - \mathrm{tr}\left(\mathbf{C}^2\right)\right] = \frac{1}{2}\left[I_1^2\left(\mathbf{U}^2\right) - \mathrm{tr}(\mathbf{U}^4)\right]$$

$$= \frac{1}{2}\left[\mathrm{tr}^2\left(\mathbf{U}^2\right) - \mathrm{tr}(\mathbf{U}^4)\right]$$

$$= \frac{1}{2}\left[\left(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\right)^2 - \mathrm{tr}(\mathbf{U}^4)\right]$$

$$= \frac{1}{2}\Big[\boxed{I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U})} + 4I_2^2(\mathbf{U})$$