1. Given that $\forall\ \mathbf{v} \in \mathbf{V}$, $\boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\boldsymbol{a} = \mathbf{b}$

We are given that $\forall\ \mathbf{v} \in \mathscr{V}$, $\boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$ this implies,

$$\boldsymbol{a} \cdot \mathbf{v} - \mathbf{b} \cdot \mathbf{v} = (\boldsymbol{a} - \mathbf{b}) \cdot \mathbf{v} = 0$$

Define the vector $\mathbf{c} \equiv \boldsymbol{a} - \mathbf{b}$. The equation becomes,

$$\mathbf{c} \cdot \mathbf{v} = \|\mathbf{c}\| \|\mathbf{v}\| \cos\theta = 0.$$

Because $\mathbf{v}$ can be any vector, it does not have to be perpendicular to $\mathbf{c}$ and we can rule out the trivial case of its being the zero vector. This leaves us with the only choice that $\|\mathbf{c}\| = 0$. And, the only vector that has zero magnitude is the zero vector. So that,

$\mathbf{c} \equiv \boldsymbol{a} - \mathbf{b} = \mathbf{o}$, or $\boldsymbol{a} = \mathbf{b}$.

2. Given that $\forall\ \mathbf{v} \in \mathbf{V}$, $\boldsymbol{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\boldsymbol{a} = \mathbf{b}$

Starting with $\mathbf{c} \equiv \boldsymbol{a} - \mathbf{b}$

We could have arrived at the same conclusion from

$$\mathbf{c} \cdot \mathbf{v} = 0$$

by choosing $\mathbf{v} = \mathbf{c}$. We can do this because the equation is valid for every vector $\mathbf{v}$. It must therefore be valid for a particular vector.

The equation then becomes,

$$\mathbf{c} \cdot \mathbf{c} = 0$$

which, again is only possible if $\mathbf{c} = \mathbf{o}$, the zero vector.

Hence $\boldsymbol{a} = \mathbf{b}$.

3. Given that $\forall\ \mathbf{v} \in V$, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$, show that $\boldsymbol{a} = \mathbf{b}$

We are given that $\forall\ \mathbf{v} \in V$, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$,

Now take a dot product with $\boldsymbol{a}$, we have that,

$$\boldsymbol{a} \cdot \mathbf{b} \times \mathbf{v} = (\boldsymbol{a} \times \mathbf{b}) \cdot \mathbf{v} = 0 = \mathbf{o} \cdot \mathbf{v}$$

for all $\mathbf{v}$ proving from Quiz 1.1 that $\boldsymbol{a} \times \mathbf{b} = \mathbf{o}$. This shows that $\boldsymbol{a}$ and $\mathbf{b}$ are collinear. We can therefore write that $\mathbf{b} = \alpha\boldsymbol{a}$

Hence, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} = \alpha\boldsymbol{a} \times \mathbf{v}$ where α is a scalar. So that

$$(\boldsymbol{a} \times \mathbf{v})(1 - \alpha) = \mathbf{o} \Rightarrow 1 = \alpha$$

showing that $\boldsymbol{a} = \mathbf{b}$ as was required.

4. Simplify the following by employing the substitution properties of the Kronecker delta

$(a) e_{ijk}\delta_{kn}, (b) e_{ijk}\delta_{is}\delta_{jm}$ $(c) e_{ijk}\delta_{is}\delta_{jm}$ $(d)\ a_{ij}\delta_{in}$ $(e)\delta_{ij}\delta_{jn}$ $(f)\delta_{ij}\delta_{jn}\delta_{ni}$

$(a)\ e_{ijn}$ $(b) e_{sjk}$ $(c) e_{smn}$ $(d) a_{nj} (e)\delta_{nj}$ $(f)\delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \cdots + \delta_{33} = 3$

5. Given that, $I_{ij} = \iiint_V \left(x^m x^m \delta_{ij} - x^i x^j\right) \rho(x^1, x^2, x^3) dx^1 dx^2 dx^3$ is the moment of inertia along thebaxis $i - j$ where $x = x^1, y = x^2, z = x^3$ and $\rho(x^1, x^2, x^3)$ is scalar density of the material find all the components of the tensor.

$$I_{11} = \iiint_V (y^2 + z^2)\, \rho(x, y, z) dxdydz, \quad I_{21} = I_{12} = \iiint_V xy\, \rho(x, y, z) dxdydz,$$

$$I_{22} = \iiint_V (z^2 + x^2)\, \rho(x, y, z) dxdydz, \quad I_{32} = I_{23} = \iiint_V yz\, \rho(x, y, z) dxdydz,$$

$$I_{31} = I_{13} = \iiint_V xy\, \rho(x, y, z) dxdydz, \quad I_{33} = \iiint_V (x^2 + y^2)\, \rho(x, y, z) dxdydz$$

6. Show that the Cylindrical Polar basis vectors,

$$\boldsymbol{e}_r(r, \phi, z) = \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}$$

$$\boldsymbol{e}_\phi(r, \phi, z) = -\sin\phi\, \boldsymbol{i} + \cos\phi\, \boldsymbol{j}$$

$$\boldsymbol{e}_z(r, \phi, z) = \boldsymbol{k}$$

constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

$$\|\boldsymbol{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$

$$\left\|\boldsymbol{e}_\phi\right\|^2 = \sin^2\phi + \cos^2\phi = 1$$

$$\|\boldsymbol{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs:

$$\boldsymbol{e}_r \cdot \boldsymbol{e}_\phi = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$
$$\boldsymbol{e}_\phi \cdot \boldsymbol{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 \ = 0$$
$$\boldsymbol{e}_z \cdot \boldsymbol{e}_r = \cos\phi \times 0 + \ \sin\phi \times 0 + 1 \times 0 = 0$$

So that they are pairwise orthogonal.

7. Show that the Spherical Polar basis vectors

$$\boldsymbol{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$$
$$\boldsymbol{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}$$
$$\boldsymbol{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}.$$

Constitute an orthonormal system. [**Hint:** Show their magnitudes are unity and they are pairwise orthogonal].

Follow the same procedure as in the above question and obtain the similar result for the spherical polar case.

8. Find the derivatives of all the basis vectors above.

9. Given that the position vector in spherical coordinates is given by $\boldsymbol{R} = \rho \boldsymbol{e}_\rho(\theta, \phi)$, where $\boldsymbol{e}_\rho = \sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$ show that the set $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \theta}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\}$ forms a basis set of orthogonal vectors. This is called the natural basis for the coordinate system. Normalize them to form an orthonormal (physical) basis.

$$\frac{\partial \boldsymbol{R}}{\partial \rho} = \boldsymbol{e}_\rho,$$

$$\frac{\partial \boldsymbol{R}}{\partial \theta} = \rho\frac{\partial \boldsymbol{e}_\rho}{\partial \theta} = \rho\frac{\partial}{\partial \theta}(\sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})$$

$$= \rho(\cos\theta\cos\phi\,\boldsymbol{i} + \cos\theta\sin\phi\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k}) = \rho\boldsymbol{e}_\theta.$$

$$\frac{\partial \boldsymbol{R}}{\partial \phi} = \rho\frac{\partial \boldsymbol{e}_\rho}{\partial \phi} = \rho\frac{\partial}{\partial \phi}(\sin\theta\cos\phi\,\boldsymbol{i} + \sin\theta\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})$$

$$= \rho(-\sin\theta\sin\phi\,\boldsymbol{i} + \sin\theta\cos\phi\,\boldsymbol{j})$$

$$= \rho\sin\theta\,\boldsymbol{e}_\phi.$$

From these, we can see that $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \theta}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\} = \{\boldsymbol{e}_\rho, \rho\boldsymbol{e}_\theta, \rho\sin\theta\,\boldsymbol{e}_\phi\}$. Obviously, the magnitudes are $\{1, \rho, \rho\sin\theta\}$ respectively. Consequently, this basis set can be normalized to $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi\}$

10. Given that the position vector in cylindrical polar coordinates is given by $\boldsymbol{R} = r\boldsymbol{e}_r + z\boldsymbol{e}_z$, where $\boldsymbol{e}_r = \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}$, and $\boldsymbol{e}_z = \boldsymbol{k}$ show that the set $\left\{\frac{\partial \boldsymbol{R}}{\partial r}, \frac{\partial \boldsymbol{R}}{\partial \phi}, \frac{\partial \boldsymbol{R}}{\partial z}\right\}$ forms a basis set of orthogonal vectors. This is called the natural basis for the coordinate system. Normalize them to form an orthonormal basis (the physical basis).

$$\frac{\partial \boldsymbol{R}}{\partial r} = \boldsymbol{e}_r,$$
$$\frac{\partial \boldsymbol{R}}{\partial \phi} = r\frac{\partial \boldsymbol{e}_r}{\partial \phi} = r\frac{\partial}{\partial \phi}(\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}) = r(-\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}) = r\boldsymbol{e}_\phi$$
$$\frac{\partial \boldsymbol{R}}{\partial z} = \boldsymbol{e}_z$$

From these, we can see that $\left\{\frac{\partial \boldsymbol{R}}{\partial \rho}, \frac{\partial \boldsymbol{R}}{\partial \phi}, \frac{\partial \boldsymbol{R}}{\partial \phi}\right\} = \{\boldsymbol{e}_r, r\boldsymbol{e}_\phi, \boldsymbol{e}_z\}$. Obviously, the magnitudes are $\{1, r, 1\}$ respectively. Consequently, this basis set can be normalized to $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$.

11. For Cartesian Coordinates, show that the natural basis coincides with the physical basis. [**Hint**: Obtain the natural basis from the set, $\left\{\frac{\partial \mathbf{R}}{\partial x}, \frac{\partial \mathbf{R}}{\partial y}, \frac{\partial \mathbf{R}}{\partial z}\right\}$. The physical basis is the normalized natural basis.]

12. Show that the contraction of a symmetric object with an antisymmetric object equals zero. For example given that $a_{mn}, m, n = 1,2,3$ is antisymmetric, Show that $a_{mn}x^m x^n = 0$.

(a) It is easily seen that $x^m x^n = x^n x^m$ hence symmetric. If a_{mn} is anti-symmetric, then the contraction $a_{mn}x^m x^n$ must necessarily vanish.

(b) Given that $a_{mn}x^m x^n = 0$ for arbitrary values of $x^n, n = 1,2,3$ then we can write,

$$a_{mn}x^m x^n = -a_{mn}x^m x^n$$

because zero is also a negative of itself. Swapping the roles of x^m and x^n on the RHS of the above, we can write,

$$\begin{aligned} a_{mn}x^m x^n &= -a_{mn}x^m x^n \\ &= -a_{mn}x^n x^m \\ &= -a_{nm}x^n x^m \end{aligned}$$

after swapping the roles of the two dummy indices. We therefore consolidate on the LHS by writing,

$$a_{mn}x^m x^n + a_{nm}x^n x^m = 0$$
$$(a_{mn} + a_{nm})x^m x^n = 0$$

Notice that the quantity in the parenthesis is always symmetric. And also note the contraction of two symmetric tensors can only vanish if one or both tensors vanish. Here, $x^m x^n$ is a product of arbitrary tensors. We are left with the fact that

$$a_{mn} + a_{nm} = 0$$

or,

$$a_{mn} = -a_{nm}$$

which is the definition of anti-symmetry.

13. The angle $0 \leq \theta \leq \pi$ between two skew lines in space is defined as the angle between their direction vectors when these vectors are placed at the origin. Show that for two lines with direction numbers a_i and $b_i, i = 1;\ 2;\ 3$ the cosine of the angle between these lines satisfies

$$\cos\theta = \frac{a_i b_i}{\sqrt{(a_i a_i)}\sqrt{(b_i b_i)}}$$

First note that the dot product of the two line vectors is,

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i = |a||b|\cos\theta = \sqrt{(a_i a_i)}\sqrt{(b_i b_i)}\cos\theta$$

From where it is obvious that,

$$\cos\theta = \frac{a_i b_i}{\sqrt{(a_i a_i)}\sqrt{(b_i b_i)}}$$

as required.

14. Let $\lambda = A_{ij}x_i x_j$ where $A_{ij} = A_{ji}$. Calculate $(a)\,\frac{\partial\lambda}{\partial x_m}, (b)\,\frac{\partial^2\lambda}{\partial x_m \partial x_k}$

$$\lambda = A_{ij}x_i x_j = A_{mk}x_m x_k$$

$$\frac{\partial\lambda}{\partial x_m} = A_{mk}x_k + A_{lk}x_l\frac{\partial x_k}{\partial x_m}$$

$$= A_{mk}x_k + A_{lk}x_l\delta_{km} = A_{mk}x_k + A_{lk}x_l\delta_{km}$$

$$= A_{mk}x_k + A_{lm}x_l = 2A_{mk}x_k, \text{ and furthermore,}$$

$$\frac{\partial^2\lambda}{\partial x_m \partial x_k} = 2A_{mk}.$$

Remember that in the above we made use of the liberty to alter the dummy indices to conform to the requirements of the derivative. The substitutionary attribute of the Kronecker delta has been used to advantage here.

15. If $A_{ij} = A_i B_j \neq 0 \; \forall \; i, j$ values and $A_{ij} = A_{ji}$ for $i, j = 1,2, \dots, N$ Show that $A_{ij} = \lambda B_i B_j$ where λ is constant. Find λ.

We are given that

$$A_{ij} = A_i B_j = A_j B_i$$

Since the composition is symmetrical. Let us contract this quantity with B_j. We obtain,

$$A_i B_j B_j = A_j B_i B_j \text{ so that,}$$

$$A_i = B_i \left(\frac{A_k B_k}{B_j B_j} \right) = B_i \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{B} \cdot \mathbf{B}} \right) = \lambda B_i$$

It is correct to divide the equation by $B_j B_j$ since it is a scalar sum and hence a scalar. Division of indexed objects is not defined and hence not permissible. Yet we can write the last equation out in full component form as,

$$A_1 = \lambda B_1, A_2 = \lambda B_2, \ldots, A_N = \lambda B_N$$

It is obvious from the above that,

$$\lambda = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \cdots = \frac{A_N}{B_N}$$

16. Let $x_i = a_{ij}\bar{x}_j \;\; i,j = 1,2,3.$ denote a change of variables from a barred system of coordinates to an unbarred system and assume that $A_i = a_{ij}A_j$ where a_{ij} are constants. $\bar{A}_{\boldsymbol{i}}$ is a function of the $\bar{x}_j$ variables and A_j is a function of the x_i variables. Calculate $\frac{\partial \bar{A}_i}{\partial \bar{x}_m}$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}_m} = \frac{\partial}{\partial \bar{x}_m}\left(a_{ij}A_j\right) \text{ recall that } a_{ij} \text{is a constant, so that,}$$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}_m} = a_{ij}\frac{\partial A_j}{\partial \bar{x}_m} = a_{ij}\frac{\partial A_j}{\partial x_k}\frac{\partial x_k}{\partial \bar{x}_m} = a_{ij}a_{km}\frac{\partial A_j}{\partial x_k}$$

The last equality coming from the fact that, $x_k = a_{km}\bar{x}_m$ so that $\frac{\partial x_k}{\partial \bar{x}_m} = a_{km}$.

Note that the dummy indices may change in any convenient way. The free indices of this expression are i and m.

17. Show that, $e^{rst}e_{ijk} = \delta^{rst}_{ijk} = \begin{vmatrix} \delta^r_i & \delta^r_j & \delta^r_k \\ \delta^s_i & \delta^s_j & \delta^s_k \\ \delta^t_i & \delta^t_j & \delta^t_k \end{vmatrix}$

Proof: Let the determinant of a^i_j be $\begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix}$. We now interchange rows or columns and obtain,

$$-\text{Det}\left(a^i_j\right) = e^{213}\text{Det}\left(a^i_j\right) = \begin{vmatrix} a^2_1 & a^2_2 & a^2_3 \\ a^1_1 & a^1_2 & a^1_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix} = \begin{vmatrix} a^1_2 & a^1_1 & a^1_3 \\ a^2_2 & a^2_1 & a^2_3 \\ a^3_2 & a^3_1 & a^3_3 \end{vmatrix}.$$

Clearly, for an arbitrary number of row interchanges, we have

$$e^{rst}\text{Det}\left(a^i_j\right) = \begin{vmatrix} a^r_1 & a^r_2 & a^r_3 \\ a^s_1 & a^s_2 & a^s_3 \\ a^t_1 & a^t_2 & a^t_3 \end{vmatrix}$$

Or column changes, we have,

$$e_{ijk}\mathrm{Det}\left(a_j^i\right) = \begin{vmatrix} a_i^1 & a_j^1 & a_k^1 \\ a_i^2 & a_j^2 & a_k^2 \\ a_i^3 & a_j^3 & a_k^3 \end{vmatrix}.$$

For an arbitrary sequence of row and column changes, we easily see that:

$$e_{ijk}e^{rst}\ \mathrm{Det}\left(a_j^i\right) = \begin{vmatrix} a_i^r & a_j^r & a_k^r \\ a_i^s & a_j^s & a_k^s \\ a_i^t & a_j^t & a_k^t \end{vmatrix} \qquad (1).$$

Now let $a_j^i = \delta_j^i$,

We immediately obtain,

$$e_{ijk}e^{rst} = \delta_{rst}^{ijk} = \begin{vmatrix} \delta_i^r & \delta_j^r & \delta_k^r \\ \delta_i^s & \delta_j^s & \delta_k^s \\ \delta_i^t & \delta_j^t & \delta_k^t \end{vmatrix} \qquad (2)$$

18. Use a symbolics processor to calculate the conjugate metric tensor given that, $\mathbf{g}_1 \cdot \mathbf{g}_2 = \mathbf{g}_2 \cdot \mathbf{g}_1 = \cos \vartheta_{12}$, $\mathbf{g}_2 \cdot \mathbf{g}_3 = \mathbf{g}_3 \cdot \mathbf{g}_3 = \cos \vartheta_{23}$ and $\mathbf{g}_3 \cdot \mathbf{g}_1 = \mathbf{g}_1 \cdot \mathbf{g}_3 = \cos \vartheta_{13}$

The matrix

$$[g_{ij}] = \begin{bmatrix} 1 & \cos \vartheta_{12} & \cos \vartheta_{13} \\ \cos \vartheta_{12} & 1 & \cos \vartheta_{23} \\ \cos \vartheta_{13} & \cos \vartheta_{23} & 1 \end{bmatrix}$$

Now the conjugate metric tensor has a matrix that is the inverse of the above. First consider the fact that the determinant of $[g_{ij}]$ obtained by direct evaluation, is:

$$\Delta \equiv |g_{ij}| = 1 - \cos^2 \vartheta_{12} - \cos^2 \vartheta_{23} - \cos^2 \vartheta_{13} + 2 \cos \vartheta_{12} \cos \vartheta_{13} \cos \vartheta_{23}$$

The Inverse matrix (using a symbolic algebra processor) is therefore,

$$[g^{ij}] = \frac{1}{\Delta} \begin{bmatrix} 1 - c^2 \vartheta_{23} & -c\vartheta_{12} + c\vartheta_{13} c\vartheta_{23} & -c\vartheta_{13} + c\vartheta_{12} c\vartheta_{23} \\ -c\vartheta_{12} + c\vartheta_{13} c\vartheta_{23} & 1 - c^2 \vartheta_{13} & -c\vartheta_{23} + c\vartheta_{12} c\vartheta_{13} \\ -c\vartheta_{13} + c\vartheta_{12} c\vartheta_{23} & -c\vartheta_{23} + c\vartheta_{12} c\vartheta_{13} & 1 - c^2 \vartheta_{12} \end{bmatrix}$$

Where for simplicity we have written, $c\vartheta \equiv \cos\vartheta$

Notice that the metric tensor is always symmetric. Why must this be so?

This symmetry is obvious from its definition: $g_{ij} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j}$ since the dot product is commutative.

19. When a coordinate system is orthogonal, $g^{ij} = g_{ij} = 0$ whenever $i \neq j$. Show that, $g_{11} = \frac{1}{g^{11}}$, $g_{22} = \frac{1}{g^{22}}$ and that $g_{33} = \frac{1}{g^{33}}$.

Proof: In the last chapter, we proved that, $g^{ij} g_{jk} = \delta^i_k$. Expanding the implied sum, we have,

$g^{i1} g_{1k} + g^{i2} g_{2k} + g^{i3} g_{3k} = \delta^i_k$. Let $i = 1$, recalling the assumption of orthogonality, we have, $g^{11} g_{1k} + g^{12} g_{2k} + g^{13} g_{3k} = \delta^1_k$ which yields $g^{11} g_{1k} = \delta^1_k$ which equals 1 when $k = 1$. We have therefore proved that $g^{11} g_{11} = \delta^1_1 = 1$. which proves the first expression. The remaining two can obviously be proved if we started with $i = 2 \; or \; i = 3$ in the above proof.

We have therefore shown that, for orthogonal systems, $g_{11} = \frac{1}{g^{11}}$, $g_{22} = \frac{1}{g^{22}}$ and that $g_{33} = \frac{1}{g^{33}}$.

Given the fact that the quadratic form,

$ds^2 = g_{ij}dx^i dx^j$ expands to $g_{11}(dx^1)^2 + g_{11}(dx^2)^2 + g_{11}(dx^3)^2$ for orthogonal systems, we adopt the convention that $g_{11} = (h_1)^2,\ g_{22} = (h_2)^2$ and that $g_{33} = (h_3)^2$. From the above arguments, this obviously means that, $g^{11} = \frac{1}{(h_1)^2}, g^{22} = \frac{1}{(h_2)^2}$ *and* $g^{33} = \frac{1}{(h_3)^2}$. We must take care to note when superscripts imply exponentiation. To avoid ambiguity, we can always use parenthesis when this is so.

20. The trilinear mapping, $[\ ,\ ,\ ]: \mathscr{V} \times \mathscr{V} \times \mathscr{V} \to \mathscr{R}$ from the product set $\mathscr{V} \times \mathscr{V} \times \mathscr{V}$ to real space is defined by: $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

In component form,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon^{ijk} a_i b_j c_k$$

Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield,

$$\begin{aligned} \epsilon^{jki} b_j c_k a_i = [\mathbf{b}, \mathbf{c}, \mathbf{a}] &= \epsilon^{ijk} b_i c_j a_k \\ = \epsilon^{kij} c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] &= \epsilon^{ijk} c_i a_j b_k \end{aligned}$$

The other results follow from antisymmetric arrangements and the nature of ϵ^{ijk}.

21. The transformation equations from the Cartesian to the oblate spheroidal coordinates ξ, η and φ are: $x = f\xi\eta \sin\varphi$, $y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}$, and $z = f\xi\eta \cos\varphi$, where f is a constant representing the half the distance between the foci of a family of confocal ellipses. Find the components of the metric tensor in this system.

The metric tensor components are:

$$g_{\xi\xi} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2$$

$$= (f\eta \sin\varphi)^2 + f^2\xi^2\left(\frac{1-\eta^2}{\xi^2-1}\right) + (f\eta\cos\varphi)^2 = f^2\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)$$

$$g_{\eta\eta} = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 = f^2\left(\frac{\xi^2-\eta^2}{1-\xi^2}\right)$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = (f\xi\eta)^2$$

$$g_{\xi\eta} = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial z}{\partial \xi}\right)\left(\frac{\partial z}{\partial \eta}\right)$$

$$= (f\eta \sin\varphi)(f\xi \sin\varphi) - \left(f\eta\sqrt{\frac{\xi^2 - 1}{1 - \eta^2}}\right)\left(f\xi\sqrt{\frac{1 - \eta^2}{\xi^2 - 1}}\right) + (f\eta\cos\varphi)(f\xi\cos\varphi)$$

$$= 0 = g_{\eta\varphi} = g_{\varphi\xi}$$

22. Show that the oblate spheroidal coordinate systems are orthogonal. Find an expression for the Laplacian of a scalar function in this system.

Example above shows that $g_{\xi\eta} = g_{\eta\varphi} = g_{\varphi\xi} = 0$. This is the required proof of orthogonality. Using the computation formula in example 11, we may write for the oblate spheroidal coordinates that,

$$\nabla^2\Phi = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{f^3\xi^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(f\xi\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\frac{\partial\Phi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(f\xi\eta\sqrt{\frac{1-\eta^2}{\xi^2-1}}\frac{\partial\Phi}{\partial\eta}\right)\right]$$
$$+\frac{\partial}{\partial\eta}\left(\frac{f(\xi^2-\eta^2)}{\xi\eta\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial\Phi}{\partial\eta}\right)$$

23. Given that, $[\mathbf{a},\mathbf{b},\mathbf{c}] \equiv \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a},\mathbf{b},\mathbf{c})$ are linearly dependent.

Suppose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that,

$$\begin{aligned}[\mathbf{a},\mathbf{b},\mathbf{c}] &= \epsilon^{ijk}a_ib_jc_k = \epsilon^{ijk}(\alpha b_i + \beta c_i)b_jc_k \\ &= \alpha\epsilon^{ijk}b_ib_jc_k + \beta\epsilon^{ijk}c_ib_jc_k \\ &= 0\end{aligned}$$

Note that $b_ib_jc_k$ is symmetric in i and j, $c_ib_jc_k$ is symmetric in i and k and ϵ^{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.

24. Show that the product of a symmetric and an antisymmetric object vanishes.

Consider the product sum, $\epsilon^{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and ϵ^{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here.

$$\epsilon^{ijk} b_i b_j c_k = -\epsilon^{jik} b_i b_j c_k = -\epsilon^{jik} b_j b_i c_k = -\epsilon^{ijk} b_i b_j c_k = 0$$

The first equality on account of the antisymmetry of ϵ^{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself.

This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.

25. Show that the product $\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \mathbf{B}$ Can be written in indicial notation as, $a_{ij} a_{jk} = b_{ik}$.

To show this, apply summation convention and see that, for $i = 1,\ k = 1,\ a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$

for $i = 1,\ k = 2,\ a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$

for $i = 1,\ k = 3,\ a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$

for $i = 2,\ k = 1,\ a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$

for $i = 2,\ k = 2,\ a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$

for $i = 2,\ k = 3,\ a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$

for $i = 3,\ k = 1,\ a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$

for $i = 3,\ k = 2,\ a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$

for $i = 3,\ k = 3,\ a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$

It is necessary to go through this manual process to gain the experience of seeing this transparently in future. Similary, $\mathbf{AA}^{\mathrm{T}} = \mathbf{B}$ can be written in indicial notation as, $a_{ij}a_{kj} = b_{ik}$ which again becomes clear after a manual expansion after invoking the summation convention.

26. Given that $\mathbf{g}_i, i = 1,2,3$ are linearly independent vectors forming a basis; and that $\mathbf{g}^j, j = 1,2,3$ are dual vectors defined by the reciprocity relationship, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$. Show that $\mathbf{g}^j = g^{ij}\mathbf{g}_i = g^{ji}\mathbf{g}_i$ and establish the relation, $g_{ij}g^{jk} = \delta_i^k$

Since the $\mathbf{g}_i$s form a basis, any other vector can be expressed in terms of them. In particular, the dual vectors $\mathbf{g}^j$ can be written in terms of the $\mathbf{g}_i$s:

$$\mathbf{g}^j = \alpha\mathbf{g}_1 + \beta\mathbf{g}_2 + \gamma\mathbf{g}_3$$

We now find the coefficients on this basis:

Dotting with $\mathbf{g}^1 \Rightarrow \ \mathbf{g}^j \cdot \mathbf{g}^1 = \alpha\mathbf{g}_1 \cdot \mathbf{g}^1 + \beta\mathbf{g}_2 \cdot \mathbf{g}^1 + \gamma\mathbf{g}_3 \cdot \mathbf{g}^1 = g^{j1} = \alpha$ after invojing the reciprocity relationships. In the same way we find that $\beta = g^{j2}$ and $\gamma = g^{j3}$so that,

$$\mathbf{g}^j = g^{j1}\mathbf{g}_1 + g^{j2}\mathbf{g}_2 + g^{j3}\mathbf{g}_3 = g^{ji}\mathbf{g}_i.$$

Similarly, using the dual as basis, we can find an expression for the $\mathbf{g}_i$s: $\mathbf{g}_i = g_{i\alpha}\mathbf{g}^\alpha$ using the same reciprocity relationships: $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k$. Using the above, we can write

$$\mathbf{g}_i \cdot \mathbf{g}^k = (g_{i\alpha}\mathbf{g}^\alpha) \cdot \left(g^{k\beta}\mathbf{g}_\beta\right) = g_{i\alpha}g^{k\beta}\mathbf{g}^\alpha \cdot \mathbf{g}_\beta = g_{i\alpha}g^{k\beta}\delta_\beta^\alpha = \delta_i^k$$

which shows that

$$g_{i\alpha}g^{k\alpha} = g_{ij}g^{jk} = \delta_i^k$$

As required. This shows that the tensors represented by g_{ij} and g^{ij} are inverses of each other. One is called the metric tensor and the other the associated metric tensor. We shall see later that these are simply the components of the unit tensor in different coordinate systems.

27. For the basis vectors $\mathbf{g}_i, i = 1,2,3$ and their duals, $\mathbf{g}^j, j = 1,2,3$ If $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$ and $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3$, show that $vV = 1$.

Given $\mathbf{g}_i, i = 1,2,3$, note that $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$ and to $\mathbf{g}_3$. It must be parallel to the vector $\mathbf{g}_2 \times \mathbf{g}_3$. A scalar constant V^{-1} must exist such that,

$$\mathbf{g}^1 = V^{-1}\,\mathbf{g}_2 \times \mathbf{g}_3$$
$$\mathbf{g}^2 = V^{-1}\,\mathbf{g}_3 \times \mathbf{g}_1$$
$$\mathbf{g}^3 = V^{-1}\,\mathbf{g}_1 \times \mathbf{g}_2$$

Since (dot the first with $\mathbf{g}_1$ to see) Now we are given that $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$. Using the above relations, we can write,

$$
\begin{aligned}
\mathbf{g}^2 \times \mathbf{g}^3 &= (V^{-1}\,\mathbf{g}_3 \times\ \mathbf{g}_1) \times (V^{-1}\,\mathbf{g}_1 \times\ \mathbf{g}_2) \\
&= V^{-2}[(\,\mathbf{g}_3 \times\ \mathbf{g}_1 \cdot\ \ \mathbf{g}_2)\mathbf{g}_1 - (\,\mathbf{g}_3 \times\ \mathbf{g}_1 \cdot\ \ \mathbf{g}_1)\mathbf{g}_2] \\
&= V^{-2}(\,\mathbf{g}_1 \times\ \mathbf{g}_2 \cdot\ \ \mathbf{g}_3)\mathbf{g}_1 = V^{-1}\mathbf{g}_1
\end{aligned}
$$

We can now write,

$$v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3 = \mathbf{g}^1 \cdot V^{-1}\mathbf{g}_1 = V^{-1}\mathbf{g}^1 \cdot \mathbf{g}_1 = V^{-1}$$

Showing that, $v\,V = 1$ as required. It is a trivial matter to show that $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3$, for, if we take a dot product of the equation, $\mathbf{g}^1 = V^{-1}\,\mathbf{g}_2 \times\ \mathbf{g}_3$, the result follows so that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \frac{1}{\mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3}$.

28. Given the position vector $\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i = \mathbf{r}(u^1, u^2, u^3)$, by transforming into the coordinate system spanned by the basis vectors, $\mathbf{g}_i$ defined by, $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i}\,du^i \equiv \mathbf{g}_i du^i$, show that $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, where g is the determinant $|g_{ij}|$.

Changing variables, we can write that,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i = \mathbf{r}(u^1, u^2, u^3)$$

So that we have new coordinates $u^k, k = 1,2,3$. In this new system, the differential of the position vector $\boldsymbol{r}$ is,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i}\, du^i \equiv \mathbf{g}_i du^i$$

the above equation, as we shall soon show, defines the natural basis vectors in the new coordinate system. The vectors $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ are not necessarily unit vectors but they form a basis of the new system provided,

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \neq 0$$

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial x^k}{\partial u^i} \mathbf{e}_k$$

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \\ \frac{\partial x^1}{\partial u^3} & \frac{\partial x^2}{\partial u^3} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \left| \frac{\partial x^k}{\partial u^i} \right| \neq 0$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} = \left(\frac{\partial x^k}{\partial u^i} \mathbf{e}_k \right) \cdot \left(\frac{\partial x^l}{\partial u^j} \mathbf{e}_l \right)$$

$$= \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\mathbf{e}_k \cdot \mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\delta_{kl} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}$$

Clearly, the determinant of g_{ij} (we shall prove later that the determinant of a product of matrices is the product of the determinants)

$$g \equiv |g_{ij}| = \left|\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}\right| = \left|\frac{\partial x^k}{\partial u^i}\right|^2 = V^2$$

This means, $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \left|\frac{\partial x^i}{\partial u^j}\right| = \sqrt{g}$. We can therefore write,

$$\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = e_{123}\sqrt{g}$$

Swapping indices 2 and 3, we have,

$$\mathbf{g}_1 \cdot \mathbf{g}_3 \times \mathbf{g}_2 = -\sqrt{g} = e_{132}\sqrt{g} = \mathbf{g}_1 \times \mathbf{g}_3 \cdot \mathbf{g}_2$$

The second equality coming from the fact that swapping the cross with the dot changes nothing. Lastly, swapping 1 and 3 in the last equation shows that,

$\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2 = -\left(-\sqrt{g}\right) = e_{312}\sqrt{g}$. These three expressions together imply that,

$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g} e_{ijk}$ as required.

29. Given that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, where g is the determinant $|g_{ij}|$. Show that, $\mathbf{g}_i \times \mathbf{g}_j \cdot \mathbf{g}_k = \sqrt{g} e_{ijk} \equiv \epsilon_{ijk}$. Conclude further that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k$

Given that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, the fact that the triple product obeys the rule, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$, combined with the fact that the triple product vanishes when any two of its vectors are collinear allow us to write that

$$\mathbf{g}_i \times \mathbf{g}_j \cdot \mathbf{g}_k = e_{ijk} \sqrt{g} \equiv \epsilon_{ijk}$$

By the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, we have that, $\mathbf{g}_1 \cdot \mathbf{g}^1 = 1, \mathbf{g}_1 \cdot \mathbf{g}^2 = 0$, $\mathbf{g}_1 \cdot \mathbf{g}^3 = 0$. It follows that $\mathbf{g}^1$ must be perpendicular to the plane of $\mathbf{g}_2$and $\mathbf{g}_3$ making it parallel to $\mathbf{g}_2 \times \mathbf{g}_3$ A scalar constant α must exist such that, $\mathbf{g}^1 = \alpha\, \mathbf{g}_2 \times \mathbf{g}_3$. Dot product of both sides with $\mathbf{g}_1$ shows that $\alpha = 1/\sqrt{g}$. Therefore,

$$\mathbf{g}_2 \times \mathbf{g}_3 = \sqrt{g} \mathbf{g}^1$$
$$\mathbf{g}_3 \times \mathbf{g}_1 = \sqrt{g} \mathbf{g}^2$$

$$\mathbf{g}_1 \times \mathbf{g}_2 = \sqrt{g}\mathbf{g}^3$$

These three results together can be expressed as, $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$.

30. Use the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$ and the fact that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \frac{1}{\mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3}. = \sqrt{g}$ to show that $\mathbf{g}^i \times \mathbf{g}^j = \frac{1}{\sqrt{g}} e^{ijk}\mathbf{g}_k = \epsilon^{ijk}\mathbf{g}_k$.

By the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, we have that, $\mathbf{g}_1 \cdot \mathbf{g}^1 = 1$, $\mathbf{g}_2 \cdot \mathbf{g}^1 = 0$, $\mathbf{g}_3 \cdot \mathbf{g}^1 = 0$. It follows that $\mathbf{g}_1$ must be perpendicular to the plane of $\mathbf{g}^2$ and $\mathbf{g}^3$ making it parallel to $\mathbf{g}^2 \times \mathbf{g}^3$. A scalar constant β must exist such that, $\mathbf{g}_1 = \beta\mathbf{g}^2 \times \mathbf{g}^3$. Dot product of both sides with $\mathbf{g}_1$ shows that $\beta = \sqrt{g}$. Therefore,

$$\mathbf{g}^2 \times \mathbf{g}^3 = \frac{1}{\sqrt{g}}\mathbf{g}_1$$

$$\mathbf{g}^3 \times \mathbf{g}^1 = \frac{1}{\sqrt{g}}\mathbf{g}_2$$

$$\mathbf{g}^1 \times \mathbf{g}^2 = \frac{1}{\sqrt{g}}\mathbf{g}_3$$

These three results together can be written as, $\mathbf{g}^i \times \mathbf{g}^j = \frac{1}{\sqrt{g}} e^{ijk} \mathbf{g}_k = \epsilon^{ijk} \mathbf{g}_k$ if we write $\epsilon^{ijk} \equiv \frac{1}{\sqrt{g}} e^{ijk}$.

31. Given that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k$, and that $\epsilon^{ij\alpha} \epsilon_{ijk} = 2\delta_i^\alpha$, Find an expression for $\mathbf{g}^k$ in terms of its dual vectors.

take a contraction with $\epsilon^{ij\alpha}$ and find the expression for $\mathbf{g}^k$

$$\begin{aligned} \epsilon^{ij\alpha} \mathbf{g}_i \times \mathbf{g}_j &= \epsilon^{ij\alpha} \epsilon_{ijk} \mathbf{g}^k \\ &= 2\delta_k^\alpha \mathbf{g}^k = 2\mathbf{g}^\alpha \end{aligned}$$

So that $\mathbf{g}^i = \frac{1}{2} \epsilon^{ijk} \mathbf{g}_j \times \mathbf{g}_k$

32. Show that the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in general coordinates is $a^i b^j \epsilon_{ijk} \mathbf{g}^k$ or $\epsilon^{ijk} a_i b_j \mathbf{g}_k$ where a^i, b^j are the respective contravariant components and a_i, b_j the covariant.

Express vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as contravariant components: $\boldsymbol{a} = a^i \mathbf{g}_i$, and $\boldsymbol{b} = b^i \mathbf{g}_i$. Using the above result, we can write that,

$$\boldsymbol{a} \times \boldsymbol{b} = (a^i \mathbf{g}_i) \times (b^j \mathbf{g}_j) = a^i b^j \mathbf{g}_i \times \mathbf{g}_j = a^i b^j \epsilon_{ijk} \mathbf{g}^k.$$

Express vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as covariant components: $\boldsymbol{a} = a_i \mathbf{g}^i$ and $\boldsymbol{b} = b_i \mathbf{g}^i$. Again, proceeding as before, we can write,

$$\boldsymbol{a} \times \boldsymbol{b} = \left(a_i \mathbf{g}^i\right) \times \left(b_j \mathbf{g}^j\right) = \epsilon^{ijk} a_i b_j \mathbf{g}_k$$

Express vectors $\boldsymbol{a}$ as contravariant components: $\boldsymbol{a} = a^i \boldsymbol{g}_i$ and $\boldsymbol{b}$ as covariant components: $\boldsymbol{b} = b_i \boldsymbol{g}^i$

$$\boldsymbol{a} \times \boldsymbol{b} = \left(a^i \mathbf{g}_i\right) \times \left(b_j \mathbf{g}^j\right) = a^i b_j \left(\mathbf{g}_i \times \mathbf{g}^j\right)$$

33. If $\forall\ \mathbf{v} \in \mathscr{V}, \mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$, Show that $\mathbf{a} = \mathbf{b}$; If $\forall\ \mathbf{v} \in \mathscr{V}, \mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$, Show that $\mathbf{a} = \mathbf{b}$

$\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} \Rightarrow (\mathbf{a} - \mathbf{b})\mathbf{v} = \mathbf{o}$. Since $\mathbf{v}$ is arbitrary, let $\mathbf{v} = \mathbf{a} - \mathbf{b}$. We therefore have

$$|\mathbf{a} - \mathbf{b}|^2 = 0$$
$$\Rightarrow\ \mathbf{a} - \mathbf{b} = \mathbf{o}$$

It is only the zero vector that has a magnitude of zero. Therefore $\mathbf{a} = \mathbf{b}$.

Secondly, we are given that $\forall\ \mathbf{v} \in \mathscr{V}, \boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$,

Now take a dot product with $\boldsymbol{a}$, we have that,

$$\boldsymbol{a} \cdot \mathbf{b} \times \mathbf{v} = (\boldsymbol{a} \times \mathbf{b}) \cdot \mathbf{v} = 0 = \mathbf{o} \cdot \mathbf{v}$$

for all $\mathbf{v}$ proving from the first part, that $\boldsymbol{a} \times \mathbf{b} = \mathbf{o}$. This shows that $\boldsymbol{a} \times \mathbf{b}$ are collinear. We can therefore write that $\mathbf{b} = \alpha \boldsymbol{a}$

Hence, $\boldsymbol{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} = \alpha \boldsymbol{a} \times \mathbf{v}$ where α is a scalar. So that

$$(\boldsymbol{a} \times \mathbf{v})(1 - \alpha) = \mathbf{o} \Rightarrow 1 = \alpha$$

showing that $\boldsymbol{a} = \mathbf{b}$ as was required.

34. Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$

$$\begin{aligned}(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} - [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{w}]\mathbf{v} \\ &= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} \\ &= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{w} \\ &= (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

Consequently,

$$\begin{aligned}[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] &= \mathbf{e} \cdot [(\mathbf{e} \times \mathbf{u}) \times (\mathbf{e} \times \mathbf{v})] \\ &= \mathbf{e} \cdot [(\mathbf{e} \otimes \mathbf{e})(\mathbf{u} \times \mathbf{v})] \\ &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{e} \otimes \mathbf{e})\mathbf{e} \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e} = [\mathbf{e}, \mathbf{u}, \mathbf{v}]\end{aligned}$$

making use of the symmetry of $(\mathbf{e} \otimes \mathbf{e})$.

35. Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, using the result, $(\mathbf{w}\times\mathbf{u})\times(\mathbf{w}\times\mathbf{v}) = (\mathbf{w}\otimes\mathbf{w})(\mathbf{u}\times\mathbf{v})$, show that $[(\mathbf{u}\times\mathbf{v}),(\mathbf{v}\times\mathbf{w}),(\mathbf{w}\times\mathbf{u})] = [\mathbf{u},\mathbf{v},\mathbf{w}]^2$

From the given result,

$$\begin{aligned}[(\mathbf{u}\times\mathbf{v}),(\mathbf{v}\times\mathbf{w}),(\mathbf{w}\times\mathbf{u})] &= -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{u}) \\ &= -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{w})(\mathbf{v}\times\mathbf{u}) \\ &= (\mathbf{u}\times\mathbf{v})\big((\mathbf{w}\cdot\mathbf{u}\times\mathbf{v})\mathbf{w}\big) \\ &= [\mathbf{u},\mathbf{v},\mathbf{w}]^2\end{aligned}$$

36. In the transformation from the (x,y,z) system to the $(r,\ \phi,Z)$ coordinate system, the position vector changed from $\mathbf{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ to $\mathbf{R} = r\boldsymbol{e}_r(\phi) + Z\boldsymbol{e}_z$. Show by partial differentiation only, that the basis vectors in respective coordinates are $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ and $\{\boldsymbol{e}_r,\boldsymbol{e}_\phi,\boldsymbol{e}_z\}$ respectively, $\boldsymbol{e}_\phi(\phi) = r\frac{\partial \boldsymbol{e}_r(\phi)}{\partial\phi}$

In the transformation from the (x,y,z) system to the $(r,\ \phi,Z)$ coordinate system, the position vector changed from $\mathbf{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ to $\mathbf{R} = r\boldsymbol{e}_r(\phi) + Z\boldsymbol{e}_z$. Show by partial differentiation only, that the basis vectors in respective coordinates are $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ and $\{\boldsymbol{e}_r,\boldsymbol{e}_\phi,\boldsymbol{e}_z\}$ respectively, $\boldsymbol{e}_\phi(\phi) =$

$r\frac{\partial \boldsymbol{e}_r(\phi)}{\partial \phi}$.

$$\mathbf{R} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Differentiating with respect to x, y and z respectively shows

$$R(\text{x\_}, \text{y\_}, \text{z\_}) = ix + jy + kz$$

$$\frac{\partial R(x, y, z)}{\partial \{x, y, z\}} = \{i, j, k\}$$

Similarly, $\mathbf{R} = r\boldsymbol{e}_r(\phi) + Z\boldsymbol{e}_z, \frac{\partial \mathbf{R}}{\partial r} = \boldsymbol{e}_r(\phi); \frac{\partial \mathbf{R}}{\partial \phi} = r\frac{\partial \boldsymbol{e}_r(\phi)}{\partial \phi} \equiv e_\phi$ and $\frac{\partial \mathbf{R}}{\partial \phi} = \boldsymbol{e}_z$

37. If the position vector in another system with coordinate variables $(\rho, \phi, \theta)$is $\mathbf{R} = \rho\boldsymbol{e}_\rho(\phi, \theta)$, use partial differentiation to find the basis vectors in this system also.

If the position vector in another system with coordinate variables $(\rho, \phi, \theta)$is $\mathbf{R} = \rho\boldsymbol{e}_\rho(\phi, \theta)$, use the same method to find the basis vectors in this system also.

Similar to the examples above,

$$\mathbf{R} = \rho\boldsymbol{e}_\rho(\phi, \theta)$$

$$\mathbf{g}_1 = \frac{\partial \mathbf{R}}{\partial \rho} = \boldsymbol{e}_\rho(\phi, \theta),$$
$$\mathbf{g}_2 = \frac{\partial \mathbf{R}}{\partial \theta} = \rho \frac{\partial \boldsymbol{e}_\rho(\phi, \theta)}{\partial \theta} \equiv \boldsymbol{e}_\theta$$
$$\mathbf{g}_3 = \frac{\partial \mathbf{R}}{\partial \phi} = \rho \frac{\partial \boldsymbol{e}_\rho(\phi, \theta)}{\partial \phi} \equiv \boldsymbol{e}_\phi$$

because we are told that $\boldsymbol{e}_\rho$ is a function of both ϕ and θ. This shows that the basis vectors, rather than being constants, vary with ϕ and θ throughout the domain. These can no longer be treated as constants like the Cartesian basis vectors.

38. The transformation from Cartesian to the other system is given explicitly as $x(r, \phi, Z) = r\cos\phi$, $y(r, \phi, Z) = r\sin\phi$ and $z(r, \phi, Z) = Z$, find explicit expression for the basis vectors $\mathbf{g}_i,\ i = 1,2,3$. Also find the reciprocal basis vectors $\mathbf{g}^j,\ j = 1,2,3$. [Hint: $2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$]

Mathematica code to make these differentiations easy. Here is how it looks like:

```
R[r_, ϕ_, Z_] := i r Cos[ϕ] + j r Sin[ϕ] + k Z

g = D[R[r, ϕ, Z], {{r, ϕ, Z}}]
{i Cos[ϕ] + j Sin[ϕ], j r Cos[ϕ] - i r Sin[ϕ], k}
```

It is clear that,

$$\mathbf{g}_1 = \frac{\partial \mathbf{R}}{\partial r}$$
$$= \boldsymbol{e}_r(\phi) = \mathbf{i}\text{Cos}[\phi] + \mathbf{j}\text{Sin}[\phi]$$
$$\mathbf{g}_2 = \frac{\partial \mathbf{R}}{\partial \phi} = r\frac{\partial \boldsymbol{e}_r(\phi)}{\partial \phi}$$
$$\equiv e_\phi = \mathbf{j}r\text{Cos}[\phi] - \mathbf{i}r\text{Sin}[\phi]$$
$$\text{and } \mathbf{g}_3 = \frac{\partial \mathbf{R}}{\partial Z} = \boldsymbol{e}_z = \mathbf{k}$$

$$2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$$

so that

$$\mathbf{g}^1 = \frac{1}{2}\epsilon^{1jk}\mathbf{g}_j \times \mathbf{g}_k$$

In this double sum, only two out of the nine terms survive; these are:

$$\mathbf{g}^1 = \frac{1}{2}\epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3 + \frac{1}{2}\epsilon^{132}\mathbf{g}_3 \times \mathbf{g}_2 = \epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3$$
$$= (\mathbf{j}r\mathrm{Cos}[\phi] - \mathbf{i}r\mathrm{Sin}[\phi]) \times \mathbf{k} = -\mathbf{i}\, r\mathrm{Cos}[\phi] - \mathbf{j}r\mathrm{Sin}[\phi]$$

39. Find the dual bases for the Cartesian system.

To find the dual contravariant basis, we use the formula,

$$2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$$

Let us call the covariant basis (it is customary to label the basis obtained by direct differentiation covariant) of the Cartesian $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the contravariant basis $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. Now, $\mathbf{g}^1 = \frac{1}{2}\epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3 + \frac{1}{2}\epsilon^{132}\mathbf{g}_3 \times \mathbf{g}_2 = \epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3$ so that, $\mathbf{I} = \mathbf{j} \times \mathbf{k} = \mathbf{i}$; $\mathbf{J} = \mathbf{k} \times \mathbf{i} = \mathbf{j}$; and $\mathbf{K} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$. This shows that for the Cartesian system, the dual bases coincide and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$or $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} = \{\mathbf{I}, \mathbf{J}, \mathbf{K}\} or\ \{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$. Both systems are orthogonal and normalized.

40. Find the reciprocal bases for the spherical coordinate systems. Are they orthogonal? Are they normalized?

The transformation equations for the spherical system are,

$$R(\rho\_, \theta\_, \phi\_) := \boldsymbol{i}\rho\sin(\theta)\cos(\phi) + \boldsymbol{j}\rho\sin(\theta)\sin(\phi) + \boldsymbol{k}\rho\cos(\theta)$$

The natural basis can be found using the Mathematica code:

```
R[ρ_, θ_, ϕ_] := i ρ Sin[θ] Cos[ϕ] + j ρ Sin[θ] Sin[ϕ] + k ρ Cos[θ]

g = ∂{{ρ,θ,ϕ}} R[ρ, θ, ϕ]
{k Cos[θ] + i Cos[ϕ] Sin[θ] + j Sin[θ] Sin[ϕ], i ρ Cos[θ] Cos[ϕ] - k ρ

KroneckerProduct[g, g] // MatrixForm
                          (k Cos[θ] + i Cos[ϕ] Sin[θ] + j Sin[θ] Sin
 (i ρ Cos[θ] Cos[ϕ] - k ρ Sin[θ] + j ρ Cos[θ] Sin[ϕ]) (k Cos[θ] + i Co
        (k Cos[θ] + i Cos[ϕ] Sin[θ] + j Sin[θ] Sin[ϕ]) (j ρ Cos[ϕ] Si

D[{ρ Cos[ϕ] Sin[θ], ρ Sin[ϕ] Sin[θ], ρ Cos[θ]}, {{ρ, θ, ϕ}}]
{{Cos[ϕ] Sin[θ], ρ Cos[θ] Cos[ϕ], -ρ Sin[θ] Sin[ϕ]}, {Sin[θ] Sin[

Transpose[%] . %
{{Cos[θ]^2 + Cos[ϕ]^2 Sin[θ]^2 + Sin[θ]^2 Sin[ϕ]^2, -ρ Cos[θ] Sin[θ] + ρ (
 {-ρ Cos[θ] Sin[θ] + ρ Cos[θ] Cos[ϕ]^2 Sin[θ] + ρ Cos[θ] Sin[θ] Sin[ϕ

Simplify[%] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin[\theta]^2 \end{pmatrix}$$

$$\mathbf{g}_1 = \frac{\partial \mathbf{R}}{\partial \rho} = \boldsymbol{e}_\rho(\phi, \theta) = \{\boldsymbol{i}\sin(\theta)\cos(\phi) + \boldsymbol{j}\sin(\theta)\sin(\phi) + \boldsymbol{k}\cos(\theta)\}$$

$$\mathbf{g}_2 = \frac{\partial \mathbf{R}}{\partial \theta} = \rho \frac{\partial \boldsymbol{e}_\rho(\phi, \theta)}{\partial \theta} \equiv e_\theta(\phi, \theta)$$

$$= \boldsymbol{i}\rho\cos(\theta)\cos(\phi) + \boldsymbol{j}\rho\cos(\theta)\sin(\phi) - \boldsymbol{k}\rho\sin(\theta)$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{R}}{\partial \phi} \equiv e_\phi(\phi, \theta) = \boldsymbol{j}\rho\sin(\theta)\cos(\phi) - \boldsymbol{i}\rho\sin(\theta)\sin(\phi)$$

Using the same formula as above, it is easy to see that $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$ for this system are $\boldsymbol{i}\sin(\theta)\cos(\phi) + \boldsymbol{j}\sin(\theta)\sin(\phi) + \boldsymbol{k}\cos(\theta)$

$$(\boldsymbol{i}\cos(\theta)\cos(\phi) + \boldsymbol{j}\cos(\theta)\sin(\phi) - \boldsymbol{k}\sin(\theta))/\rho$$

and

41. Show that curl $\mathbf{u} \times \mathbf{v} = (\mathbf{v} \cdot \text{grad}\,\mathbf{u}) + (\mathbf{u} \cdot \text{div}\,\mathbf{v}) - (\mathbf{v} \cdot \text{div}\,\mathbf{u}) - (\mathbf{u} \cdot \text{grad}\,\mathbf{v})$

Taking the associated (covariant) vector of the expression for the cross product in the last example, it is straightforward to see that the LHS in indicial notation is,

$$\epsilon^{lmi}\left(\epsilon_{ijk}u^j v^k\right)_{,m}$$

Expanding in the usual way, noting the relation between the alternating tensors and the Kronecker deltas,

$$\begin{aligned}
\epsilon^{lmi}\left(\epsilon_{ijk}u^j v^k\right)_{,m} &= \delta^{lmi}_{jki}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right) \\
&= \delta^{lm}_{jk}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right) \\
&= \begin{vmatrix} \delta^l_j & \delta^m_j \\ \delta^l_k & \delta^m_k \end{vmatrix}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right) \\
&= \left(\delta^l_j\delta^m_k - \delta^l_k\delta^m_j\right)\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right) \\
&= \delta^l_j\delta^m_k u^j{}_{,m}v^k - \delta^l_j\delta^m_k u^j v^k{}_{,m} + \delta^l_k\delta^m_j u^j{}_{,m}v^k \\
&\quad - \delta^l_k\delta^m_j u^j v^k{}_{,m} \\
&= u^l{}_{,m}v^m - u^m{}_{,m}v^l + u^l v^m{}_{,m} - u^m v^l{}_{,m}
\end{aligned}$$

Which is the result we seek in indicial notation.

42. . In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i \mathbf{e}_i$. Show that (a) $x_{,j} = \frac{x_j}{x}$, (b) $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$, (c) $x_{,ii} = \frac{2}{x}$, (d) If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ $U_{,ii} = 0$ and $\mathrm{div}\left(\frac{\mathbf{r}}{x}\right) = \frac{2}{x}$. (in the notation here comma denotes differentiation with respect to the coordinate in question: $x_{,2} = \frac{\partial x}{\partial x^2}$ the derivative wrt the second coordinate.)

$$(a)\quad x = \sqrt{x_i x_i}$$

$$x_{,j} = \frac{\partial\sqrt{x_i x_i}}{\partial x_j} = \frac{\partial\sqrt{x_i x_i}}{\partial(x_i x_i)} \times \frac{\partial(x_i x_i)}{\partial x_j} = \frac{1}{2\sqrt{x_i x_i}}\left[x_i \delta_{ij} + x_i \delta_{ij}\right] = \frac{x_j}{x}.$$

$$(b)\ x_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial\sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right) = \frac{x\frac{\partial x_i}{\partial x_j} - x_i\frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2}$$

$$= \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$$

$$(c)\quad x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3} = \frac{3}{x} - \frac{(x)^2}{(x)^3} = \frac{2}{x}.$$

(d) $U = \frac{1}{x}$ so that

$$U_{,j} = \frac{\partial \frac{1}{x}}{\partial x_j} = \frac{\partial \frac{1}{x}}{\partial x} \times \frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$

Consequently,

$$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right) = \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i \frac{\partial}{\partial x_j}(x^3)}{x^6}$$

$$= \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial(x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6} = \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6}$$

$$= \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

$$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5} = \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$

$$\operatorname{div}\left(\frac{\mathbf{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x}x_{j,j} + \left(\frac{1}{x}\right)_{,j} = \frac{3}{x} + x_j\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\frac{dx}{dx_j}\right)$$

$$= \frac{3}{x} + x_j \left[-\left(\frac{1}{x^2}\right) \frac{x_j}{x} \right] = \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$