The Inverse Method

2-Dimensional Elasticity

Methods of Solution

* The governing equations of 2-D elasticity include the following:
* 1. Balance Equations (Momentum & Angular Momentum) giving us the Cauchy's equations of Motion.
* 2. Compatibility Equations. Solving the problem of finding six strains out of three displacements in a consistent manner.

Methods of Solution

* 3. Stress-Strain Relations – essentially a generalization of the Hooke's law for multi-dimensional situations.

These can be expressed in terms of strains – (Navier's equations) or interms of stresses – (Beltrami-Michell)

They are partial differential equations even in the simplest cases.

It is a huge task to attempt to solve them directly even though there are examples of such.

Working From Answer to Question

* One surprisingly fruitful approach can best be described as “Working from answer to question” or the Inverse method.
* In this method, you find a solution to the governing equations and then search for the specific problem that that solution applies to.
* Funny as it looks, this is one of the most important ways to obtain solutions to Elasticity problems.

Inverse Method for Rectangular Domains

* In two dimensions, we have shown that the Governing equations when there are no body forces can be expressed in the biharmonic equations:

$$\nabla^4 \phi = 0$$

* In 2-D Cartesian coordinates, this becomes,

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Polynomial Solutions

Clearly,

$$\phi = \phi(x, y)$$

* And the simplest functions we shall consider for $\phi(x, y)$ are polynomials in x and y: that is, functions that can generally be represented as,

$$\phi(x, y) = A_{00} + A_{10}x + A_{01}y + A_{11}xy + A_{20}x^2 + A_{02}y^2 + \cdots$$

Or,

$$\phi(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} A_{mn}x^m y^n$$

The first three terms are inconsequential in the generation of stresses and are usually left out. $\phi(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn}x^m y^n$.

Polynomial Solutions

We then obtain the stresses from the Airy relations,

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \text{ and } \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Substituting the polynomials in the bi harmonic equations we obtain, after differentiating four times with respect to each variable and twice with respect to each respectively, that,

Polynomial Powers

* $\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = \sum_{m=4}^{M}\sum_{n=0}^{N} m(m-1)(m-2)(m-3)A_{mn}x^{m-4}y^n$

$$+2\sum_{m=2}^{M}\sum_{n=2}^{N} m(m-1)n(n-1)A_{mn}x^{m-2}y^{n-2}$$

$$+\sum_{m=0}^{M}\sum_{n=4}^{N} n(n-1)(n-2)(n-3)A_{mn}x^{m}y^{n-4} = 0$$

* $= \sum_{m=2}^{M}\sum_{n=2}^{N}(m+2)(m+1)m(m-1)A_{m+2,n-2}x^{m-2}y^{n-2}$

$$+2\sum_{m=2}^{M}\sum_{n=2}^{N} m(m-1)n(n-1)A_{mn}x^{m-2}y^{n-2}$$

$$+\sum_{m=2}^{M}\sum_{n=2}^{N}(n+2)(n+1)n(n-1)A_{m-2,n+2}x^{m-2}y^{n+2} = 0$$

* So that for each mn pair, the typical term,

$$(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2} = 0$$

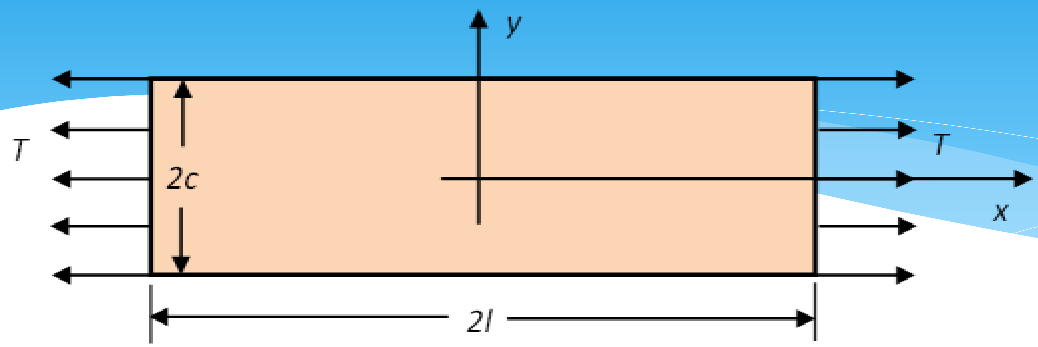
Polynomial Solutions

* In specific cases, we find the relations for these constants that will make the function satisfy the Biharmonic equations.
* Method produces polynomial stress distributions, and thus would not satisfy general boundary conditions. However, using Saint-Venant's principle we can replace a non-polynomial condition with a statically equivalent polynomial loading. This formulation is most useful for problems with rectangular domains, and is commonly based on inverse solution concept where we assume a polynomial solution form and then try to find what problem it will solve.

St Venant’s Principle

* The stresses and strains in a body at points that are sufficiently remote from points of application of load depends only on the static resultant of the loads and not on the distribution of loads, or,
* "... the difference between the effects of two different but statically equivalent loads becomes very small at sufficiently large distances from load."

1. Beam in Uniaxial Tension



Boundary Conditions

$\sigma_x(\pm l, y) = T, \sigma_y(x, \pm c) = 0,$
$\tau_{xy}(\pm l, y) = \tau_{xy}(x, \pm c) = 0$

In this case, as seen above, the boundary conditions are all constants; It makes sense to select $M = N = 2$.

$$\phi = A_{02} y^2$$

From Airy equations, we have,

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 2A_{02};\ \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0;\ \tau_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

These stresses are constant throughout the body. In particular, at $x = \pm l$, $\sigma_x = 2A_{02} = T \Rightarrow A_{02} = \frac{T}{2}$

1. Displacement Fields.

* The displacement fields we obtain depends on what kind of 2-D situation we are dealing with. If we assume this is a thin lamina with the third dimension that is small, we assume plane stress so that,

$$e_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right] = \frac{T}{E};\ e_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] = -\frac{\nu T}{E};\ e_z = -\nu\frac{T}{E};\ e_{xy} = 0$$

$$e_x = \frac{\partial u}{\partial x} = \frac{T}{E} \Rightarrow u = \frac{T}{E}x + f(y); e_y = \frac{\partial v}{\partial y} \Rightarrow v = -\nu\frac{T}{E}y + g(x);$$

$$e_{xy} = 0 \Rightarrow \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial f(y)}{\partial y} + \frac{\partial g(x)}{\partial x} = 0$$

so that each function is linear. Hence, $g(x) = a_1x + a_2;\ f(y) = -a_1y + a_3$

1. Rigid Body Displacements

Three constants require three conditions. If we can further assume that the body is prevented from moving or rotating at the origin, then $u(0,0) = v(0,0) = 0$, so that,

$$u(x,y)]_{(0,0)} = \frac{T}{E}(0) - a_1(0) + a_3 \Rightarrow \; a_3 = 0, \text{and}$$

$$v(x,y)]_{(0,0)} = -\nu\frac{T}{E}(0) + a_1(0) + a_2 = 0 \Rightarrow \; a_2 = 0$$

The vanishing of the rotation about the z −axis means that,

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

The rotation vector is the vector cross of the rotation tensor: $-\frac{1}{2}\epsilon^{ijk}w_{jk}$

Therefore, $\frac{df(y)}{dy} - \frac{dg(x)}{dx} = 0$, or $a_1 = -a_1 \Rightarrow \; a_1 = 0$. Under these conditions,

$$u = \frac{T}{E}x \text{ and } v = -\nu\frac{T}{E}y$$

Rotation Tensor, Vector

* The second order tensor $\nabla\mathbf{u}$ can be broken into two components:

$$\nabla\mathbf{u} = \boldsymbol{\epsilon} + \boldsymbol{\omega}$$

Where, the symmetric part, $\boldsymbol{\epsilon}$ is the small-strain tensor, and the antisymmetric part, $\boldsymbol{\omega}$ is the rotation tensor. Clearly,

$$\nabla^{\mathrm{T}}\mathbf{u} = \boldsymbol{\epsilon} - \boldsymbol{\omega}$$

So that,

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla^{\mathrm{T}}\mathbf{u}\right) \textbf{ and } \boldsymbol{\omega} = \frac{1}{2}\left(\nabla\mathbf{u} - \nabla^{\mathrm{T}}\mathbf{u}\right)$$

Rotation Tensor

- In component terms, the rotation tensor is,

$$\omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

Since $\boldsymbol{\omega} = \frac{1}{2}\left(\nabla \mathbf{u} - \nabla^{\mathrm{T}} \mathbf{u}\right)$ is antisymmetric, there exists an axial vector $\mathbf{w}$ defined as the vector cross of this tensor:

$$\boldsymbol{\omega} = \mathbf{w} \times$$

Again, in index notation,

$$\omega_{ij} = -\epsilon_{ijk} w^k$$

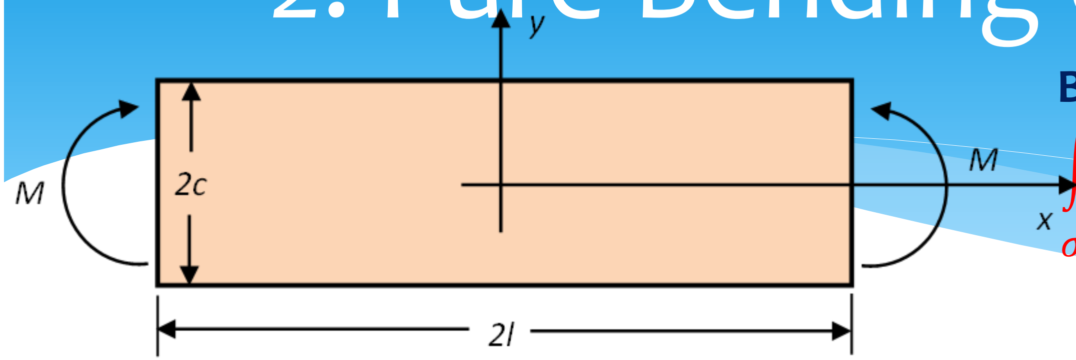
Which can be easily inverted to obtain Slide 63 Continuum Mechanics,

$$w^i = -\frac{1}{2}\epsilon^{ijk}\omega_{jk}$$

In particular, the rotation about z −axis in the previous slide is

$$w^3 = -\frac{1}{2}\epsilon^{3jk}\omega_{jk} = -\frac{1}{2}\left(\epsilon^{312}\omega_{12} - \epsilon^{321}\omega_{21}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

2. Pure Bending of a Beam



Boundary Conditions

$$\int_{-c}^{c} \sigma_x(\pm l, y)dy = 0\,, \int_{-c}^{c} \sigma_x(\pm l, y)ydy = -M$$

$$\sigma_y(x, \pm c) = 0, \tau_{xy}(\pm l, y) = \tau_{xy}(x, \pm c) = 0$$

This is a well known solution from elementary theory. The stress distribution along the x-plane is linear. We are therefore led by this consideration to try a third-order Airy function,

$$\phi = A_{03}y^3$$

Clearly, $\sigma_x = \frac{\partial^2\phi}{\partial y^2} = 6A_{03}y;\ \sigma_y = \frac{\partial^2\phi}{\partial x^2} = 0;\ \tau_{xy} = \frac{\partial^2\phi}{\partial x\partial y} = 0$. Assuming plane stress, the displacements are calculated as before:

$$\int_{-c}^{c} \sigma_x(\pm l, y)dy = \int_{-c}^{c} 6A_{03}ydy = [3A_{03}y^2]_{-c}^{c} = 3A_{03}y^2 - 3A_{03}y^2 = 0$$

identically

2. Strains & Displacements

$$\int_{-c}^{c} 6A_{03}y^2 dy = 2A_{03}y^3\Big|_{-c}^{c} = 4A_{03}c^3 = -M$$

So that, $A_{03} = -\frac{M}{4c^3}$. Consequently, $\sigma_x = -\frac{3M}{2c^3}y$. Integrating the strain-displacement equations as before, using the plane stress relations, we easily see that,

$$e_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right] = -\frac{3M}{2Ec^3}y;\ e_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] = \frac{3\nu My}{2Ec^3};\ e_{xy} = 0$$

$$e_x = \frac{\partial u}{\partial x} = \frac{3M}{2Ec^3}y \Rightarrow u = -\frac{3M}{2Ec^3}yx + f(y); e_y = \frac{\partial v}{\partial y} \Rightarrow v = \frac{3\nu My^2}{4Ec^3} + g(x);$$

In terms of the moment of area, $I = \frac{2c^3}{3}$ for unit thickness we have,

$$u = -\frac{M}{EI}xy + f(y);\ v = \frac{\nu My^2}{2EI} + g(x)$$

2. Strains & Displacements

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{M}{EI}x + f'(y) + g'(x) = 0$$

So that, $-\frac{M}{EI}x + g'(x) = -f'(y) = a_1$ where a_1 is an arbitrary constant.

Clearly,

$$g(x) = \frac{M}{2EI}x^2 + a_1 x + a_3, \text{ and}$$
$$f(y) = -a_1 y + a_2$$

As usual, we find the three constants by specifying kinematic conditions. If we assume that there is no vertical displacement at either end and also there is no horizontal displacement at the left end, $v(\pm l, 0) = 0$ and $u(-l, 0) = 0$ respectively,

2. Displacement Field

$$u(x,y)\big|_{(-l,0)} = -\frac{M}{EI}xy - a_1 y + a_2\Big|_{(-l,0)} = a_2 = 0$$

$$v(x,y)\big|_{(-l,0)} = \frac{\nu M y^2}{2EI} + \frac{M}{2EI}x^2 + a_1 x + a_3\Big|_{(-l,0)}$$

$$\Rightarrow a_3 = -\frac{\nu M l^2}{2EI}$$

And subtracting the equation for $v(x,y)|_{(l,0)}$ from this yields $a_1 = 0$

Consequently,

$$u(x,y) = -\frac{M}{EI}xy; \text{and } v(x,y) = \frac{M}{2EI}\left(x^2 + \nu y^2 - l^2\right)$$

2. Comparison with Elementary Theory

Elasticity

$$\sigma_x = -\frac{M}{I}y; \sigma_y = \tau_{xy} = 0$$

$$u(x, y) = -\frac{M}{EI}xy;$$

$$v(x, y) = \frac{M}{2EI}\left(x^2 + \nu y^2 - l^2\right)$$

Strength of Materials

$$\sigma_x = -\frac{M}{I}y; \sigma_y = \tau_{xy} = 0$$

$$v(x, 0) = \frac{M}{2EI}\left(x^2 - l^2\right)$$

The elementary solution gives the values on the elastic. At this line the solutions are identical.

Elasticity & Strength of Materials

- The elementary solution gives the values on the elastic
- We note the consistency of the two solutions when they provide values.

Other Polynomials

Consider

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

In this case, by our usual notation, we have made

$$A_{22} = \frac{a_2}{2}, A_{11} = b_2 \text{ and } A_{02} = \frac{c_2}{2}$$

To satisfy the biharmonic equation,

$$\nabla^2\nabla^2\phi = \frac{\partial^4\phi}{\partial x^4} + \frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0,$$

We must have,

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = c_2, \sigma_y = \frac{\partial^2\phi}{\partial x^2} = a_2, \text{ and } \tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -b_2$$

These stresses are constant throughout the body.

Other Polynomials

Now consider,

$$\phi_3 = \frac{a_3}{3 \times 2}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{3 \times 2}y^3$$

Again,

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = c_3 x + d_3 y, \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = a_3 x + b_3 y,$$

$$\text{and } \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -b_3 x - c_3 y$$

If we take a stress function

$$\phi_4 = \frac{a_4}{4 \times 3}x^4 + \frac{b_4}{3 \times 2}x^3 y + \frac{c_4}{2}x^2 y^2 + \frac{d_4}{3 \times 2}xy^3 + \frac{e_4}{4 \times 3}y^4$$

Substituting into the equation,

$$\nabla^2 \nabla^2 \phi = \frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$2a_4 + 4c_4 + 2e_4 = 0$$

Other Polynomials

So that,

$$e_4 = -(2c_4 + a_4)$$

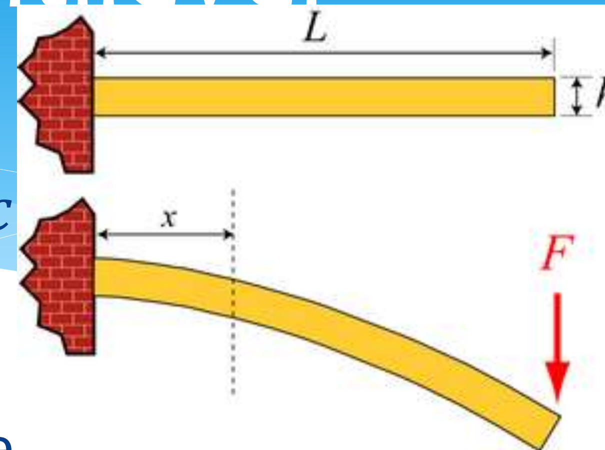
The generated stress components In this case are:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4)$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -b_4 x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

3. Bending of a Cantilever



* Consider a cantilever loaded at the end by a point load F. Assume $h = 2c$
* If we assume that the surfaces are otherwise free of tractions.

Let $c_4 = a_4 = b_4 = e_4 = 0$ in the above stress function and allowing $a_2 = c_2 = 0$,

We have,

$$\phi = \phi_2 + \phi_4 = \frac{d_4}{3 \times 2} xy^3 + b_2 xy$$

And for the stresses, $\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = d_4 xy$, $\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0$, $\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{d_4}{2} y^2 - b_2$

Cantilever

* The condition that

$$\tau_{xy}(x, \pm c) = 0 \Rightarrow -b_2 - \frac{d_4}{2}c^2 = 0$$

Giving, $d_4 = -\frac{2b_2}{c^2}$.

To satisfy the condition that the load at the end must equal the applied load F, we can demand, $-\int_{-c}^{c} \tau_{xy} dy \ = -\int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = F$

Therefore $b_2 = \frac{3F}{4c}$

We can therefore write,

$$\sigma_x = -\frac{Fx}{I}, \sigma_y = 0, \text{and } \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{3F}{4c}\left(1 - \frac{1}{c^2} y^2\right)$$

Reading Assignment

* Articles 19-26 in Timoshenko and Goodier gives details of classical solutions in Cartesian coordinates.
* Pages 158-168 in Sadd's book treat some of these same solutions.
* Read these and prepare for Polar coordinates solutions in the next class.
* Three weeks time, Exams