2-D Elasticity

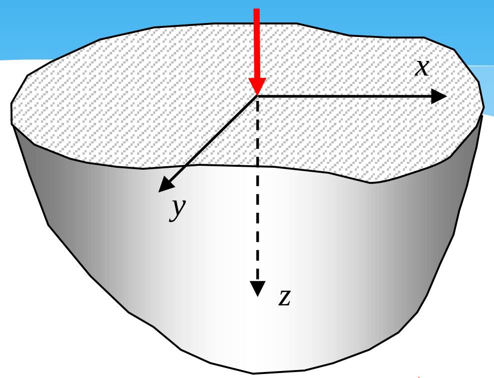
Plane Stress, Plane Strain, Anti-plane Stress

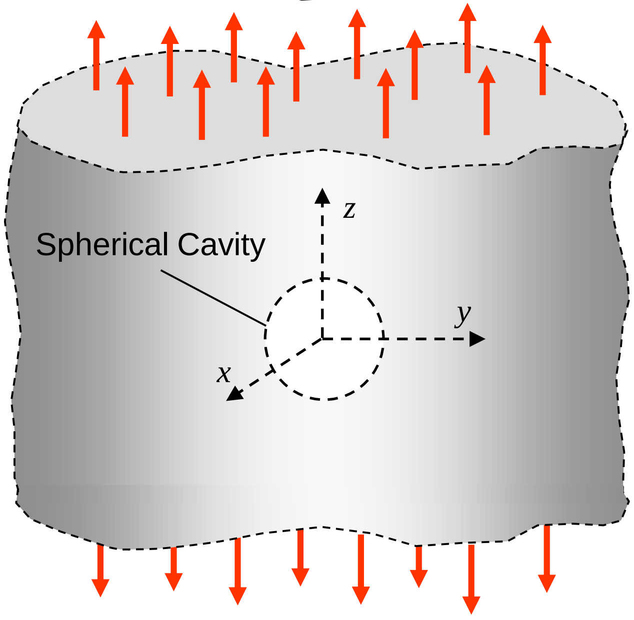
Simplifying Assumptions

* The general elastostatic or elastodynamic problem is, in general, 3-dimensional in nature.
* These problems are difficult to solve except when simplifying assumptions are made.
* Such assumptions include symmetry considerations, vanishing or negligibility of components of stress tensor, strain tensor or the displacement vector.

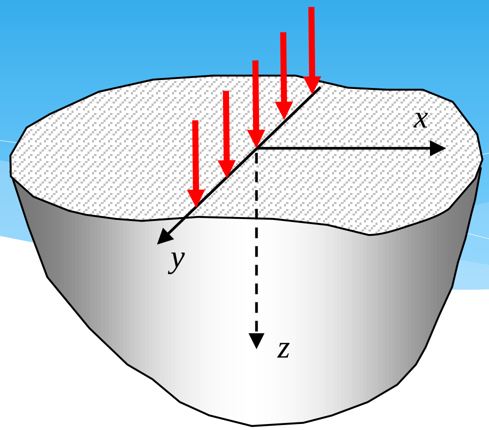
Two vs Three Dimensional Problems

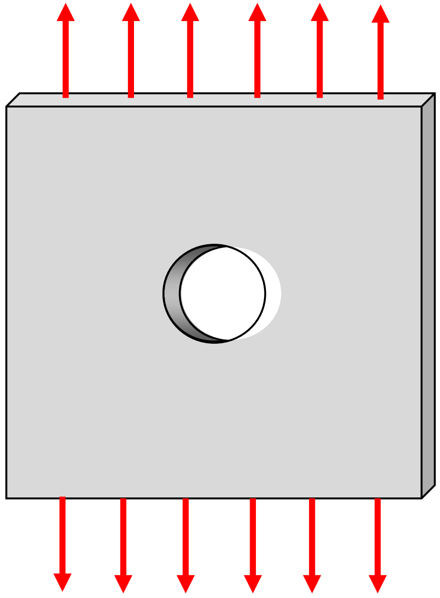
Three-Dimensional





Two-Dimensional





2-D Cases

* Four such assumptions lead us to special cases that are 2-dimensional. These are
  1. Plane Strain
  2. Plane Stress
  3. Generalised Case, and
  4. Anti-plane strain

  Before discussing the 2-D case, we consider the final governing equation needed to fully define elastodynamic problems. These are the compatibility equations.

Compatibility

* The small strain – displacement partial differential equations,

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

after noting the symmetry of strain is a set of six equations. There is no problem with these if we are given displacements and we are to calculate the strains.

* Consider the converse problem: If we are given the strains, and we are to find the displacements.
* Note that we now have six equations and we are to find three unknowns!

Compatibility Equations

* Is this possible? Can we always find the three strains uniquely for any specification of strains? What if a set of three strains gives one set of answers; can we prescribe another set of three strains and get the same displacements? Will these be compatible?
* As a simple mathematical example of this problem, consider the two partial differential equations for a single unknown function:

Compatibility

Consider a function $z = z(x, y)$ depending on two independent variables x and y. If the following two differential equations are satisfied, $\frac{\partial z}{\partial x} = y$, and $\frac{\partial z}{\partial y} = 2x - y$ can we then find a consistent expression for the function $z(x, y)$? If we differentiate each expression respectively wrt y and x, we have

$$\frac{\partial^2 z}{\partial y \partial x} = 1, \text{and } \frac{\partial^2 z}{\partial x \partial y} = 2$$

Which shows inconsistency everywhere. This goes to show we cannot consistently select the second PDE independently after the first. The converse is also true.

Integrability & Compatibility

In general therefore, for such a simple system of equations to be consistently integrable, if in general,

$$\frac{\partial z}{\partial x} = f(x, y) \text{ and } \frac{\partial z}{\partial y} = g(x, y)$$

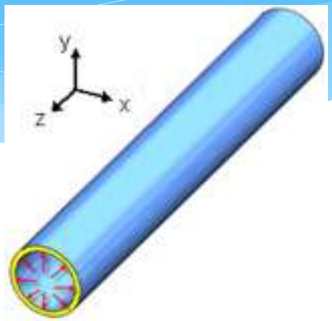
the functions f and g cannot be specified independently as we did above. To guarantee consistency (compatibility), we must impose the extra condition that,

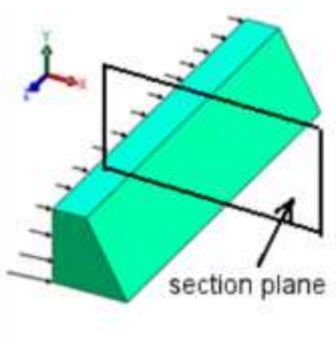
$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

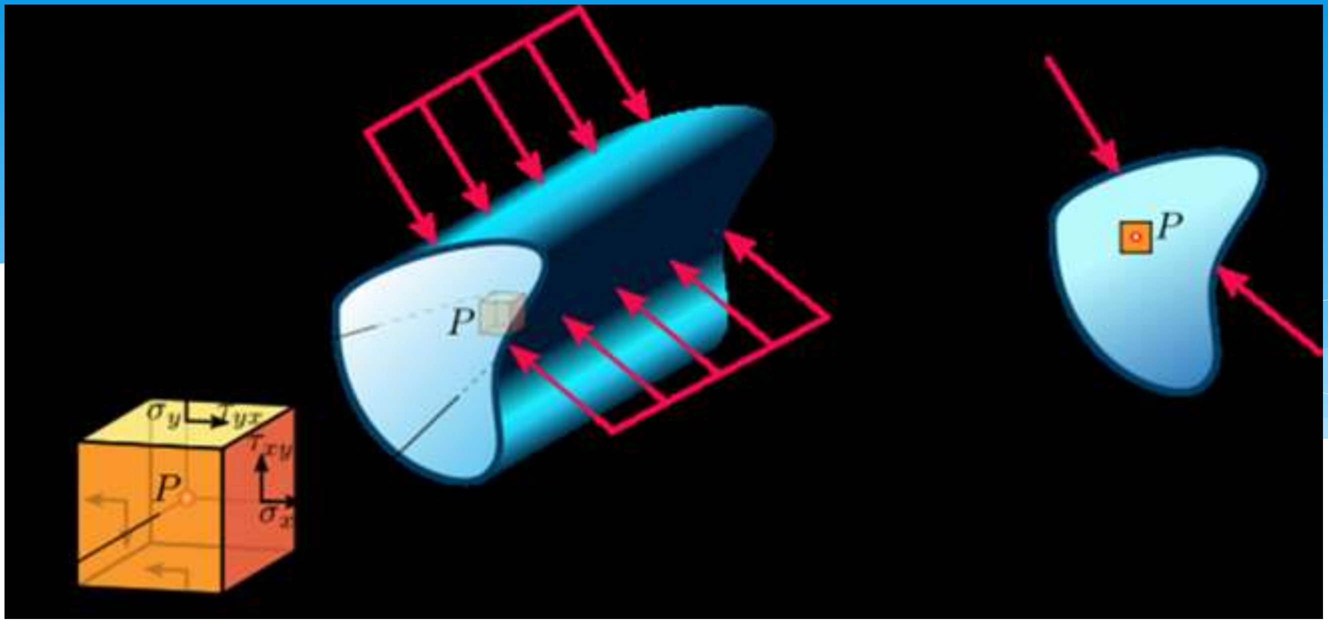
Which is the compatibility equation in this context. We will introduce compatibility later in two dimensional elastic problems.

Plane Strain

- Consider an infinite pipe transporting a fluid under pressure as shown, We can assume that the each slice of the pipe is similar to all others in the loading conditions.
- Another example is a retaining wall such as a dam retaining a large amount of water as shown. This is another example of a case where the longitudinal dimension is much larger than the others and the stress situation in each section is again repeated for all sections. We may assume the longitudinal dimension is infinite.







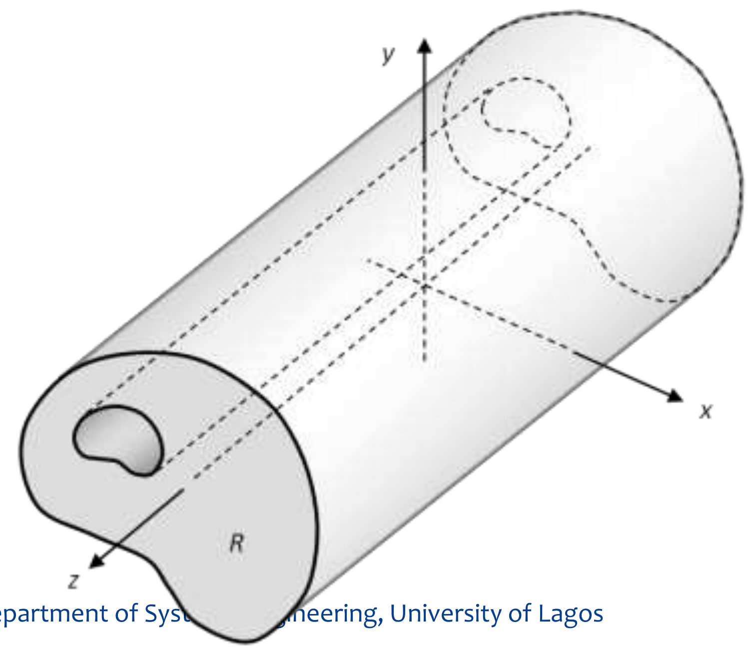
* The same situation is what occurs in a long culvert or any other cylindrical shape where the generator is of a very large dimension compared to the others.
* Furthermore, the external tractions have no components in the logitudinal direction.
* For these cases, we can assume that the third component, $u_3 (= w = 0)$ of strain is zero.

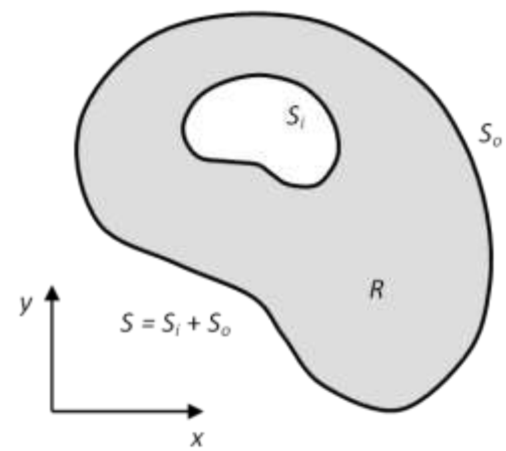
Plane Strain

- Mathematically, we define Plane Strain situation in terms of displacements as follows:

$$u = u(x, y), v = v(x, y), w = 0$$

Is the state of plane strain in the $x - y$ plane.





Plane Strain

All cross sections will have identical displacements and the 3-D problem now reduces to a 2-D problem.

In Cartesian coordinates, the strains can be calculated as,

$$\epsilon_{11} = e_x = \frac{\partial u}{\partial x}$$

$$\epsilon_{22} = e_y = \frac{\partial v}{\partial y}$$

$$\epsilon_{12} = e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

With

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0 = e_z = e_{xz} = e_{zy}$$

Generalized Hooke's Law

The Constitutive equations, $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\text{tr}\boldsymbol{\varepsilon}$ can be written for this case when we note that, $\text{tr}\boldsymbol{\varepsilon} = e_x + e_y$ so that,

$$\sigma_x = 2\mu e_x + \lambda(e_x + e_y)$$

$$\sigma_y = 2\mu e_y + \lambda(e_x + e_y)$$

$$\sigma_z = \lambda(e_x + e_y), \tau_{xy} = 2\mu e_{xy}, \tau_{xz} = \tau_{xz} = 0.$$

* Adding the first two equations, we have,

$$\sigma_x + \sigma_y = 2\mu(e_x + e_y) + 2\lambda(e_x + e_y)$$
$$= 2(\lambda + \mu)\,(e_x + e_y)$$

* Now, from the last linear elastic equation,

$$\sigma_z = \lambda(e_x + e_y)$$
$$= \frac{\lambda}{2(\lambda + \mu)}\,(\sigma_x + \sigma_y)$$
$$= \nu\,(\sigma_x + \sigma_y)$$

Generalized Hooke's Law

Where we have defined $\nu \equiv \frac{\lambda}{2(\lambda+\mu)}$ as the Poisson Ratio in terms of the Lame's constants.

We can invert the linear constitutive equations $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\mathrm{tr}\boldsymbol{\varepsilon}$ using the following Mathematica code:

```
Solve[σx == (2 μ + λ) ϵx + λ ϵy + λ ϵz && σy == λ ϵx + (2 μ + λ) ϵy + λ ϵz &&
  σz == λ ϵx + λ ϵy + (2 μ + λ) ϵz, {ϵx, ϵy, ϵz}]
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$$\left\{\left\{\epsilon_x \to -\frac{-2\lambda\sigma_x - 2\mu\sigma_x + \lambda\sigma_y + \lambda\sigma_z}{2\mu(3\lambda+2\mu)},\ \epsilon_y \to -\frac{\lambda\sigma_x - 2\lambda\sigma_y - 2\mu\sigma_y + \lambda\sigma_z}{2\mu(3\lambda+2\mu)},\ \epsilon_z \to -\frac{\lambda\sigma_x + \lambda\sigma_y - 2\lambda\sigma_z - 2\mu\sigma_z}{2\mu(3\lambda+2\mu)}\right\}\right\}$$

$$e_x = \frac{\lambda+\mu}{\mu(2\mu+3\lambda)}\sigma_x - \frac{\lambda}{2\mu(2\mu+3\lambda)}(\sigma_y+\sigma_z)$$

$$e_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y+\sigma_z)\right)$$

Generalized Hooke's Law

The constant $\mathrm{E} \equiv \frac{\mu(2\mu+3\lambda)}{\lambda+\mu}$ is the Elasticity modulus. The other strains can be written as:

$$e_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_z + \sigma_x)\right) \text{ and } e_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$$

Furthermore, the equilibrium equations,

$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = 0$ reduce to the pair,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

and

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

while the third equation is identically satisfied trivially.

Compatibility of Strain

* There are two non-trivial displacements and three non-trivial strains. As we have previously discussed, the latter cannot be specified arbitrarily for consistency. In order to achieve this consistency, we differentiate the third strain-displacement equation to obtain,

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2}$$

* With the last equation obtained from substituting the first two strain-displacement equations.

Compatibility of Strain

This equation can be expressed in terms of stresses by rewriting the constitutive equations in terms of strains

$$e_x = \frac{1+\nu}{E}\left[(1-\nu)\sigma_x - \nu\sigma_y\right];\; e_y = \frac{1+\nu}{E}\left[(1-\nu)\sigma_y - \nu\sigma_x\right];\; e_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

Note that in terms of the Lame's coefficients, the modulus of elasticity and Poisson ratio are related as follows:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}; \text{and } \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

Substituting the above into

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2}$$

we obtain,

Compatibility of Strain

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

The two equilibrium equation and the compatibility equation must be satisfied within the boundary of the material and, on the boundary, we must have that the traction (stress) vector satisfies $\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$, where $\boldsymbol{\sigma}$ is the stress tensor at the bounding surface and $\mathbf{n}$ is the outward boundary unit vector on the surface.

This can be written in simpler terms when we note that, on the boundary,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x^{(b)} & \tau_{xy}^{(b)} \\ \tau_{xy}^{(b)} & \sigma_y^{(b)} \end{pmatrix}$$

And that, the surface unit normal $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$ so that, we must satisfy

Boundary Conditions

$$\mathbf{T}^{(\mathbf{n})} = \begin{pmatrix} T_x^{(n)} \\ T_y^{(n)} \end{pmatrix} = \boldsymbol{\sigma}\mathbf{n} = \begin{pmatrix} \sigma_x^{(b)} & \tau_{xy}^{(b)} \\ \tau_{xy}^{(b)} & \sigma_y^{(b)} \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix}$$

So that, on the boundary,

$$T_x^{(n)} = \sigma_x^{(b)} n_x + \tau_{xy}^{(b)} n_y$$

And

$$T_y^{(n)} = \tau_{xy}^{(b)} n_x + \sigma_y^{(b)} n_y$$

Navier's Equations

In terms of displacements, Beginning from the general Navier's equations of Elastodynamics:

$$(\lambda + \mu)\text{grad}(\text{div } \mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{b} = \varrho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Observe that the displacement vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

With $u = u(x, y), v = v(x, y)$, and $w = 0$,

$$\text{div } \mathbf{u} = \frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y}$$

So that, for equilibrium, the two surviving equations are:

$$(\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y}\right) + \mu\nabla^2 u = 0$$

$$(\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u(x, y)}{\partial x} + \frac{\partial v(x, y)}{\partial y}\right) + \mu\nabla^2 v = 0$$

Navier's Equations

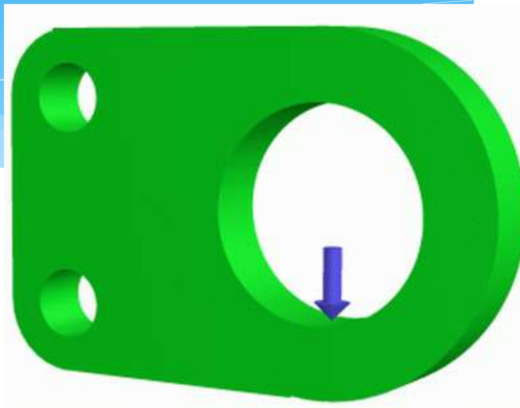
* The last Navier's Equations,

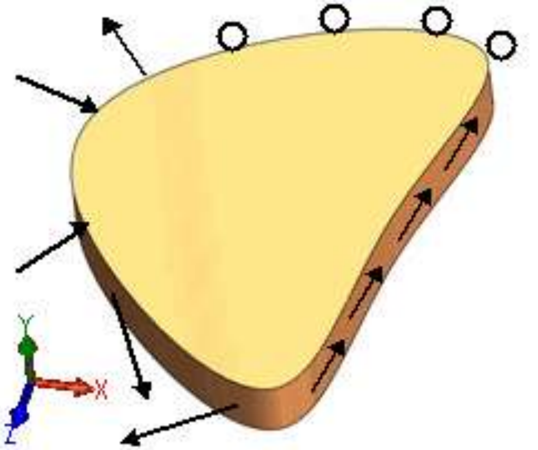
$$(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u(x,y)}{\partial x}+\frac{\partial v(x,y)}{\partial y}\right)+\mu\nabla^2 w=0$$

vanishes identically term by term on the account of $u(x,y)$ and $v(x,y)$ being independent of z, and the fact that $w=0$, making the last LHS term vanish also.

Plane Stress

* Consider a thin bracket held in position by two bolts and loaded vertically as shown. We assume that the bracket is so thin that there is no variation in the load distribution along its thickness
* Another thin member may be loaded with supports as shown. In these and similar cases, when it is sufficiently accurate to assume that all the stresses are distributed uniformly along the thickness and they are independent of the third coordinate
* These cases are said to be in plane stress.





Plane Stress

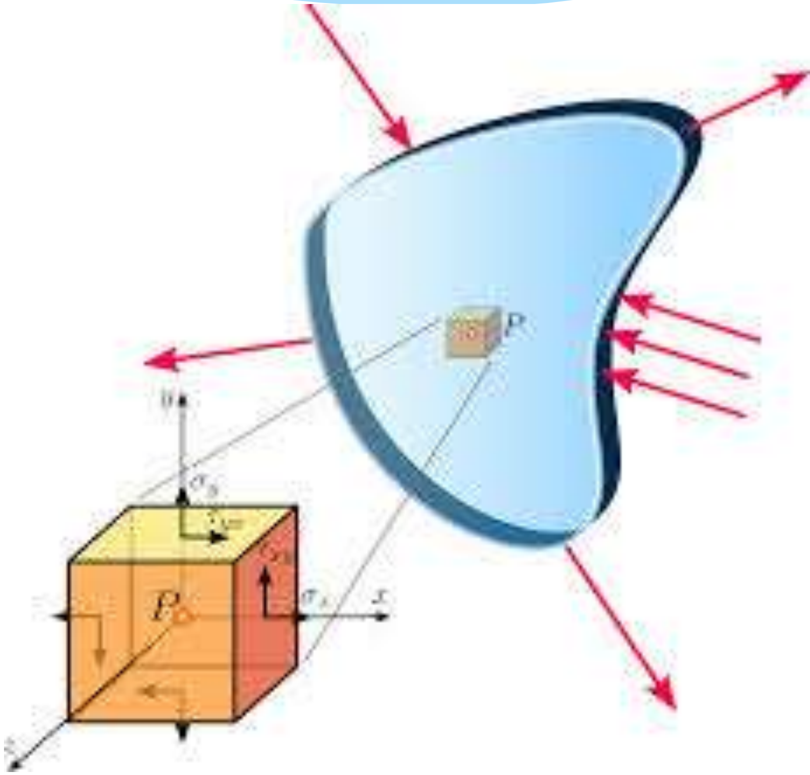
* In the plane stress case, we have the following representation:

$$\sigma_x = \sigma_x(x, y);\ \sigma_y = \sigma_y(x, y);$$

$$\tau_{xy} = \tau_{xy}(x, y);$$

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

Notice that the difference between this state and plane strain are enormous. The former is defined by specification on displacements while plane stress is defined by the stresses as shown above.



Body Forces

- Plane stress does not admit body forces or any tractions in the third axis.
- In order to obtain the Plane Stress constitutive equations in terms of
- It is easier to arrive at the plane stress constitutive equations from the Elastic Modulus/Poisson Ratio E, ν pair:

$$e_x = \frac{1}{E}[\sigma_x - \nu\sigma_y];\ e_y = \frac{1}{E}[\sigma_y - \nu\sigma_x]$$

$$e_z = \frac{-\nu}{E}[\sigma_x + \sigma_y];\ e_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

All other strains e_{yz}, e_{zx}vanish.

Strain-Displacement Relations

* The following strain-displacement relations apply in plane stress:

$$\epsilon_{11} = e_x = \frac{\partial u}{\partial x};\ \epsilon_{22} = e_y = \frac{\partial v}{\partial y};\ \epsilon_{33} = e_z = \frac{\partial w}{\partial z}$$

$$\epsilon_{12} = e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

With $\epsilon_{13} = \epsilon_{23} = 0 = e_{xz} = e_{zy}$.

Furthermore, we can conclude easily, based on the variation of the stresses, that

$$e_x = e_x(x, y);\ e_y = e_y(x, y);$$

$$e_{xy} = e_{xy}(x, y);$$

Equilibrium

Furthermore, under plane stress, just as in the plane strain case the equilibrium equations,

div $\boldsymbol{\sigma} + \boldsymbol{b} = 0$ reduce to the pair,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

and

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

And the third equation states that F_z, the third body force, vanishes identically.

Compatibility

* In terms of strain, the compatibility equations are the same as before:

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2}$$

When we express this in terms of stresses, we obtain a slightly different equation (elastic constants only):

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$

The difference between the two equations is NOT trivial!

Navier's Equations

As before, in general 3-D, Navier's Equations,

$$(\lambda + \mu)\text{grad}(\text{div } \mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{b} = \varrho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

In plane stress, $e_x + e_y = \frac{1}{E}[\sigma_x + \sigma_y - \nu\sigma_x - \nu\sigma_y] = \frac{1-\nu}{E}[\sigma_x + \sigma_y]$

$$e_z = \frac{-\nu}{E}[\sigma_x + \sigma_y] = \frac{-\nu}{1-\nu}[e_x + e_y]$$

The trace of the strain tensor, tr $\boldsymbol{\epsilon}$ can be obtained from

$$\text{tr } \boldsymbol{\epsilon} = e_x + e_y + e_z = \frac{1-2\nu}{1-\nu}[e_x + e_y] = \frac{2\mu}{\lambda + 2\mu}[e_x + e_y]$$

Note that the divergence of the displacement vector, div $\mathbf{u} = \text{tr } \boldsymbol{\epsilon}$ the trace of the small strain tensor. Consequently, the Navier elastostatic equations become,

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial}{\partial x}\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y}\right) + \mu\nabla^2 u = 0$$

$$\frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial}{\partial y}\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y}\right) + \mu\nabla^2 v = 0$$

?? $\frac{2\mu(\lambda+\mu)}{\lambda+2\mu} = \frac{E}{2(1-\nu^2)}$??

Zero Body Forces

If the body forces vanish, the two compatibility equations, plane stress or plane strain become identical even when expressed in terms of stresses:

$$\nabla^2(\sigma_x + \sigma_y) = 0$$

That is, the trace of the stress tensor is harmonic.

Generalized Plane Stress

- The plane strain situation is truly 2-D because for the field quantities – stresses, strains and displacements – are 2-D in their functional dependencies.
- For plane stress, the antiplane quantities do not completely shed the third dimension dependency unless an averaging process is assumed. Details of this averaging process are presented in the classical works of Timoshenko & Goodier (Art 78) and in section 7.3 of our recommended text.

Anti-Plane Strain

- Another 2-D situation is called anti-plane strain (also known in many textbooks as anti-plane shear.
- The strain components for this is $u = 0, v = 0$, and $w = w(x, y)$. The first two strains vanish while the third is only dependent on x, y.

Consequently, $e_x = \frac{\partial u}{\partial x} = 0, e_y = \frac{\partial v}{\partial y} = 0, e_z = \frac{\partial w(x,y)}{\partial z} = 0$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{1}{2}\frac{\partial w}{\partial y}$$

$$e_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\frac{\partial w}{\partial x}$$

Anti-Plane Stresses

The Constitutive equations for anti-plane strain, $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\text{tr}\boldsymbol{\varepsilon}$ can be written for this case when we note that, $\text{tr}\boldsymbol{\varepsilon} = 0$ so that,

$$\sigma_x = 2\mu e_x = 0, \sigma_y = 2\mu = e_y = 0, \sigma_z = 2\mu e_z = 0$$

$$\tau_{xy} = 2\mu e_{xy} = 0, \tau_{yz} = 2\mu e_{yz} = \mu\frac{\partial w}{\partial y}$$

$$\tau_{xz} = 2\mu e_{xz} = \mu\frac{\partial w}{\partial x}$$

Equilibrium Equations

* For anti-plane strain, equilibrium equations,

$$\text{div}\,\boldsymbol{\sigma} + \mathbf{b} = 0$$

The first two result in: $F_x = F_y = 0$; while the third becomes,

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + F_z = 0.$$

Navier's Equations

In terms of displacements, Beginning from the general Navier's equations of Elastodynamics:

$$(\lambda + \mu)\text{grad}(\text{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{b} = \varrho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Observe that the displacement vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

With $u = 0, v = 0, and\ w = w(x, y)$, $\mathbf{u} = 0\mathbf{i} + 0\mathbf{j} + w\mathbf{k}$

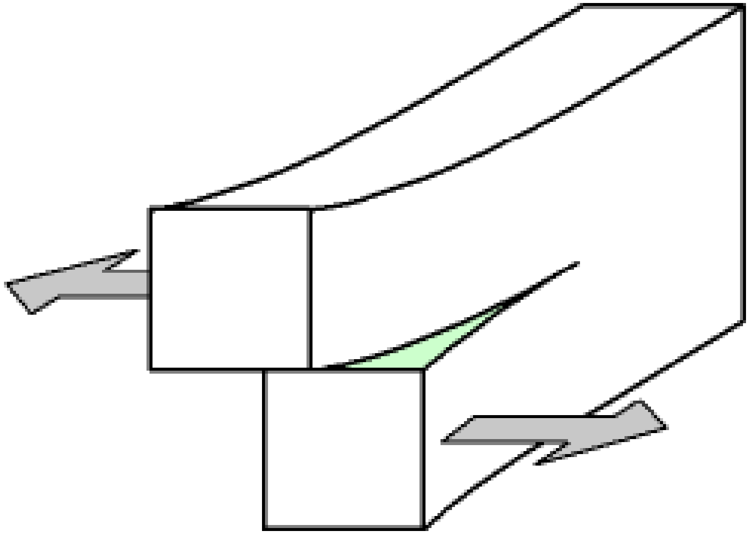
$$\text{div}\,\mathbf{u} = \frac{\partial w(x, y)}{\partial z} = 0.$$

$$\mu\nabla^2 w + F_z = \varrho\frac{\partial^2 w}{\partial t^2}$$

On the vanishing of the two other body force components by the equilibrium equations.

Anti-Plane Strain

* The Classical Mode 3 Fracture is a practical example of a anti-plane shear
* This is of practical importance in seismic applications



Stress Function

* The preeminent method for dealing with 2-D problems of elasticity is by assuming the existence of a function that can generate the stresses by way of partial differentiation. Such a function is called a stress function. For 2-D problems, we examine the stress function named after British Astronomer, Airy:
* The method reduces the general formulation to a single governing equation in terms of a single unknown. The resulting equation is then solvable by several methods of applied mathematics, and thus many analytical solutions to problems of interest can be found.
* This scheme is based on the general idea of developing a representation for the stress field that will automatically satisfy equilibrium by using the relations

Airy Stress Function

* Consider the function, $\phi = \phi(x, y)$ such that the stresses,

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \text{and } \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

A substitution of the Airy Stress function into the 2-D equilibrium equations shows that the equations for plane strain as well as plane stress are automatically satisfied.

Substituting into the compatibility equations, we have,

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Which is the Biharmonic equation.

2-D Solution Strategy

* The stress function method reduces the solution of the 2-D elastostatic problem to the finding of a biharmonic function $\phi = \phi(x, y)$.

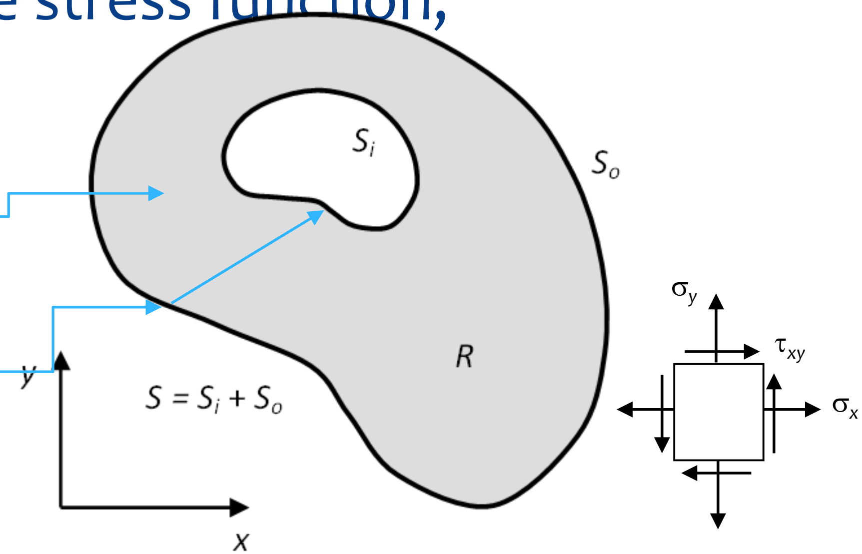
Stress Function Solution

- We have now reduced the solution of the 2-D elastostatic problem to finding a biharmonic function. This function is to be determined in the two-dimensional region R bounded by the boundary S as shown. Appropriate boundary conditions over S are necessary to complete the solution. Traction boundary conditions would involve the specification of second derivatives of the stress function;

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

$$T_x^{(n)} = \sigma_x^{(b)} n_x + \tau_{xy}^{(b)} n_y = \frac{\partial^2 \phi}{\partial y^2} n_x - \frac{\partial^2 \phi}{\partial x \partial y} n_y$$

$$T_y^{(n)} = \tau_{xy}^{(b)} n_x + \sigma_y^{(b)} n_y = -\frac{\partial^2 \phi}{\partial x \partial y} n_x + \frac{\partial^2 \phi}{\partial x^2} n_y$$



Polar Formulation
Plane Strain

In Polar coordinates, as before, the third displacement, u_z component vanishes. $u_r = u_r(r,\theta), u_\theta = u_\theta(r,\theta)$. Consequently, the strain components are:

$$e_r = \frac{\partial u_r}{\partial r}, e_\theta = \frac{1}{r}\left(\frac{u_r}{r} + \frac{\partial u_\theta}{\partial \theta}\right), e_{r\theta} = \frac{1}{2r}\left(\frac{\partial u_r}{\partial \theta} + r\frac{\partial u_\theta}{\partial \theta} - u_\theta\right)$$

With the other strain components vanishing: , $e_z = e_{rz} = e_{\theta z} = 0.$

Clearly the trace of the strain tensor, $\text{tr}\,\boldsymbol{\epsilon} = e_r + e_\theta$. We can now write the components of the stress-strain relations, $\sigma = 2\mu\boldsymbol{\epsilon} + \boldsymbol{\lambda}\mathbf{1}\,\text{tr}\,\boldsymbol{\epsilon}$

$\sigma_r = 2\mu e_r + \lambda(e_r + e_\theta), \sigma_\theta = 2\mu e_\theta + \lambda(e_r + e_\theta), \sigma_z = \lambda(e_r + e_\theta)$

$\tau_{r\theta} = 2\mu e_{r\theta}, \tau_{rz} = \tau_{\theta z} = 0$

Polar Formulation
Plane Stress

- As in the Cartesian case, It is easier to arrive at the plane stress constitutive equations from the Elastic Modulus/Poisson Ratio E, ν pair:

$$e_r = \frac{1}{E}[\sigma_r - \nu\sigma_\theta]; \; e_\theta = \frac{1}{E}[\sigma_\theta - \nu\sigma_r]$$

$$e_z = \frac{-\nu}{E}[\sigma_r + \sigma_\theta]; \; e_{r\theta} = \frac{1+\nu}{E}\tau_{r\theta}$$

All other strains $e_{rz}, e_{z\theta}$vanish.

Clearly the trace of the strain tensor, $\text{tr}\,\boldsymbol{\epsilon} = e_r + e_\theta$. We can now write the components of the stress-strain relations, $\sigma = 2\mu\boldsymbol{\epsilon} + \boldsymbol{\lambda 1}\,\text{tr}\,\boldsymbol{\epsilon}$

$$\sigma_r = 2\mu e_r + \lambda(e_r + e_\theta), \sigma_\theta = 2\mu e_\theta + \lambda(e_r + e_\theta), \sigma_z = \lambda(e_r + e_\theta)$$

$$\tau_{r\theta} = 2\mu e_{r\theta}, \tau_{rz} = \tau_{\theta z} = 0$$

Polar Formulation

Equilibrium Equations

In 2-D, the Polar representation of equilibrium equations,

$$\text{div}\,\boldsymbol{\sigma} + \mathbf{b} = 0$$

Is easily shown (Use Mathematica) as,

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + F_\theta = 0$$

With zero body forces, these are automatically satisfied if we choose,

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

Polar Stress Function

$$\frac{\partial \sigma_r}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right)$$

$$\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\right)$$

$$\frac{\sigma_r - \sigma_\theta}{r} = \frac{1}{r}\left(\frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{\partial^2 \phi}{\partial r^2}\right)$$

Neglecting body forces, the first equilibrium equation is,

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

Substituting and expanding, we have,

Polar Stress Function

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\right)+\frac{1}{r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{\partial^2\phi}{\partial r^2}\right)$$

$$= -\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^3\phi}{\partial r\partial\theta^2}-\frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}\right)+\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r^3}\frac{\partial^2\phi}{\partial\theta^2}$$

$$-\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}=$$

$$-\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}-\frac{2}{r^3}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial^3\phi}{\partial r\partial\theta^2}+\frac{1}{r^3}\frac{\partial^2\phi}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^3\phi}{\partial r\partial\theta^2}+\frac{1}{r^2}\frac{\partial\phi}{\partial r}+\frac{1}{r^3}\frac{\partial^2\phi}{\partial\theta^2}-\frac{1}{r}\frac{\partial^2\phi}{\partial r^2}$$

$$=0$$

It can similarly be shown that the second equilibrium equation also vanish:

$$\frac{\partial \tau_{r\theta}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\right) = \frac{\partial}{\partial r}\left(-\frac{1}{r^2}\frac{\partial \phi}{\partial \theta} + \frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta}\right)$$

$$= -\frac{2}{r^3}\frac{\partial \phi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial r \partial \theta} - \frac{1}{r}\frac{\partial^3 \phi}{\partial r^2 \partial \theta}$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial^3 \phi}{\partial r^2 \partial \theta} \text{ and } \frac{2\tau_{r\theta}}{r} = \frac{2}{r^3}\frac{\partial \phi}{\partial \theta} - \frac{2}{r^2}\frac{\partial^2 \phi}{\partial r \partial \theta}$$

So that, as expected, $\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} = 0$. And for zero body forces, compatibility leads as before to the biharmonic equation,

$$\nabla^2\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = \nabla^2\nabla^2\phi = 0$$

Polar Airy Stress Functions

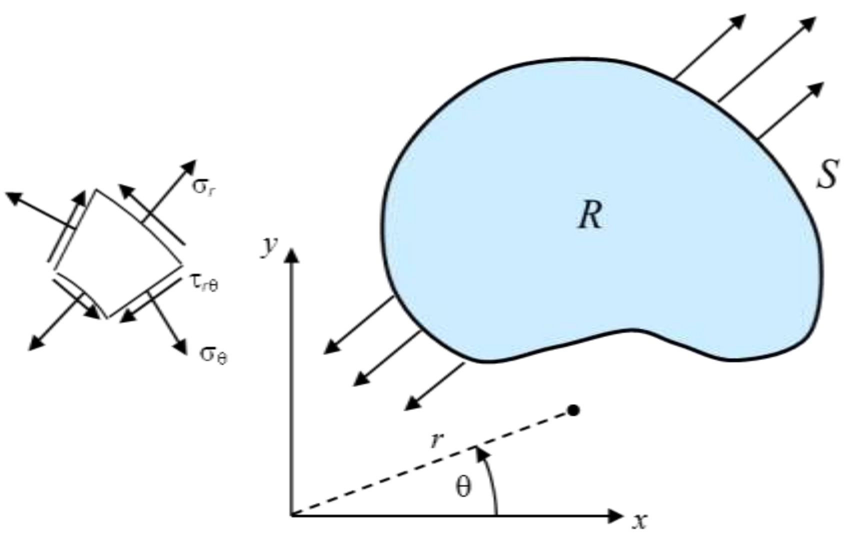
The Airy function generated stresses,

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \tau_{r\theta} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial \theta}\right)$$

Can be substituted into the equilibrium equations $\nabla^2(\sigma_r + \sigma_\theta) = 0$, so that, :

$$\nabla^2\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = \nabla^2\nabla^2\phi = 0$$



On the boundary of the body, the traction conditions will be specified in terms of given functions of r and θ so that,

$$T_r = f_1(r, \theta), T_\theta = f_2(r, \theta)$$

Exercize

1. Begin with the linear elastic constitutive equation,

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\text{tr}\boldsymbol{\varepsilon}$$

and plane strain displacement vector, $u = u(x, y), v = v(x, y), w = 0$, obtain the strain-stress relations for plane strain: $e_x = \frac{1+\nu}{E}\left((1-\nu)\sigma_x - \nu\sigma_y\right)$, $e_y = \frac{1+\nu}{E}\left((1-\nu)\sigma_y - \nu\sigma_x\right)$, and $e_{xy} = \frac{1+\nu}{E}\tau_{xy}$.

2. What is a compatibility Equation? Explain when the governing equations of an elatic problem must be augmented by such equations. How many Compatibility equations are needed for (a) Plane Stress, (b) Plane Strain, (c) Anti-Plane Strain, (d) 3-D stress system.

(e) Obtain the equation of compatibility for plane strain.

3. Express the compatibility equations for plane stress and plane strain in terms of stresses.

4. Show that the following stress system satisfies the conditions of plane strain: $\sigma_x = kxy, \sigma_y = kx, \sigma_z = \nu kx(1+y). \tau_{xy} = -\frac{1}{2}ky^2, \tau_{zy} = \tau_{xz} = 0$ where k is a constant.

6. (a) Show that the divergence $\mathrm{div}\,\mathbf{u} = \mathrm{tr}\,\boldsymbol{\epsilon}$ for small strains. Is it also true for Eulerian strain? Why?

(b) Find the dilatation of plane strain in terms of derivatives of displacements. (Dilatation is the trace of the strain.)

c. Show that the expressions for the Navier's equation for plane strain are $(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y}\right) + \mu\nabla^2 u = 0$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y}\right) + \mu\nabla^2 v = 0$$

(d) Show that the z-component of the Navier equations vanisg identically for a system in $x - y$ plane strain.

7. Use Cauchy stress law to find the expression for the surface tractions $\mathbf{T}^{(\mathbf{n})} = \begin{pmatrix} T_x^{(n)} \\ T_y^{(n)} \end{pmatrix}$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x^{(b)} & \tau_{xy}^{(b)} \\ \tau_{xy}^{(b)} & \sigma_y^{(b)} \end{pmatrix}$$

Given that the surface unit normal $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$.

8. State the conditions for plane stress. Beginning with the stress-strain relations, $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\mathrm{tr}\boldsymbol{\varepsilon}$

Show that in plane stress, the strain-stress relations are,

$$e_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right];\ e_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right]$$

$$e_z = \frac{-\nu}{E}\left[\sigma_x + \sigma_y\right];\ e_{xy} = \frac{1+\nu}{E}\tau_{xy}$$

All other strains e_{yz}, e_{zx} vanish.

9. Show that the dilatation under plane stress can be written as,

$$\text{tr}\,\boldsymbol{\epsilon} = e_x + e_y + e_z = \frac{1-2\nu}{1-\nu}\left[e_x + e_y\right] = \frac{2\mu}{\lambda + 2\mu}\left[e_x + e_y\right]$$

Hence obtain the plane stress expression for the Navier governing equations in terms of displacements for plane stress.

10. State the conditions for anti-plane stress. Show that under these conditions that the dilatation of stress is zero.

(c) Find an expression for the corresponding Navier equations.

(d) Write the three equlirium equations for anti-plane strain.

11 (a) Write out the expression for stresses derived from the Airy Stress Function. (b) Show that the Airy Stress functions are Biharmonic. (c) Write the expression for the boundary conditions for using Airy Stress Function.

12. Given the equilibrium equation, $\text{div}\,\boldsymbol{\sigma} + \mathbf{b} = 0$, Use Mathematica to find the Polar representation for these equations.

13. The stresses generated by Airy Stress function in Polar coordinates are,

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

Show that these satisfy equilibrium equations.