A Bit of History

* Until the 17th century the understanding of stress was largely intuitive and empirical
* Ancient and medieval architects did develop some geometrical methods and simple formulas to compute the proper sizes of pillars and beams, but the scientific understanding of stress became possible only after the necessary mathematical tools of differential and integral calculus were invented in the 17th and 18th centuries:
* The concept of stress as it is understood today can be traced back to Cauchy's work of 1827.
* He made the notion of stress precise by surmising that a body responds to external loading by transmitting forces internally throughout the body via a tensor-valued field that is now called the Cauchy Stress Tensor.

Theory of Stress & Heat Flux

Stress: Scalar, Vector & Tensor
Cauchy Stress Principle, Conjugate Stress Tensors
Fourier-Stokes Heat Flux Theorem

Stress

What is "Stress"?

- Stress is a measure of ***force intensity*** either within or on the bounding surface of a body subjected to loads.
- The Continuum Model takes a **macroscopic** approach:
  - **Measurable aggregate behavior** rather than the microscopic, atomistic activities that may in fact have led to them, and consequently, the
  - Standard **results of calculus** applicable in the case of limiting values of this quotient as the areas to which the forces are applied become very small.

What is Stress?

- The simple answer of force per unit area raises the following questions:
  - What force?
  - As the size and shape of the material in question changes as motion evolves: What area:
  - What direction? Which surface? Which location?
- A more rigorous definition of stress is required to settle these and other matters of importance.

Three Stresses

In defining stress, care must be taken to note that sometimes we are talking about a

Tensor – the *Stress Tensor*.

- This completely characterizes the stress state at a particular location. That such a tensor exists is proved as Cauchy's theorem – a fundamental law in Continuum Mechanics.

Vector - the *Traction – Stress Vector*

- Intensity of resultant forces on a particular surface in a specific direction. This is, roughly speaking, what we have in mind when we say that stress is "force per unit area."

Scalar – the *scalar magnitude* of the traction vector or some other scalar function of the stress tensor.

Two Forces

"… a distinction is established between two types of forces which we have called 'body forces' and 'surface tractions', the former being conceived as due to a direct action at a distance, and the latter to contact action." **AEH Love**

It is convenient to examine these forces by categorizing them as follows:

Body forces $\boldsymbol{b}$ (force per unit mass);

* These are forces originating from sources (fields of force usually) outside of the body that act on the volume (or mass) of the body.

Surface forces i.e.: $\boldsymbol{f}$ (force per unit area of surface across they which they act)

Body Forces

Body forces *b* (force per unit mass); These are forces originating from sources - usually outside of the body

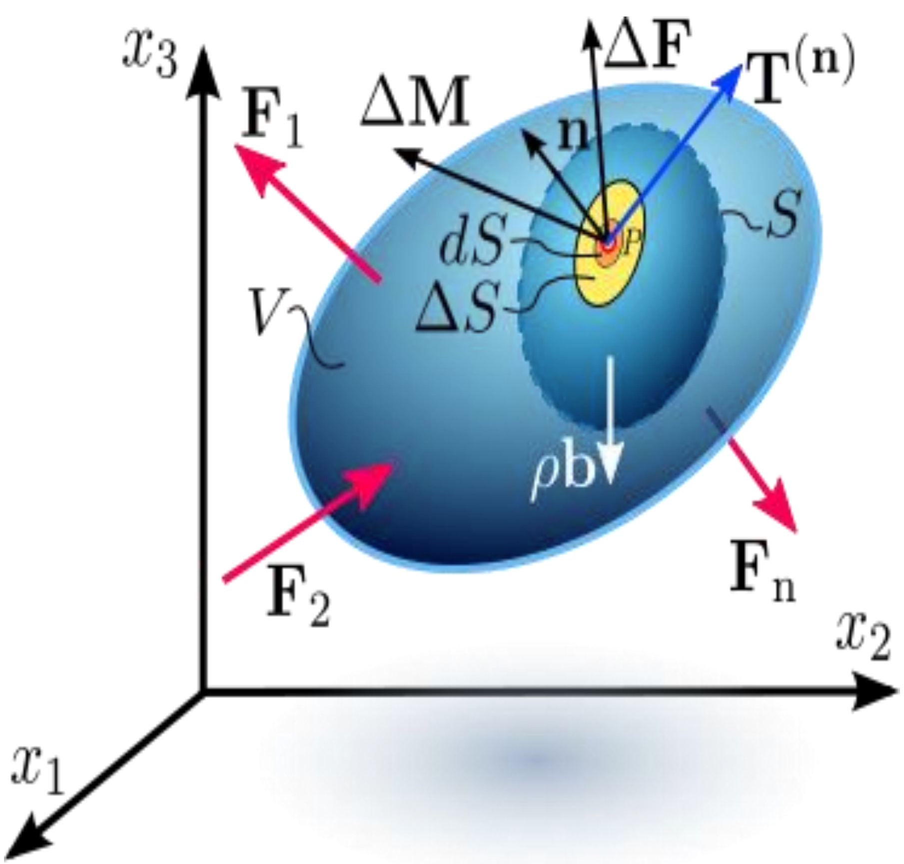
- Fields of force
- Act on the volume (or mass) of the body.

- These forces arise from the placement of the body in force fields
  - **Definition:** A field of force is a *Euclidean Point Space* in which a force function is specified at every point
  - *Examples:* gravitational, electrical, magnetic or inertia
- As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed.
- Body forces are therefore assumed to be continuously distributed over the entire volume of the material.

Surface Forces

- Force per unit area of surface across they which they act and are distributed in some fashion over a surface element of the body
- Element could be part of the bounding surface, or an arbitrary element of surface within the body;
  - Examples: Shear stresses, normal stresses such as hydrostatic pressure, Wind loading, contact with another solid etc.

Forces on an Element



- Forces on an element V surrounding a point $P(x_1, x_2, x_3)$ in a body acted upon by the forces $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$.
- The resultant body force per unit volume is $\boldsymbol{b}(x_1, x_2, x_3)$.
- Consider an infinitesimal area element oriented in such a way that the unit outward normal to its surface is $\mathbf{n}(x_1, x_2, x_3)$.
- If the resultant force on the surface ΔS of ΔV is $\Delta \boldsymbol{F}$, and this results in a traction intensity which will in general vary over ΔS.
- Surface traction vector on this elemental surface be $\boldsymbol{T}$, it is convenient to label this traction $\boldsymbol{T}^{(\boldsymbol{n})}$. The superscript $\boldsymbol{n}$ emphasises the fact that this traction is the resultant on the *surface whose outward normal* is $\boldsymbol{n}$.

Surface Forces on an Element

* We can write that,

$$\boldsymbol{T}^{(\boldsymbol{n})} = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{F}}{\Delta S} = \frac{d\boldsymbol{F}}{dS}$$

IMPORTANT NOTE: *The traction $\boldsymbol{T}^{(\boldsymbol{n})}$ gets its direction, not from $\boldsymbol{n}$ but from $\boldsymbol{F}$. $\boldsymbol{n}$ signifies the orientation of the surface on which it acts.*

Or, conversely that

$$\Delta \boldsymbol{F} = \int_{\Delta S} \boldsymbol{T}^{(\boldsymbol{n})} dS$$

In general, $\boldsymbol{T}^{(\boldsymbol{n})} = \boldsymbol{T}^{(\boldsymbol{n})}(x_1, x_2, x_3)$ and $\boldsymbol{n} = \boldsymbol{n}(x_1, x_2, x_3)$ as the surface itself is not necessarily a plane. It is only as the limit is approached that $\boldsymbol{n}$ is a fixed direction for the elemental area and $\Delta \boldsymbol{F}$ and $\boldsymbol{T}^{(\boldsymbol{n})}$ are in the same direction.

Body Forces on an Element

Furthermore, the body force per unit mass is $\varrho \boldsymbol{b}$. The density $\varrho = \varrho(x_1, x_2, x_3)$ as it varies over the whole body. If the resultant body force in the volume element ΔV is $\Delta \boldsymbol{P}$, we can compute the body force per unit volume

$$\boldsymbol{b} = \frac{1}{\varrho} \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{P}}{\Delta V} = \frac{1}{\varrho} \frac{d\boldsymbol{P}}{dV}$$

so that the body force on the element of volume

$$\Delta \boldsymbol{P} = \int_{\Delta V} \varrho \boldsymbol{b} dV \, .$$

Surface Traction

Vector intensity of the vector force on the surface as the surface area approaches a limit.

* It is defined for a specific surface with an orientation defined by the outward normal $\boldsymbol{n}$.
* This implies immediately that the traction at a given point is dependent upon the orientation of the surface. It is a vector that has different values at the same point depending upon the orientation of the surface we are looking at.

Surface Traction

It is a function of the coordinate variables.

- It is therefore proper to write,

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{T}(\mathbf{n}, x_1, x_2, x_3) \equiv \mathbf{T}^{(\mathbf{n})}(x_1, x_2, x_3)$$

to make these dependencies explicitly obvious

- In general, $\mathbf{T}$ and $\mathbf{n}$ are not in the same direction; that is, there is an angular orientation between the resultant ***stress vector*** and the surface outward normal.

Surface Traction is expressed as the vector sum of its projection $t_n \equiv \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n}$ along the normal $\mathbf{n}$ and $t_s \equiv \left\| \mathbf{T}^{(\mathbf{n})} - t_n \mathbf{n} \right\|$ on the surface itself.

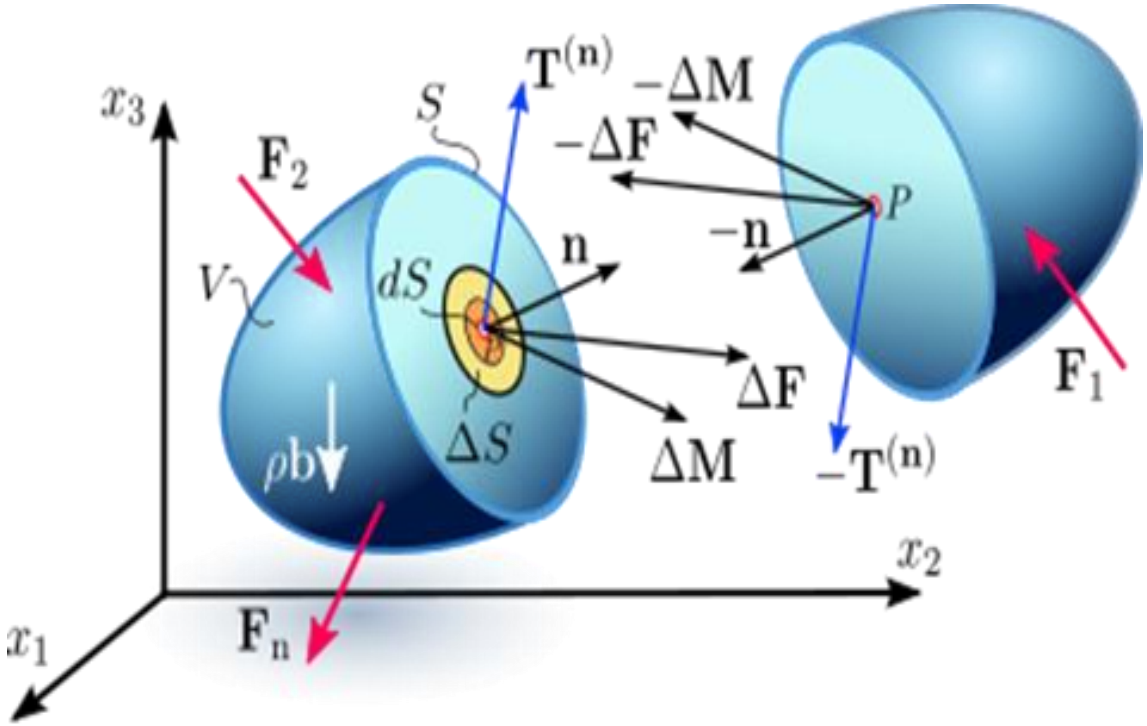
Normal & Shearing Stresses

* It is easy to show that this shear stress vector is the surface projection of the resultant $(\mathbf{I}-\mathbf{n}\otimes\mathbf{n})\mathbf{T}^{(\mathbf{n})}$. These normal and shearing components of the stress vector are called the *normal* and *shear tractions* respectively. They are the scalar magnitudes of the normal and tangential components of the reaction on any surface of interest.

Euler-Cauchy Stress Principle

- The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equipollent to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field $\mathbf{T}^{(\mathbf{n})}$. In view of Newton's third law of action and reaction, this principle can be expressed compactly in the equation,

$$\mathbf{T}^{(-\mathbf{n})} = -\mathbf{T}^{(\mathbf{n})}$$



x_3
$\mathbf{T}^{(\mathbf{n})}$
$-\Delta\mathbf{M}$
S
$-\Delta\mathbf{F}$
$\mathbf{F}_2$
P
$-\mathbf{n}$
$\mathbf{n}$
V
dS
$\Delta\mathbf{F}$
$\mathbf{F}_1$
ΔS
$\rho\mathbf{b}$
$\Delta\mathbf{M}$
$-\mathbf{T}^{(\mathbf{n})}$
x_2
x_1
$\mathbf{F}_n$

Cauchy's Theorem

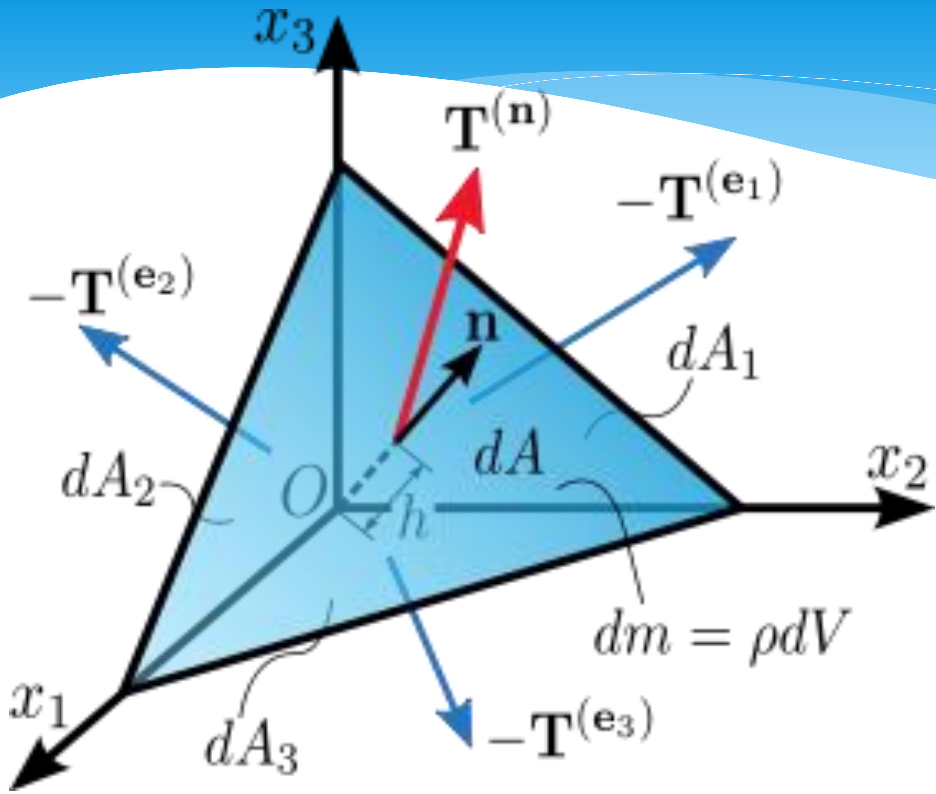
* *Provided the stress vector* $\mathbf{T}^{(\mathbf{n})}$ *acting on a surface with outwardly drawn unit normal* $\mathbf{n}$ *is a continuous function of the coordinate variables, there exists a second-order tensor field* $\boldsymbol{\sigma}(\mathbf{x})$*, independent of* $\mathbf{n}$*, such that* $\mathbf{T}^{(\mathbf{n})}$ *is a linear function of* $\mathbf{n}$ *such that:*

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$$

* The tensor $\boldsymbol{\sigma}$ in the above relationship is the tensor of proportionality and it is called *Cauchy Stress Tensor*. It is also the "true stress" tensor for reasons that will become clear later.

Cauchy Stress Theorem

- To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal base area dA oriented in an arbitrary direction specified by a normal vector $\mathbf{n}$ (Figure 6.3). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane $\mathbf{n}$. The stress vector on this plane is denoted by $\mathbf{T}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\boldsymbol{T}^{(\boldsymbol{e}_1)}, \boldsymbol{T}^{(\boldsymbol{e}_2)}$ and $\boldsymbol{T}^{(\boldsymbol{e}_3)}$ From equilibrium of forces, Newton's second law of motion, we have
$$\varrho\left(\frac{h}{3}dA\right)\boldsymbol{a} = \mathbf{T}^{(\mathbf{n})}dA - \mathbf{T}^{(\mathbf{e}_1)}dA_1 - \mathbf{T}^{(\mathbf{e}_2)}dA_2 - \mathbf{T}^{(\mathbf{e}_3)}dA_3$$
$$= \mathbf{T}^{(\mathbf{n})}dA - \mathbf{T}^{(\mathbf{e}_\mathrm{i})}dA_i$$



where the right-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ϱ is the density, $\boldsymbol{a}$ is the acceleration, and h is the height of the tetrahedron, considering the plane $\boldsymbol{n}$ as the base.

$$\varrho\left(\frac{h}{3}dA\right)\boldsymbol{a} = \boldsymbol{T}^{(\boldsymbol{n})}dA - \boldsymbol{T}^{(\boldsymbol{e}_1)}dA_1 - \boldsymbol{T}^{(\boldsymbol{e}_2)}dA_2 - \boldsymbol{T}^{(\boldsymbol{e}_3)}dA_3$$

$$= \boldsymbol{T}^{(\boldsymbol{n})}dA - \boldsymbol{T}^{(\boldsymbol{e}_i)}dA_i$$

Cauchy Theorem

The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face:

$$dA_i = (\boldsymbol{n} \cdot \boldsymbol{e}_i) dA = n_i dA$$

and then substituting into the equation to cancel out dA:

$$\mathbf{T}^{(\mathbf{n})} dA - \mathbf{T}^{(\boldsymbol{e}_i)} n_i dA = \rho \left(\frac{h}{3} dA \right) \boldsymbol{a}$$

To consider the limiting case as the tetrahedron shrinks to a point, $h \rightarrow 0$, that is the height of the tetrahedron approaches zero. As a result, the right-hand-side of the equation approaches 0, so the equation becomes,

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{T}^{(\boldsymbol{e}_i)} n_i$$

Interpretation

We are now to interpret the components $\boldsymbol{T}^{(\boldsymbol{e}_i)}$ in this equation. Consider $\boldsymbol{T}^{(\boldsymbol{e}_1)}$ the value of the resultant stress traction on the first coordinate plane. Resolving this along the coordinate axes, we have,

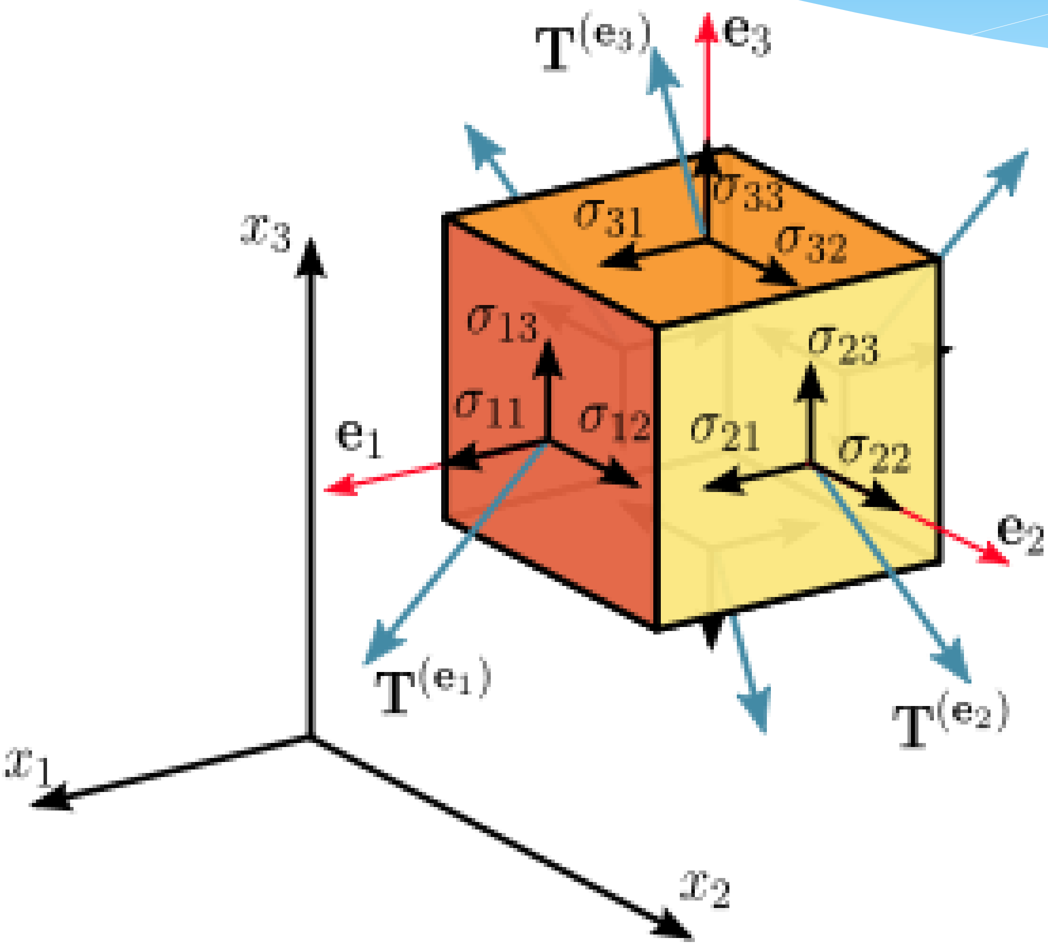
$$\begin{aligned}\boldsymbol{T}^{(\boldsymbol{e}_1)} &= \left[\boldsymbol{e}_1 \cdot \boldsymbol{T}^{(\boldsymbol{e}_1)}\right]\boldsymbol{e}_1 + \left[\boldsymbol{e}_2 \cdot \boldsymbol{T}^{(\boldsymbol{e}_1)}\right]\boldsymbol{e}_2 + \left[\boldsymbol{e}_3 \cdot \boldsymbol{T}^{(\boldsymbol{e}_1)}\right]\boldsymbol{e}_3 \\ &= \sigma_{11}\boldsymbol{e}_1 + \sigma_{12}\boldsymbol{e}_2 + \sigma_{13}\boldsymbol{e}_3 \\ &= \sigma_{1j}\boldsymbol{e}_j\end{aligned}$$

Where the scalar quantity σ_{1j} is defined by the above equation as,

$$\sigma_{1j} = \boldsymbol{e}_j \cdot \boldsymbol{T}^{(\boldsymbol{e}_1)}$$

or in general, we write the components as,

$\sigma_{ij} = \boldsymbol{e}_j \cdot \boldsymbol{T}^{(\boldsymbol{e}_i)}, i = 1,2,3$


$\mathbf{T}^{(\mathbf{e}_3)}$
$\mathbf{e}_3$
σ_{33}
σ_{31}
σ_{32}
x_3
σ_{13}
σ_{23}
σ_{11}
σ_{12}
σ_{21}
$\mathbf{e}_1$
σ_{22}
$\mathbf{e}_2$
$\mathbf{T}^{(\mathbf{e}_1)}$
$\mathbf{T}^{(\mathbf{e}_2)}$
x_1
x_2

Interpretation

* Above figure is a graphical depiction of this definition where we can see that $\sigma_{ij} = \boldsymbol{e}_j \cdot \boldsymbol{T}^{(\boldsymbol{e}_i)}$ is the scalar component of the stress vector on the i coordinate plane in the j direction. For any coordinate plane therefore, we may write, $\boldsymbol{T}^{(\boldsymbol{e}_i)} = \sigma_{ij}\boldsymbol{e}_j$, so that the stress or traction vector on an arbitrary plane determined by its orientation in the outward normal $\boldsymbol{n}$,
$$\boldsymbol{T}^{(\boldsymbol{n})} = \boldsymbol{T}^{(\boldsymbol{e}_i)} n_i = \sigma_{ij}\boldsymbol{e}_j n_i$$
* Which is another way of saying that the component of the vector $\boldsymbol{T}^{(\boldsymbol{n})}$ along the j coordinate direction is $\sigma_{ij} n_i$ which is the contraction, $\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} = \boldsymbol{T}^{(\mathbf{n})}$. This proves Cauchy Theorem.

Normal Stress Again

* Obviously, σ_{ij} are the components of the stress tensor in the coordinate system of computation that we have used so far. The Cauchy law, being a vector equation remains valid in all coordinate systems. We will then have to compute the different values of the stress tensor in the system of choice when for any reason we choose to work in not-Cartesian coordinates.
* Earlier on, we introduced the normal $t_n \equiv \boldsymbol{T}^{(\boldsymbol{n})} \cdot \boldsymbol{n}$ and shearing $\boldsymbol{T}^{(\boldsymbol{n})} - t_n \boldsymbol{n}$ components of the stress vector. We can compute these values now in terms of the scalar components of the stress tensor. Using Cauchy theorem, we have that,

$$\sigma = t_n \equiv \boldsymbol{T}^{(\boldsymbol{n})} \cdot \boldsymbol{n} = (\boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n}) \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\sigma}\boldsymbol{n}$$

Shear Stress

* This double contraction defines the scalar value which is the magnitude of the stress vector acting in the direction of the normal to the plane. The other important scalar quantity – the magnitude of the corresponding projection of the traction vector to the surface itself is obtained from Pythagoras theorem:
* $\tau = \left\|\boldsymbol{T}^{(\boldsymbol{n})} - t_n \boldsymbol{n}\right\| = \left\|\boldsymbol{T}^{(\boldsymbol{n})} - \sigma \boldsymbol{n}\right\|$

Tensor Bases

* The arguments that proved Cauchy theorem could have been based on non-Cartesian coordinate systems. The stress equation must remain unchanged however and the stress tensor characterizing the state of stress at a point remains an invariant. As it is with any second-order tensor, its components in general coordinates will be obtained from,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \sigma^i_{.j}\mathbf{g}_i \otimes \mathbf{g}^j = \sigma_j^{.i}\mathbf{g}^j \otimes \mathbf{g}_i$$

* Because the Cauchy stress is based on areas in the deformed body, it is a spatial quantity and appropriate to an Eulerian kinematic formulation. Whenever it is more convenient to work in Lagrangian coordinates, a stress tensor based on this may become more appropriate.

Nominal Stress

From last chapter, we recall that the vector current area in a deformed body $d\boldsymbol{a} = J\boldsymbol{F}^{-T}d\boldsymbol{A}$ where $d\boldsymbol{A}$ is its image in the material coordinates. The resultant force acting on an area bounded by ΔS in the deformed coordinates can be obtained, using Cauchy stress theorem as,

$$dP = \int_{\Delta S} \boldsymbol{\sigma} d\boldsymbol{a} = \int_{\Delta S_0} J\boldsymbol{\sigma}\left(\boldsymbol{F}^{-T}d\boldsymbol{A}\right)$$

$$= \int_{\Delta S_0} J\boldsymbol{\sigma}\boldsymbol{F}^{-T}d\boldsymbol{A}$$

$$= \int_{\Delta S_0} \boldsymbol{N}^{T}d\boldsymbol{A}$$

* where $\boldsymbol{N} \equiv J\boldsymbol{F}^{-1}\boldsymbol{\sigma}$ is called the *Nominal Stress* Tensor

First Piola-Kirchhoff Stress

- The transpose of the nominal stress is called the *First Piola-Kirchhoff Stress Tensor* $\boldsymbol{s}$. Consequently,

$$\boldsymbol{s} \equiv \boldsymbol{N}^T = J\boldsymbol{\sigma}^T \boldsymbol{F}^{-T}$$

This transformation applied to $\boldsymbol{\sigma}$ to produce $\boldsymbol{s}$, when applied as in this or any other case to any tensor is called a *Piola transformation*.

- In the above equation, we have used the yet-to be proved fact that the Cauchy stress tensor is symmetric. This will be established later.
- The components of the $\boldsymbol{s}$ stress are the forces acting on the deformed configuration, per unit undeformed area. They are thought of as acting on the undeformed solid.

Kirchhoff Stress

- Recall that the ratio of elemental volumes in the spatial to material, coordinates

$$\frac{dv}{dV} = J = \det \boldsymbol{F}$$

where $\boldsymbol{F}$ is the deformation gradient of the transformation. It therefore follows that the Kirchhoff stress $\boldsymbol{\tau}$, defined by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}$$

is no different from Cauchy Stress Tensor during isochoric (or volume-preserving) deformations and motions. It is used widely in numerical algorithms in metal plasticity (where there is no change in volume during plastic deformation).

Second Piola-Kirchhoff Stress

- The second Piola Kirchhoff Stress, $\boldsymbol{\Xi}$ is useful in Conjugate stress analysis. It is defined by,

$$\boldsymbol{N} \equiv J\boldsymbol{F}^{-1}\boldsymbol{\sigma} = \boldsymbol{\Xi}\boldsymbol{F}^{T}$$

- In terms of the Cauchy stress $\boldsymbol{\sigma}$ we can write,

$$\boldsymbol{\Xi} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}.$$

- In terms of the nominal stress tensor, $\boldsymbol{\Xi} = \boldsymbol{N}\boldsymbol{F}^{-T}$.
- Note that the First Piola-Kirchhoff Stress Tensor is not symmetric. On the other hand, the 2nd P-K tensor is symmetric just like the Cauchy tensor. Furthermore if the material rotates without a change in stress state (rigid rotation), the components of the 2nd Piola-Kirchhoff stress tensor will remain constant, irrespective of material orientation. It is also important as an energy conjugate to the Lagrange strain.

Extremal Values & Principal Planes

- As a second order symmetric tensor, the Cauchy Stress Tensor has positive definite quadratic forms when operated on by real vectors. Its eigenvalues are real and its principal invariants are:
- $I_1(\boldsymbol{\sigma}) = \mathrm{tr}(\boldsymbol{\sigma}) = \sigma_{ij}\mathbf{g}^i \cdot \mathbf{g}^j = \sigma^{ij}\mathbf{g}_i \cdot \mathbf{g}_j = \sigma^{i}_{.j}\mathbf{g}_i \cdot \mathbf{g}^j = \sigma_j^{.i}\mathbf{g}^j \cdot \mathbf{g}_i = \sigma_i^{.i}$
- $I_2(\boldsymbol{\sigma}) = \dfrac{[\mathrm{tr}(\boldsymbol{\sigma})]^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)}{2} = \dfrac{1}{2}\left[\sigma_i^i \sigma_j^j - \sigma_j^i \sigma_i^j\right]$
- (Half of the square of the trace of the tensor minus the trace of the square of the tensor $\boldsymbol{\sigma}$. The third and last scalar invariant of a tensor $\boldsymbol{\sigma}$ is its determinant if it exists.)
- $I_3(\boldsymbol{\sigma}) = \det(\boldsymbol{\sigma}) = e^{ijk}\sigma_i^1 \sigma_j^2 \sigma_k^3$

Stress Systems

- Because of its simplicity, working and thinking in the principal coordinate system is often very useful when considering the state of the elastic medium at a particular point. Principal stresses are often expressed in the following equation for evaluating stresses in the x and y directions or axial and bending stresses on a part. The principal normal stresses can then be used to calculate the *Von Mises* stress and ultimately the safety factor and margin of safety

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Using just the part of the equation under the square root is equal to the maximum and minimum shear stress for plus and minus. This is shown as:

$$\tau_1, \tau_2 = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

We are now in a position to recalculate the normal and shearing components of the traction vector $\boldsymbol{T}^{(\boldsymbol{n})}$

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{\sigma n} = \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix}\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$= \sigma_1 (n_1)^2 + \sigma_2 (n_2)^2 + \sigma_3 (n_3)^2$$

And the shear stress is the scalar magnitude of,

$$\boldsymbol{T}^{(\boldsymbol{n})} - t_n\boldsymbol{n} = \boldsymbol{\sigma n} - \boldsymbol{n}\cdot\boldsymbol{\sigma n} = (\boldsymbol{\sigma} - \boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}$$

$$= \left(\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} - \begin{bmatrix} n_1\sigma_1 & 0 & 0 \\ 0 & n_2\sigma_2 & 0 \\ 0 & 0 & n_3\sigma_3 \end{bmatrix}\right)\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\therefore \tau = \sqrt{(\sigma_1 n_1)^2 + (\sigma_2 n_2)^2 + (\sigma_3 n_3)^2 - [\sigma_1 (n_1)^2 + \sigma_2 (n_2)^2 + \sigma_3 (n_3)^2]^2}$$

* From these, we can write in a more compact form that,

$$\sigma^2 + \tau^2 = (\sigma_1 n_1)^2 + (\sigma_2 n_2)^2 + (\sigma_3 n_3)^2$$

Mohr Circle

* The values of normal and shear stresses given in terms of the principal coordinates can be solved (using *Mathematica*) after noting that, $(n_1)^2 + (n_2)^2 + (n_3)^2 = 0$ as shown below to obtain the Mohr circle of stress in three dimensions.
* The values of the square of the direction cosines follow the three circles

$(n_1)^2 \propto \sigma^2 + \tau^2 - \sigma\sigma_3 + \sigma_2(\sigma_3 - \sigma)(n_2)^2 \propto \sigma^2 + \tau^2 - \sigma\sigma_3 + \sigma_1(\sigma_3 - \sigma)$

$(n_3)^2 \propto \sigma^2 + \tau^2 - \sigma\sigma_2 + \sigma_1(\sigma_2 - \sigma)$

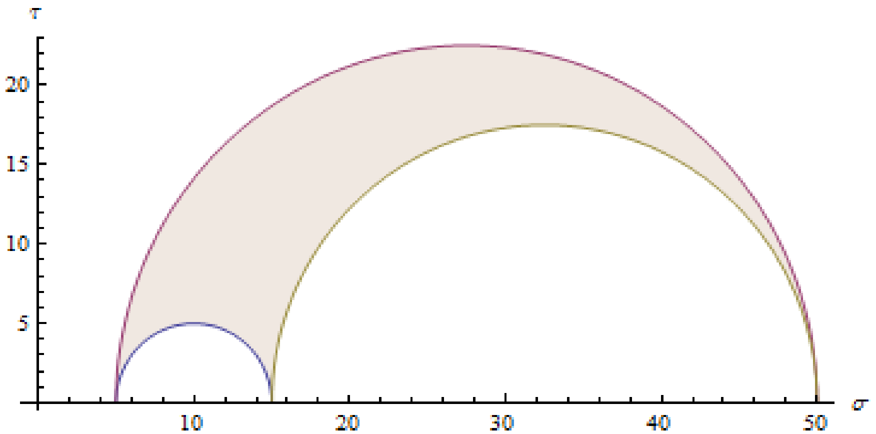
* In the figure, we used the symbols, $\alpha_1 = (n_1)^2, \alpha_2 = (n_2)^2$and $\alpha_3 = (n_3)^2$ with principal stress values of $\sigma_1 = 50, \sigma_2 = 15, \sigma_3 = 5.$

```
Simplify[
 Solve[{-σ + α1 σ1 + α2 σ2 + α3 σ3 == 0, -σ^2 - τ^2 + α1 σ1^2 + α2 σ2^2 + α3 σ3^2 == 0,
   α1 + α2 + α3 == 1}, {α1, α2, α3}]]
```

$$\left\{\left\{\alpha_1 \to \frac{\sigma^2 + \tau^2 - \sigma\,\sigma_3 + \sigma_2\,(-\sigma + \sigma_3)}{(\sigma_1 - \sigma_2)\,(\sigma_1 - \sigma_3)},\ \alpha_2 \to -\frac{\sigma^2 + \tau^2 - \sigma\,\sigma_3 + \sigma_1\,(-\sigma + \sigma_3)}{(\sigma_1 - \sigma_2)\,(\sigma_2 - \sigma_3)},\ \alpha_3 \to \frac{\sigma^2 + \tau^2 - \sigma\,\sigma_2 + \sigma_1\,(-\sigma + \sigma_2)}{(\sigma_1 - \sigma_3)\,(\sigma_2 - \sigma_3)}\right\}\right\}$$

```
Simplify[
  Solve[{-σ + α1 σ1 + α2 σ2 + α3 σ3 == 0, -σ^2 - τ^2 + α1 σ1^2 + α2 σ2^2 + α3 σ3^2 == 0,
    α1 + α2 + α3 == 1}, {α1, α2, α3}]] /. {σ1 → 50, σ2 → 15, σ3 → 5}
```

```
Plot[{y[1, σ], y[2, σ], y[3, σ]}, {σ, 0, 50}, AspectRatio → 1/2,
 Filling → {{1 → {2}}, {3 → {2}}}, FillingStyle → LightBrown,
 AxesLabel → {σ, τ}]
```



States of Stress

We present here examples of states of stress as an illustration:

1. Hydrostatic Pressure
2. Uniaxial Tension
3. Equal Biaxial tension
4. Pure Shear

Hydrostatic Pressure

$$\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{1} = -pg^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j) = -p(\mathbf{g}_i \otimes \mathbf{g}^i)$$

In a Cartesian system, we have, $\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{1} = -p(\mathbf{e}_i \otimes \mathbf{e}_i)$.

For a surface whose outward normal is the unit vector $\mathbf{n}$, the traction

$$\begin{aligned}\mathbf{T}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} = -pg^{ij}(\mathbf{g}_i \otimes \mathbf{g}_j)\mathbf{n} \\ &= -pg^{ij}\mathbf{g}_i(\mathbf{n} \cdot \mathbf{g}_j) = -pg^{ij}n_j\mathbf{g}_i \\ &= -pn^i\mathbf{g}_i = -p\mathbf{n}\end{aligned}$$

Furthermore, the scalar normal traction

$$\sigma = \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} = -p\mathbf{n} \cdot \mathbf{n} = -p$$

And the shear stress: the magnitude of the vector difference between the traction and the vector normal traction.

$$\tau = \left\| \mathbf{T}^{(\mathbf{n})} - \sigma\mathbf{n} \right\| = 0$$

Uniaxial Tension

Define uniaxial tension as a state where there is a normal traction t in a given direction (unit vector $\boldsymbol{\alpha}$)and zero traction in directions perpendicular to it. The Cauchy stress for this is $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})$. The traction on a surface with unit vector $\mathbf{n}$ is,

$$\begin{aligned} \mathbf{T}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha})\boldsymbol{n} \\ &= t\boldsymbol{\alpha}(\boldsymbol{n} \cdot \boldsymbol{\alpha}) = t\boldsymbol{\alpha} \cos \phi \end{aligned}$$

From which we can see that, under uniaxial stress, the traction is always directed along the vector $\boldsymbol{\alpha}$ no matter what the orientation of the surface might be.

Of course, when $\phi = \pi/2$, traction is zero.

Biaxial Traction

Consider the stress tensor, $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + \boldsymbol{\beta} \otimes \boldsymbol{\beta})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are perpendicular directions. The traction on an arbitrary plane $\mathbf{n}$

$$\begin{aligned}\mathbf{T}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\alpha} + \boldsymbol{\beta} \otimes \boldsymbol{\beta})\mathbf{n} \\ &= t\boldsymbol{\alpha}(\mathbf{n} \cdot \boldsymbol{\alpha}) + t\boldsymbol{\beta}(\mathbf{n} \cdot \boldsymbol{\beta}) = t\boldsymbol{\alpha} \cos\phi + t\boldsymbol{\beta} \sin\phi\end{aligned}$$

The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{t, t, 0\}$. $\boldsymbol{\sigma}(\mathbf{x})$ this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = t\mathbf{u}_1 \otimes \mathbf{u}_1 + t\mathbf{u}_2 \otimes \mathbf{u}_2$$

Pure Shear

Consider the stress tensor, $\boldsymbol{\sigma}(\mathbf{x}) = t(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha})$ where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are perpendicular directions. The traction on an arbitrary plane $\mathbf{n}$

$$\begin{aligned}\mathbf{T}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\boldsymbol{x})\mathbf{n} = t(\boldsymbol{\alpha} \otimes \boldsymbol{\beta} + \boldsymbol{\beta} \otimes \boldsymbol{\alpha})\mathbf{n} \\ &= t\boldsymbol{\alpha}(\mathbf{n} \cdot \boldsymbol{\beta}) + t\boldsymbol{\beta}(\mathbf{n} \cdot \boldsymbol{\alpha}) = t\boldsymbol{\alpha} \sin\phi + t\boldsymbol{\beta}\cos\phi\end{aligned}$$

The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{t, -t, 0\}$. Furthermore, $\boldsymbol{\sigma}$ in this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = t\mathbf{u}_1 \otimes \mathbf{u}_1 - t\mathbf{u}_2 \otimes \mathbf{u}_2$$

Theory of Heat Fluxes

Fourier Stokes Heat Flux Theorem

Heat Fluxes

Cauchy's postulated the existence of a stress tensor on the basis of which the load intensity arising from mechanical forces (body and surface forces) can be elegantly quantified in a consistent manner.

The counterpart of this for thermal exchanges with the surroundings is the Fourier-Stokes heat flux theorem.

Fourier-Stokes Theorem

Consider a spatial volume $\mathscr{B}_t$ with boundary $\partial\mathscr{B}_t$. Let the outwardly drawn normal to the surface be the unit vector n. Fourier Stokes heat Flux Principle states that $\exists\mathbf{q}(\mathbf{x},t)$ – vector field such that, the heat flow out of the volume is

$$\begin{aligned} h(\mathbf{x},t,\partial\mathscr{B}_t) &= h(\mathbf{x},t,\mathbf{n}) \\ &= -\boldsymbol{q}(\mathbf{x},t)\cdot\mathbf{n} \end{aligned}$$

$\boldsymbol{q}(\mathbf{x},t)$ is called the heat flux through the surface.

Fourier-Stokes Theorem

Heat flow into the spatial volume $\mathscr{B}_t$ volume is

$$\int_{\partial \mathscr{B}_t} \boldsymbol{q}(\mathbf{x}, t) \cdot \mathbf{n}\, da = \int_{\partial \mathscr{B}} J \boldsymbol{q}(\mathbf{x}, t) \cdot \mathbf{F}^{-\mathrm{T}} \mathbf{N}\, dA$$

$$= \int_{\partial \mathscr{B}} J \mathbf{F}^{-1} \boldsymbol{q}(\mathbf{x}, t) \cdot d\boldsymbol{A}$$

$\mathbf{Q}(\boldsymbol{X}, t)$is a Piola transformation of the spatial heat flux. That is,

$$\mathbf{Q}(\boldsymbol{X}, t) = J \mathbf{F}^{-1} \boldsymbol{q}(\mathbf{x}, t)$$