Osezua Again!

1. Thank you for the several worked examples on both the Tensor Calculus slide and the 50 questions. I am wondering if you could throw a little more light on:

$$\frac{\partial I_2(\boldsymbol{S})}{\partial \boldsymbol{S}} = \frac{1}{2}\frac{\partial}{\partial S_i^j}\left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta\right]\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta_\alpha^i \delta_j^\alpha S_\beta^\beta + \delta_\beta^i \delta_j^\beta S_\alpha^\alpha - \delta_\beta^i \delta_j^\alpha S_\alpha^\beta - \delta_\alpha^i \delta_j^\beta S_\beta^\alpha\right]\mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2}\left[\delta_j^i S_\beta^\beta + \delta_j^i S_\alpha^\alpha - S_i^j - S_i^j\right]\mathbf{g}_i \otimes \mathbf{g}^j = \cdots$$

The differential of the second invariant of a tensor w.r.t the tensor; how the several tensor components when divided by the tensor differential (in line 1) produced the several Kronecker Deltas with the tensor components (in line 2).

2. Can you also elaborate on the definition of the curl of a vector or tensor (especially a tensor).

oafak Replies:

I begin by correcting your question. Tensor theory does not make a provision for the “division” by a tensor. What is defined is the derivative which comes with a quotient law that, as we have seen, is deeply rooted in the Gateaux generalization of the differential. The interpretation of a derivative as a division is tempting; please avoid that temptation as much as you can. The origin of that temptation is the first lesson you had in differential calculus when the derivative was presented to you as the limit of a quotient. That is followed by the representation of a derivative as a quotient of dy over dx. Please get over that and always remember the actual definition as presented in the notes. Also read over the definition of the Frechét derivative.

The substantive issue here was adequately dealt with in my response to the the question asked by Atinsola. In that response, published under “Please follow Atinsola’s example”, I argued conclusively that,

$$\frac{\partial T^{\alpha}_{\beta}}{\partial T^{i}_{j}} = \delta^{j}_{\beta}\delta^{\alpha}_{i}$$

anytime the tensor $\mathbf{T}$ is symmetrical. There is no need to repeat that argument here. A repeated application of that simple rule, recalling that you are dealing with the difference of two separate products in $\frac{\partial}{\partial S^{j}_{i}}\left[S^{\alpha}_{\alpha}S^{\beta}_{\beta} - S^{\alpha}_{\beta}S^{\beta}_{\alpha}\right]$ which after applying the product rule to the above result immediately leads to what was earlier obtained. I emphasize again that here, you are not even performing a Frechét derivative here. Apart from the Kronecker Deltas generated, we are really dealing with partial derivatives. The only thing that makes it look strange is that the variables are indexed. I am confident that you will be satisfied after you have read my full response to Atinsola.

The Curl of a Vector

Now look at the issue of curl of a vector. Recall that the cross product of two vectors $\mathbf{u}\ (= u_i\mathbf{g}^i)$ and $\mathbf{v}\ (= v_i\mathbf{g}^i)$is,

$$\mathbf{u} \times \mathbf{v} = \epsilon^{ijk} u_j v_k \mathbf{g}_i$$

Imagine now a vector differential operator $\boldsymbol{\nabla} = \partial_i \mathbf{g}^i$. We can write the curl as the "cross product operation" of this differential operator on $\mathbf{v}$ in these two steps:

$$\boldsymbol{\nabla} \times \mathbf{v} = \epsilon^{ijk} \partial_j v_k \mathbf{g}_i$$

provided of course, we understand this to mean the taking of the covariant derivative of the value operated upon. *This matter is not trivial*! It is only as we take the covariant derivative that the bases can be treated as if they were constants. Otherwise, we have to differentiate the bases in addition. However, as we take covariant derivatives, the extra terms are automatically supplied. In this case, we therefore write that, $\boldsymbol{\nabla} \times \mathbf{v} = \epsilon^{ijk}(v_k)_{,j}\, \mathbf{g}_i$ or simply $\epsilon^{ijk} v_{k,j}\, \mathbf{g}_i$ as we have it in the notes. So we can write,

$$\operatorname{curl} \mathbf{v} = \boldsymbol{\nabla} \times \mathbf{v} = \epsilon^{ijk} \partial_j v_k \mathbf{g}_i = \epsilon^{ijk} v_{k,j}\, \mathbf{g}_i.$$

The Curl of a Tensor.

This definition follows the same line as before except the twist that you take the transpose before applying the differential operator. Everything else remains the same but for the fact that you now have two bases as you are dealing now with a tensor rather than a vector:

$$\operatorname{curl}\mathbf{T} = \boldsymbol{\nabla} \times \mathbf{T}^{\mathrm{T}} = \epsilon^{ijk}\partial_j T_{lk}\mathbf{g}_i \otimes \mathbf{g}^l = \epsilon^{ijk} T_{lk},_j \mathbf{g}_i \otimes \mathbf{g}^l$$

You follow these simple steps a while before you gain the confidence to apply the curl directly.

Two things to keep in mind are, firstly, that this is no longer simply a vector product but it is rather the operation of a differential operator. Secondly, the differentiation operation here is a covariant differentiation which of course degenerates into the usual partial differentiation when we are dealing with a constant orthonormal base system such as the Cartesian coordinate system.

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