On Saturday, August 9, 2014, Akande Olugbenga sent in this question:

Good day Sir,

Your effort exerted in making sure we understand this paradoxically interesting course is highly valued and appreciated. Sir, please throw more light on the solution to the question below why and what concept actually made the last step equals zero. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

In particular, why is $\epsilon_{ijk} u^i u^j v^k = 0$?

oafak Replies:

We begin by showing the equivalency of the two expressions supplied by Olugbenga. First observe that,

$$\mathbf{u}\times\ \mathbf{v}=\epsilon_{ijk}u^j v^k\mathbf{g}^i=\epsilon^{ijk}u_j v_k\mathbf{g}_i$$

each representation is correct so we can chose which is more convenient at any particular time. The vector $\mathbf{u}=u^l\mathbf{g}_l=u_l\mathbf{g}^l$. Again, we can select either one of these as we like. Let us choose the first representation in each case and argue that

$$\begin{aligned}\mathbf{u}\cdot(\mathbf{u}\times\ \mathbf{v})&=u^l\mathbf{g}_l\cdot\left(\epsilon_{ijk}u^j v^k\mathbf{g}^i\right)=u^l\epsilon_{ijk}u^j v^k\mathbf{g}_l\cdot\mathbf{g}^i\\&=u^l\epsilon_{ijk}u^j v^k\delta_l^i=u^i\epsilon_{ijk}u^j v^k\\&=\epsilon_{ijk}u^i u^j v^k\end{aligned}$$

Even if you had chosen the second vector representation for $\mathbf{u} = u_l \mathbf{g}^l$, the answer would have remained unchanged. In that case, we would have,

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= u_l \mathbf{g}^l \cdot \left(\epsilon_{ijk} u^j v^k \mathbf{g}^i\right) = u_l \epsilon_{ijk} u^j v^k \mathbf{g}^l \cdot \mathbf{g}^i \\ &= u_l \epsilon_{ijk} u^j v^k g^{li} = u^i \epsilon_{ijk} u^j v^k \\ &= \epsilon_{ijk} u^i u^j v^k \end{aligned}$$

As the index-raising property of the metric tensor would have performed the magic of raising the index because $u_l g^{li} = u^i$. We therefore only need now to show that the last expression vanishes. To do this, recall that u^i, u^j, and v^k are all scalar variables. Their order of multiplication is therefore immaterial. That means swapping i and j does not change the expression. But swapping these same indices in the tensor component ϵ_{ijk} immediately negates the sign! You therefore have the situation where a symmetric quantity, ie $\left(u^i u^j\right)$ and an antisymmetric quantity ϵ_{ijk} are in a product. The result of such a product is always zero. You will have to always watch out for such situations and should reach this conclusion immediately. I will spend the rest of this answer by proving this to be true. Consider

$$\begin{aligned} \epsilon_{ijk}u^i u^j &= \epsilon_{ijk}u^j u^i \\ &= -\epsilon_{jik}u^j u^i \\ &= -\epsilon_{ijk}u^i u^j = 0 \end{aligned}$$

The first equality coming from swapping components u^i, u^j. The second from swapping the indices i and j in ϵ_{ijk} which negates the quantity. The third equality coming from letting index i play the role of j and vice versa.

The result of zero coming from the recognition that a scalar quantity that equals its own negative must be zero!

A smart student does not need to be proving this every time. It is a principle that the product of a symmetric quantity with an anti-symmetric quantity in the same indices will give a result of zero.