1. Divergence of a product: Given that φ is a scalar field and $\mathbf{v}$ a vector field, show that $\operatorname{div}(\varphi\mathbf{v}) = (\operatorname{grad}\varphi)\cdot\mathbf{v} + \varphi \operatorname{div}\mathbf{v}$

$$\begin{aligned}\operatorname{grad}(\varphi\mathbf{v}) &= \left(\varphi v^i\right)_{,j}\mathbf{g}_i\otimes\mathbf{g}^j \\ &= \varphi_{,j}\, v^i\mathbf{g}_i\otimes\mathbf{g}^j + \varphi v^i{}_{,j}\,\mathbf{g}_i\otimes\mathbf{g}^j \\ &= \mathbf{v}\otimes(\operatorname{grad}\varphi) + \varphi \operatorname{grad}\mathbf{v}\end{aligned}$$

Now, $\operatorname{div}(\varphi\mathbf{v}) = \operatorname{tr}\big(\operatorname{grad}(\varphi\mathbf{v})\big)$. Taking the trace of the above, we have:

$$\operatorname{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\operatorname{grad}\varphi) + \varphi \operatorname{div}\mathbf{v}$$

2. Show that $\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) = (\operatorname{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\operatorname{grad}\mathbf{v})^{\mathrm{T}}\mathbf{u}$

$\mathbf{u}\cdot\mathbf{v} = u^i v_i$ is a scalar sum of components.

$$\begin{aligned}\operatorname{grad}(\mathbf{u}\cdot\mathbf{v}) &= \left(u^i v_i\right)_{,j}\mathbf{g}^j \\ &= u^i{}_{,j}\, v_i\mathbf{g}^j + u^i v_{i,j}\,\mathbf{g}^j\end{aligned}$$

Now $\operatorname{grad}\mathbf{u} = u^i{}_{,j}\,\mathbf{g}_i\otimes\mathbf{g}^j$ swapping the bases, we have that,

$$(\operatorname{grad}\mathbf{u})^{\mathrm{T}} = u^i{}_{,j}\left(\mathbf{g}^j\otimes\mathbf{g}_i\right).$$

Writing $\mathbf{v} = v_k\mathbf{g}^k$, we have that, $(\operatorname{grad}\mathbf{u})^{\mathrm{T}}\mathbf{v} = u^i{}_{,j}\, v_k\left(\mathbf{g}^j\otimes\mathbf{g}_i\right)\mathbf{g}^k = u^i{}_{,j}\, v_k\mathbf{g}^j\delta_i^k = u^i{}_{,j}\, v_i\mathbf{g}^j$

It is easy to similarly show that $u^i v_{i},_j \mathbf{g}^j = (\text{grad}\,\mathbf{v})^{\text{T}}\mathbf{u}$. Clearly,

$$\begin{aligned}\text{grad}(\mathbf{u}\cdot\mathbf{v}) &= \left(u^i v_i\right),_j \mathbf{g}^j = u^i,_j v_i \mathbf{g}^j + u^i v_i,_j \mathbf{g}^j \\ &= (\text{grad}\,\mathbf{u})^{\text{T}}\mathbf{v} + (\text{grad}\,\mathbf{v})^{\text{T}}\mathbf{u}\end{aligned}$$

As required.

3. Show that $\text{grad}(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\,\times)\text{grad}\,\mathbf{v} - (\mathbf{v}\,\times)\text{grad}\,\mathbf{u}$

$$\mathbf{u}\times\mathbf{v} = \epsilon^{ijk} u_j v_k \mathbf{g}_i$$

Recall that the gradient of this vector is the tensor,

$$\begin{aligned}\text{grad}(\mathbf{u}\times\mathbf{v}) &= \left(\epsilon^{ijk} u_j v_k\right),_l \mathbf{g}_i \otimes \mathbf{g}^l \\ &= \epsilon^{ijk} u_j,_l v_k \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} u_j v_k,_l \mathbf{g}_i \otimes \mathbf{g}^l \\ &= -\epsilon^{ikj} u_j,_l v_k \mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk} u_j v_k,_l \mathbf{g}_i \otimes \mathbf{g}^l \\ &= -(\mathbf{v}\,\times)\text{grad}\,\mathbf{u} + (\mathbf{u}\,\times)\text{grad}\,\mathbf{v}\end{aligned}$$

4. Show that $\text{div}\,(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}$

We already have the expression for $\text{grad}(\mathbf{u}\times\mathbf{v})$ above; remember that

$$\begin{aligned}\text{div}\,(\mathbf{u}\times\mathbf{v}) &= \text{tr}[\text{grad}(\mathbf{u}\times\mathbf{v})] \\ &= -\epsilon^{ikj} u_j,_l v_k \mathbf{g}_i \cdot \mathbf{g}^l + \epsilon^{ijk} u_j v_k,_l \mathbf{g}_i \cdot \mathbf{g}^l\end{aligned}$$

$$= -\epsilon^{ikj} u_{j,l}\, v_k \delta_i^l + \epsilon^{ijk} u_j v_{k,l}\, \delta_i^l$$
$$= -\epsilon^{ikj} u_{j,i}\, v_k + \epsilon^{ijk} u_j v_{k,i} = \mathbf{v} \cdot \text{curl}\, \mathbf{u} - \mathbf{u} \cdot \text{curl}\, \mathbf{v}$$

5. Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that curl $(\phi\mathbf{v}) = \phi$ curl $\mathbf{v} + (\text{grad}\, \phi) \times \mathbf{v}$.

$$\begin{aligned} \text{curl}\, (\phi\mathbf{v}) &= \epsilon^{ijk} (\phi v_k)_{,j}\, \mathbf{g}_i \\ &= \epsilon^{ijk} \left(\phi_{,j}\, v_k + \phi v_{k,j}\right) \mathbf{g}_i \\ &= \epsilon^{ijk} \phi_{,j}\, v_k \mathbf{g}_i + \epsilon^{ijk} \phi v_{k,j}\, \mathbf{g}_i \\ &= (\nabla\phi) \times \mathbf{v} + \phi\, \text{curl}\, \mathbf{v} \end{aligned}$$

6. Show that div $(\mathbf{u} \otimes \mathbf{v}) = (\text{div}\, \mathbf{v})\mathbf{u} + (\text{grad}\, \mathbf{u})\mathbf{v}$

$\mathbf{u} \otimes \mathbf{v}$ is the tensor, $u^i v^j \mathbf{g}_i \otimes \mathbf{g}_j$. The gradient of this is the third order tensor,

$$\text{grad}\, (\mathbf{u} \otimes \mathbf{v}) = \left(u^i v^j\right)_{,k} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k$$

And by divergence, we mean the contraction of the last basis vector:

$$\begin{aligned} \text{div}\, (\mathbf{u} \otimes \mathbf{v}) &= \left(u^i v^j\right)_{,k} (\mathbf{g}_i \otimes \mathbf{g}_j) \mathbf{g}^k \\ &= \left(u^i v^j\right)_{,k} \mathbf{g}_i \delta_j^k = \left(u^i v^j\right)_{,j} \mathbf{g}_i \\ &= u^i{}_{,j}\, v^j \mathbf{g}_i + u^i v^j{}_{,j}\, \mathbf{g}_i \end{aligned}$$

$$= (\text{grad } \mathbf{u})\mathbf{v} + (\text{div } \mathbf{v})\mathbf{u}$$

7. For a scalar field ϕ and a tensor field $\mathbf{T}$ show that $\text{grad } (\phi\mathbf{T}) = \phi\text{grad } \mathbf{T} + \mathbf{T} \otimes \text{grad}\phi$. Also show that $\text{div } (\phi\mathbf{T}) = \phi \text{ div } \mathbf{T} + \mathbf{T}\text{grad}\phi$

$$\begin{aligned}
\text{grad}(\phi\mathbf{T}) &= \left(\phi T^{ij}\right)_{,k} \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\
&= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k \\
&= \mathbf{T} \otimes \text{grad}\phi + \phi\text{grad } \mathbf{T}
\end{aligned}$$

Furthermore, we can contract the last two bases and obtain,

$$\begin{aligned}
\text{div}(\phi\mathbf{T}) &= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^k \\
&= \left(\phi_{,k} T^{ij} + \phi T^{ij}{}_{,k}\right)\mathbf{g}_i \delta_j^k \\
&= T^{ik} \phi_{,k} \mathbf{g}_i + \phi T^{ik}{}_{,k} \mathbf{g}_i \\
&= \mathbf{T}\text{grad}\phi + \phi \text{ div } \mathbf{T}
\end{aligned}$$

8. For two arbitrary vectors, $\mathbf{u}$ and $\mathbf{v}$, show that $\text{grad}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times)\text{grad}\mathbf{v} - (\mathbf{v} \times)\text{grad}\mathbf{u}$

$$\text{grad}(\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk} u_j v_k\right)_{,l} \mathbf{g}_i \otimes \mathbf{g}^l$$

$$
\begin{aligned}
&= \left(\epsilon^{ijk} u_{j,l}\, v_k + \epsilon^{ijk} u_j v_{k,l}\right) \mathbf{g}_i \otimes \mathbf{g}^l \\
&= \left(u_{j,l}\, \epsilon^{ijk} v_k + v_{k,l}\, \epsilon^{ijk} u_j\right) \mathbf{g}_i \otimes \mathbf{g}^l \\
&= -(\mathbf{v}\,\times)\text{grad}\mathbf{u} + \ (\mathbf{u}\,\times)\text{grad}\mathbf{v}
\end{aligned}
$$

9. For a vector field $\mathbf{u}$**,** show that $\text{grad}(\mathbf{u}\,\times)$ is a third ranked tensor. Hence or otherwise show that $\text{div}(\mathbf{u}\,\times) = -\text{curl}\,\mathbf{u}$**.**

The second–order tensor $(\mathbf{u}\,\times)$ is defined as $\epsilon^{ijk} u_j \mathbf{g}_i \otimes \mathbf{g}_k$. Taking the covariant derivative with an independent base, we have

$$
\text{grad}(\mathbf{u}\,\times) = \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \otimes \mathbf{g}_k \otimes \mathbf{g}^l
$$

This gives a third order tensor as we have seen. Contracting on the last two bases,

$$
\begin{aligned}
\text{div}(\mathbf{u}\,\times) &= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \otimes \mathbf{g}_k \cdot \mathbf{g}^l \\
&= \epsilon^{ijk} u_{j,l}\, \mathbf{g}_i \delta_k^l \\
&= \epsilon^{ijk} u_{j,k}\, \mathbf{g}_i \\
&= -\text{curl}\,\mathbf{u}
\end{aligned}
$$

10. Show that $\text{div}\,(\phi\mathbf{1}) = \text{grad}\,\phi$

Note that $\phi\mathbf{1} = \left(\phi g_{\alpha\beta}\right) \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$. Also note that

$$\text{grad}\,\phi\mathbf{1} = \left(\phi g_{\alpha\beta}\right)_{,i}\mathbf{g}^{\alpha}\otimes\mathbf{g}^{\beta}\otimes\mathbf{g}^{i}$$

The divergence of this third order tensor is the contraction of the last two bases:

$$\begin{aligned}\text{div}\,(\phi\mathbf{1}) = \text{tr}(\text{grad}\,\phi\mathbf{1}) &= \left(\phi g_{\alpha\beta}\right)_{,i}\left(\mathbf{g}^{\alpha}\otimes\mathbf{g}^{\beta}\right)\mathbf{g}^{i} = \left(\phi g_{\alpha\beta}\right)_{,i}\mathbf{g}^{\alpha}g^{\beta i}\\ &= \phi_{,i}\,g_{\alpha\beta}g^{\beta i}\mathbf{g}^{\alpha}\\ &= \phi_{,i}\,\delta^{i}_{\alpha}\mathbf{g}^{\alpha} = \phi_{,i}\,\mathbf{g}^{i} = \text{grad}\,\phi\end{aligned}$$

11. Show that $\text{curl}\,(\phi\mathbf{1}) = (\,\text{grad}\,\phi)\times$

Note that $\phi\mathbf{1} = \left(\phi g_{\alpha\beta}\right)\mathbf{g}^{\alpha}\otimes\mathbf{g}^{\beta}$, and that $\text{curl}\,\boldsymbol{T} = \epsilon^{ijk}T_{\alpha k,j}\;\mathbf{g}_i\otimes\mathbf{g}^{\alpha}$ so that,

$$\begin{aligned}\text{curl}\,(\phi\mathbf{1}) &= \epsilon^{ijk}(\phi g_{\alpha k})_{,j}\,\mathbf{g}_i\otimes\mathbf{g}^{\alpha}\\ &= \epsilon^{ijk}\left(\phi_{,j}\,g_{\alpha k}\right)\mathbf{g}_i\otimes\mathbf{g}^{\alpha} = \epsilon^{ijk}\phi_{,j}\,\mathbf{g}_i\otimes\mathbf{g}_k\\ &= (\,\text{grad}\,\phi)\times\end{aligned}$$

12. Show that $\text{curl}\,(\mathbf{v}\times) = (\text{div}\,\mathbf{v})\mathbf{1} - \text{grad}\,\mathbf{v}$

$$(\mathbf{v}\times) = \epsilon^{\alpha\beta k}v_{\beta}\,\mathbf{g}_{\alpha}\otimes\mathbf{g}_k$$

$$\text{curl}\,\boldsymbol{T} = \epsilon^{ijk}T_{\alpha k,j}\;\mathbf{g}_i\otimes\mathbf{g}^{\alpha}$$

so that

$$\text{curl}\,(\mathbf{v}\times) = \epsilon^{ijk}\epsilon^{\alpha\beta k}v_{\beta,j}\;\mathbf{g}_i\otimes\mathbf{g}_{\alpha}$$

$$= \left(g^{i\alpha}g^{j\beta} - g^{i\beta}g^{j\alpha}\right) v_{\beta},_j \ \mathbf{g}_i \otimes \mathbf{g}_\alpha$$
$$= v^j,_j \mathbf{g}^\alpha \otimes \mathbf{g}_\alpha - v^i,_j \mathbf{g}_i \otimes \mathbf{g}^j$$
$$= (\text{div } \mathbf{v})\mathbf{1} - \text{ grad } \mathbf{v}$$

13. Show that div $(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl } \mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v}$

$$\text{div } (\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk}u_j v_k\right),_i$$

Noting that the tensor ϵ^{ijk} behaves as a constant under a covariant differentiation, we can write,

$$\text{div } (\mathbf{u} \times \mathbf{v}) = \left(\epsilon^{ijk}u_j v_k\right),_i$$
$$= \epsilon^{ijk}u_j,_i \, v_k + \epsilon^{ijk}u_j v_k,_i$$
$$= \mathbf{v} \cdot \text{curl } \mathbf{u} - \mathbf{u} \cdot \text{curl } \mathbf{v}$$

14. Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that curl $(\phi\mathbf{v}) = \phi \text{ curl } \mathbf{v} + (\nabla\phi) \times \mathbf{v}$.

$$\text{curl } (\phi\mathbf{v}) = \epsilon^{ijk}(\phi v_k),_j \, \mathbf{g}_i$$
$$= \epsilon^{ijk}\left(\phi,_j \, v_k + \phi v_k,_j \right)\mathbf{g}_i$$
$$= \epsilon^{ijk}\phi,_j \, v_k \mathbf{g}_i + \epsilon^{ijk}\phi v_k,_j \, \mathbf{g}_i$$

$$= (\nabla\phi) \times \mathbf{v} + \phi \,\mathrm{curl}\, \mathbf{v}$$

15. Show that $\mathrm{curl}\,(\mathrm{grad}\,\phi) = \mathbf{o}$

For any tensor $\mathbf{v} = v_\alpha \mathbf{g}^\alpha$

$$\mathrm{curl}\,\mathbf{v} = \epsilon^{ijk} v_{k,j}\ \mathbf{g}_i$$

Let $\mathbf{v} =\ \mathrm{grad}\,\phi$. Clearly, in this case, $v_k = \phi_{,k}$ so that $v_{k,j} = \phi_{,kj}$. It therefore follows that,

$$\mathrm{curl}\,(\mathrm{grad}\,\phi) = \epsilon^{ijk}\phi_{,kj}\,\mathbf{g}_i = \mathbf{0}.$$

The contraction of symmetric tensors with unsymmetric led to this conclusion. Note that this presupposes that the order of differentiation in the scalar field is immaterial. This will be true only if the scalar field is continuous – a proposition we have assumed in the above.

16. Show that $\mathrm{curl}\,(\mathrm{grad}\,\mathbf{v}) = \mathbf{0}$

For any tensor $\mathbf{T} = \mathrm{T}_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

$$\mathrm{curl}\,\mathbf{T} = \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

Let $\mathbf{T} = \text{grad}\,\mathbf{v}$. Clearly, in this case, $\mathrm{T}_{\alpha\beta} = v_{\alpha},_{\beta}$ so that $T_{\alpha k},_{j} = v_{\alpha},_{kj}$. It therefore follows that,

$$\text{curl}\,(\text{grad}\,\mathbf{v}) = \epsilon^{ijk} v_{\alpha},_{kj}\,\mathbf{g}_i \otimes \mathbf{g}^{\alpha} = \mathbf{0}.$$

The contraction of symmetric tensors with unsymmetric led to this conclusion. Note that this presupposes that the order of differentiation in the vector field is immaterial. This will be true only if the vector field is continuous – a proposition we have assumed in the above.

17. Show that $\text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}} = \text{grad}(\text{curl}\,\mathbf{v})$

From previous derivation, we can see that, $\text{curl}\,\mathbf{T} = \epsilon^{ijk} T_{\alpha k},_{j}\ \mathbf{g}_i \otimes \mathbf{g}^{\alpha}$. Clearly,

$$\text{curl}\,\mathbf{T}^{\mathrm{T}} = \epsilon^{ijk} T_{k\alpha},_{j}\ \mathbf{g}_i \otimes \mathbf{g}^{\alpha}$$

so that $\text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}} = \epsilon^{ijk} v_{k},_{\alpha j}\,\mathbf{g}_i \otimes \mathbf{g}^{\alpha}$. But $\text{curl}\,\mathbf{v} = \epsilon^{ijk} v_{k},_{j}\ \mathbf{g}_i$. The gradient of this is,

$$\text{grad}(\text{curl}\,\mathbf{v}) = \left(\epsilon^{ijk} v_{k},_{j}\right),_{\alpha}\mathbf{g}_i \otimes \mathbf{g}^{\alpha} = \epsilon^{ijk} v_{k},_{j\alpha}\,\mathbf{g}_i \otimes \mathbf{g}^{\alpha} = \text{curl}\,(\text{grad}\,\mathbf{v})^{\mathrm{T}}$$

18. Show that $\text{div}\,(\text{grad}\,\phi \times \text{grad}\,\vartheta) = 0$

$$\text{grad}\,\phi \times \text{grad}\,\theta = \epsilon^{ijk} \phi,_{j}\,\theta,_{k}\,\mathbf{g}_i$$

The gradient of this vector is the tensor,

$$\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta) = \left(\epsilon^{ijk}\phi,_j\,\theta,_k\right),_l\,\mathbf{g}_i \otimes \mathbf{g}^l$$
$$= \epsilon^{ijk}\phi,_{jl}\,\theta,_k\,\mathbf{g}_i \otimes \mathbf{g}^l + \epsilon^{ijk}\phi,_j\,\theta,_{kl}\,\mathbf{g}_i \otimes \mathbf{g}^l$$

The trace of the above result is the divergence we are seeking:

$$\text{div}\,(\text{grad}\,\phi \times \text{grad}\,\theta) = \text{tr}[\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta)]$$
$$= \epsilon^{ijk}\phi,_{jl}\,\theta,_k\,\mathbf{g}_i \cdot \mathbf{g}^l + \epsilon^{ijk}\phi,_j\,\theta,_{kl}\,\mathbf{g}_i \cdot \mathbf{g}^l$$
$$= \epsilon^{ijk}\phi,_{jl}\,\theta,_k\,\delta_i^l + \epsilon^{ijk}\phi,_j\,\theta,_{kl}\,\delta_i^l$$
$$= \epsilon^{ijk}\phi,_{ji}\,\theta,_k + \epsilon^{ijk}\phi,_j\,\theta,_{ki} = 0$$

Each term vanishing on account of the contraction of a symmetric tensor with an antisymmetric.

19. Show that $\text{curl curl}\,\mathbf{v} = \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$

Let $\mathbf{w} = \text{curl}\,\mathbf{v} \equiv \epsilon^{ijk}v_{k,j}\,\mathbf{g}_i$. But $\text{curl}\,\mathbf{w} \equiv \epsilon^{\alpha\beta\gamma}w_{\gamma,\beta}\,\mathbf{g}_\alpha$. Upon inspection, we find that $w_\gamma = g_{\gamma i}\epsilon^{ijk}v_{k,j}$ so that

$$\text{curl}\,\mathbf{w} \equiv \epsilon^{\alpha\beta\gamma}\left(g_{\gamma i}\epsilon^{ijk}v_{k,j}\right),_\beta\,\mathbf{g}_\alpha = g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}v_{k,j\beta}\,\mathbf{g}_\alpha$$

Now, it can be shown (**see below**) that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j}g^{\beta k} - g^{\alpha k}g^{\beta j}$ so that,

$$\text{curl}\,\mathbf{w} = \left(g^{\alpha j} g^{\beta k} - g^{\alpha k} g^{\beta j}\right) v_{k},_{j\beta}\,\mathbf{g}_\alpha$$
$$= v^\beta,_{j\beta}\,\mathbf{g}^j - v^\alpha,_{jj}\,\mathbf{g}_\alpha = \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$$

20. Show that $g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g^{\alpha j} g^{\beta k} - g^{\alpha k} g^{\beta j}$

Note that

$$g_{\gamma i}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk} = g_{\gamma i}\begin{vmatrix} g^{i\alpha} & g^{i\beta} & g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix} = \begin{vmatrix} g_{\gamma i}g^{i\alpha} & g_{\gamma i}g^{i\beta} & g_{\gamma i}g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$
$$= \begin{vmatrix} \delta^\alpha_\gamma & \delta^\beta_\gamma & \delta^\gamma_\gamma \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$
$$= \delta^\alpha_\gamma \begin{vmatrix} g^{j\beta} & g^{j\gamma} \\ g^{k\beta} & g^{k\gamma} \end{vmatrix} - \delta^\beta_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\gamma} \\ g^{k\alpha} & g^{k\gamma} \end{vmatrix} + \delta^\gamma_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$
$$= \begin{vmatrix} g^{j\beta} & g^{j\alpha} \\ g^{k\beta} & g^{k\alpha} \end{vmatrix} - \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} + 3\begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} = \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$
$$= g^{\alpha j} g^{\beta k} - g^{\alpha k} g^{\beta j}$$

21. Given that $\varphi(t) = |\mathbf{A}(t)|$, Show that $\dot{\varphi}(t) = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$

$$\varphi^2 \equiv \mathbf{A} : \mathbf{A}$$

Now,

$$\frac{d}{dt}(\varphi^2) = 2\varphi \frac{d\varphi}{dt} = \frac{d\mathbf{A}}{dt} : \mathbf{A} + \mathbf{A} : \frac{d\mathbf{A}}{dt} = 2\mathbf{A} : \frac{d\mathbf{A}}{dt}$$

as inner product is commutative. We can therefore write that

$$\frac{d\varphi}{dt} = \frac{\mathbf{A}}{\varphi} : \frac{d\mathbf{A}}{dt} = \frac{\mathbf{A}}{|\mathbf{A}(t)|} : \dot{\mathbf{A}}$$

as required.

22. Given a tensor field $\mathbf{T}$, obtain the vector $\mathbf{w} \equiv \mathbf{T}^{\mathrm{T}}\boldsymbol{v}$ and show that its divergence is $\mathbf{T} : (\nabla \mathbf{v}) + \mathbf{v} \cdot \operatorname{div} \mathbf{T}$

The divergence of $\boldsymbol{w}$ is the scalar sum , $(\boldsymbol{T}_{ji}\boldsymbol{v}^j)_{,i}$. Expanding the product covariant derivative we obtain,

$$\begin{aligned} \operatorname{div}(\mathbf{T}^{\mathrm{T}}\mathbf{v}) &= (T_{ji}v^j)_{,i} = T_{ji,i}\, v^j + T_{ji}v^j{}_{,i} \\ &= (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^{\mathrm{T}} \operatorname{grad} \mathbf{v}) \\ &= (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \mathbf{T} : (\operatorname{grad} \mathbf{v}) \end{aligned}$$

Recall that scalar product of two vectors is commutative so that

$$\text{div}\,(\mathbf{T}^{\mathrm{T}}\mathbf{v}) = \mathbf{T}:(\text{grad}\,\mathbf{v}) + \mathbf{v}\cdot\text{div}\,\mathbf{T}$$

23. For a second-order tensor $\mathbf{T}$ define curl $\mathbf{T} \equiv \epsilon^{ijk} T_{\alpha k},_j\ \mathbf{g}_i \otimes \mathbf{g}^\alpha$ show that for any constant vector $\boldsymbol{a}$, $(\text{curl}\,\mathbf{T})\,\boldsymbol{a} = \text{curl}\,(\mathbf{T}^{\mathrm{T}}\boldsymbol{a})$

Express vector $\boldsymbol{a}$ in the invariant form with covariant components as $\boldsymbol{a} = a^\beta \mathbf{g}_\beta$.

It follows that

$$\begin{aligned}(\text{curl}\,\mathbf{T})\,\boldsymbol{a} &= \epsilon^{ijk} T_{\alpha k},_j\ (\mathbf{g}_i \otimes \mathbf{g}^\alpha)\boldsymbol{a} \\ &= \epsilon^{ijk} T_{\alpha k},_j\ a^\beta (\mathbf{g}_i \otimes \mathbf{g}^\alpha)\mathbf{g}_\beta \\ &= \epsilon^{ijk} T_{\alpha k},_j\ a^\beta \mathbf{g}_i \delta_\beta^\alpha \\ &= \epsilon^{ijk} (T_{\alpha k}),_j\ \mathbf{g}_i a^\alpha \\ &= \epsilon^{ijk} (T_{\alpha k} a^\alpha),_j\ \mathbf{g}_i\end{aligned}$$

The last equality resulting from the fact that vector $\boldsymbol{a}$ is a constant vector. Clearly,

$$(\text{curl}\,\mathbf{T})\,\boldsymbol{a} = \text{curl}\,(\mathbf{T}^{\mathrm{T}}\boldsymbol{a})$$

24. For any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\text{curl}\,(\mathbf{u}\otimes\mathbf{v}) = [(\text{grad}\,\mathbf{u})\mathbf{v}\,\times]^T + (\text{curl}\,\mathbf{v})\otimes\boldsymbol{u}$ where $\mathbf{v}\,\times$ is the skew tensor $\epsilon^{ikj}v_k\,\mathbf{g}_i\otimes\mathbf{g}_j$.

Recall that the curl of a tensor $\boldsymbol{T}$ is defined by $\text{curl}\,\boldsymbol{T} \equiv \epsilon^{ijk}T_{\alpha k,j}\;\mathbf{g}_i\otimes\mathbf{g}^\alpha$.

Clearly therefore,

$$\begin{aligned}
\text{curl}\,(\boldsymbol{u}\otimes\boldsymbol{v}) &= \epsilon^{ijk}(u_\alpha v_k)_{,j}\;\mathbf{g}_i\otimes\mathbf{g}^\alpha = \epsilon^{ijk}\left(u_{\alpha,j}\,v_k + u_\alpha v_{k,j}\right)\mathbf{g}_i\otimes\mathbf{g}^\alpha \\
&= \epsilon^{ijk}u_{\alpha,j}\,v_k\,\mathbf{g}_i\otimes\mathbf{g}^\alpha + \epsilon^{ijk}u_\alpha v_{k,j}\;\mathbf{g}_i\otimes\mathbf{g}^\alpha \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\right)\otimes\left(u_{\alpha,j}\,\mathbf{g}^\alpha\right) + \left(\epsilon^{ijk}v_{k,j}\;\mathbf{g}_i\right)\otimes(u_\alpha\mathbf{g}^\alpha) \\
&= \left(\epsilon^{ijk}v_k\,\mathbf{g}_i\otimes\mathbf{g}_j\right)\left(u_{\alpha,\beta}\,\mathbf{g}^\beta\otimes\mathbf{g}^\alpha\right) + \left(\epsilon^{ijk}v_{k,j}\;\mathbf{g}_i\right)\otimes(u_\alpha\mathbf{g}^\alpha) \\
&= -(\mathbf{v}\,\times)(\text{grad}\,\mathbf{u})^{\boldsymbol{T}} + (\text{curl}\,\mathbf{v})\otimes\mathbf{u} = [(\text{grad}\,\mathbf{u})\mathbf{v}\,\times]^{\boldsymbol{T}} + (\text{curl}\,\mathbf{v})\otimes\mathbf{u}
\end{aligned}$$

upon noting that the vector cross is a skew tensor.

25. Show that $\text{curl}\,(\mathbf{u}\times\mathbf{v}) = \text{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$

The vector $\mathbf{w} \equiv \mathbf{u}\times\mathbf{v} = w_{\boldsymbol{k}}\mathbf{g}^{\boldsymbol{k}} = \epsilon_{k\alpha\beta}u^\alpha v^\beta\mathbf{g}^{\boldsymbol{k}}$ and $\text{curl}\,\mathbf{w} = \boldsymbol{\epsilon}^{\boldsymbol{ijk}}w_{k,j}\,\mathbf{g}_i$.

Therefore,

$$\begin{aligned}
\text{curl}\,(\mathbf{u}\times\mathbf{v}) &= \boldsymbol{\epsilon}^{\boldsymbol{ijk}}w_{k,j}\,\mathbf{g}_i \\
&= \boldsymbol{\epsilon}^{\boldsymbol{ijk}}\epsilon_{k\alpha\beta}\left(u^\alpha v^\beta\right)_{,j}\mathbf{g}_i
\end{aligned}$$

$$= \left(\delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha\right)(u^\alpha v^\beta)_{,j}\, \mathbf{g}_i$$
$$= \left(\delta^i_\alpha \delta^j_\beta - \delta^i_\beta \delta^j_\alpha\right)(u^\alpha{}_{,j}\, v^\beta + u^\alpha v^\beta{}_{,j}\,)\mathbf{g}_i$$
$$= \left[u^i{}_{,j}\, v^j + u^i v^j{}_{,j} - (u^j{}_{,j}\, v^i + u^j v^i{}_{,j}\,)\right]\mathbf{g}_i$$
$$= \left[(u^i v^j)_{,j} - (u^j v^i)_{,j}\,\right]\mathbf{g}_i$$
$$= \mathrm{div}(\mathbf{u} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{u})$$

since $\mathrm{div}(\mathbf{u} \otimes \mathbf{v}) = (u^i v^j)_{,\alpha}\, \mathbf{g}_i \otimes \mathbf{g}_j \cdot \mathbf{g}^\alpha = (u^i v^j)_{,j}\, \mathbf{g}_i$.

26. Given a scalar point function ϕ and a second-order tensor field $\mathbf{T}$, show that $\text{curl}\ (\phi \mathbf{T}) = \phi \text{ curl } \mathbf{T} + \big((\nabla\phi)\ \times\big)\mathbf{T}^{\mathrm{T}}$ where $[(\nabla\phi)\ \times]$ is the skew tensor $\epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k$

$$\text{curl}\ (\phi \boldsymbol{T}) \equiv \epsilon^{ijk}(\phi T_{\alpha k})_{,j}\ \ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$
$$= \epsilon^{ijk}\left(\phi_{,j}\, T_{\alpha k} + \phi T_{\alpha k},_j\,\right) \mathbf{g}_i \otimes \mathbf{g}^\alpha$$
$$= \epsilon^{ijk}\phi_{,j}\, T_{\alpha k}\, \mathbf{g}_i \otimes \mathbf{g}^\alpha + \phi\epsilon^{ijk} T_{\alpha k},_j\ \ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$
$$= \left(\epsilon^{ijk}\phi_{,j}\, \mathbf{g}_i \otimes \mathbf{g}_k\right)\left(T_{\alpha\beta}\mathbf{g}^\beta \otimes \mathbf{g}^\alpha\right) + \phi\epsilon^{ijk} T_{\alpha k},_j\ \ \mathbf{g}_i \otimes \mathbf{g}^\alpha$$
$$= \phi \text{ curl } \mathbf{T} + \big((\nabla\phi)\ \times\big)\mathbf{T}^{\mathrm{T}}$$

27. For a second-order tensor field $\boldsymbol{T}$, show that $\text{div}(\text{curl}\ \mathbf{T}) = \text{curl}(\text{div}\ \mathbf{T}^{\text{T}})$

Define the second order tensor S as

$$\text{curl}\ \mathbf{T} \equiv \epsilon^{ijk} T_{\alpha k,j}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha = S^i_{.\alpha} \mathbf{g}_i \otimes \mathbf{g}^\alpha$$

The gradient of $\boldsymbol{S}$ is $S^i_{.\alpha,\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta = \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \otimes \mathbf{g}^\beta$

Clearly,

$$\begin{aligned}\text{div}(\text{curl}\ \boldsymbol{T}) &= \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i \otimes \mathbf{g}^\alpha \cdot \mathbf{g}^\beta = \epsilon^{ijk} T_{\alpha k,j\beta}\ \mathbf{g}_i\ g^{\alpha\beta} \\ &= \epsilon^{ijk} T^\beta{}_{k,j\beta}\ \mathbf{g}_i = \text{curl}(\text{div}\ \mathbf{T}^{\text{T}})\end{aligned}$$

28. Show that if $\boldsymbol{\varphi}$ defined in the space spanned by orthogonal coordinates $\boldsymbol{x^i}$, then $\boldsymbol{\nabla^2 \left(x^i \varphi\right) = 2 \frac{\partial \varphi}{\partial x^i} + x^i \nabla^2 \varphi}$.

By definition, $\nabla^2\left(x^i \varphi\right) = g^{jk}\left(x^i \varphi\right)_{,jk}$. Expanding, we have

$$\begin{aligned} g^{jk}\left(x^i\varphi\right)_{,jk} &= g^{jk}\left(x^i{}_{,j}\varphi + x^i\varphi_{,j}\right)_{,k} = g^{jk}\left(\delta^i_j\varphi + x^i\varphi_{,j}\right)_{,k} \\ &= g^{jk}\left(\delta^i_j\varphi_{,k} + x^i{}_{,k}\varphi_{,j} + x^i\varphi_{,jk}\right) \\ &= g^{jk}\left(\delta^i_j\varphi_{,k} + \delta^i_k\varphi_{,j} + x^i\varphi_{,jk}\right) \\ &= g^{ik}\varphi_{,k} + g^{ij}\varphi_{,j} + x^i g^{jk}\varphi_{,jk} \end{aligned}$$

When the coordinates are orthogonal, this becomes,

$$\frac{2}{(h_i)^2}\frac{\partial \Phi}{\partial x^i} + x^i \nabla^2 \Phi$$

where we have suspended the summation rule and h_i is the square root of the appropriate metric tensor component.

29. In Cartesian coordinates, If the volume V is enclosed by the surface S, the position vector $\boldsymbol{r} = x^i \mathbf{g}_i$ and $\boldsymbol{n}$ is the external unit normal to each surface element, show that $\frac{1}{6}\int_S \nabla(\boldsymbol{r} \cdot \boldsymbol{r}) \cdot \boldsymbol{n} dS$ equals the volume contained in V.

$$\boldsymbol{r} \cdot \boldsymbol{r} = x^i x^j \mathbf{g}_i \cdot \mathbf{g}_j = x^i x^j g_{ij}$$

By the Divergence Theorem,

$$\int_S \nabla(\boldsymbol{r}\cdot\boldsymbol{r})\cdot \boldsymbol{n} dS = \int_V \nabla\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{r})]dV = \int_V \partial_l\left[\partial_k\left(x^i x^j g_{ij}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\, dV$$

$$= \int_V \partial_l\left[g_{ij}\left(x^i{}_{,k}\, x^j + x^i x^j{}_{,k}\right)\right]\mathbf{g}^l\cdot\ \mathbf{g}^k\, dV$$

$$= \int_V g_{ij} g^{lk}\left(\delta^i_k x^j + x^i \delta^j_k\right)_{,l}\ dV = \int_V 2 g_{ik} g^{lk} x^i{}_{,l}\ dV = \int_V 2\delta^l_i \delta^i_l\, dV$$

$$= 6\int_V dV$$

30. For any Euclidean coordinate system, show that $\mathbf{div\ u}\times\mathbf{v} = \mathbf{v\ curl\ u} - \mathbf{u\ curl\ v}$

Given the contravariant vector u^i and v^i with their associated vectors u_i and v_i, the contravariant component of the above cross product is $\epsilon^{ijk}u_j v_k$ .The required divergence is simply the contraction of the covariant x^i derivative of this quantity:

$$\left(\epsilon^{ijk}u_j v_k\right)_{,i} = \epsilon^{ijk}u_{j,i}v_k + \epsilon^{ijk}u_j v_{k,i}$$

where we have treated the tensor ε^{ijk} as a constant under the covariant derivative.

Cyclically rearranging the RHS we obtain,

$$\left(\epsilon^{ijk}u_j v_k\right)_{,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jki}v_{k,i} = v_k\epsilon^{kij}u_{j,i} + u_j\epsilon^{jik}v_{k,i}$$

where we have used the anti-symmetric property of the tensor ϵ^{ijk}. The last expression shows clearly that

$$\text{div}\,\mathbf{u}\times\mathbf{v} = \mathbf{v}\,\text{curl}\,\mathbf{u} - \mathbf{u}\,\text{curl}\,\mathbf{v}$$

as required.

31. For a general tensor field $\boldsymbol{T}$ show that, $\text{curl}(\text{curl}\,\boldsymbol{T}) = [\nabla^2(\text{tr}\,\boldsymbol{T}) - \text{div}(\text{div}\,\boldsymbol{T})]\boldsymbol{I} + \text{grad}(\text{div}\,\boldsymbol{T}) + \left(\text{grad}(\text{div}\,\boldsymbol{T})\right)^{\text{T}} - \text{grad}\left(\text{grad}\,(\text{tr}\boldsymbol{T})\right) - \nabla^2\boldsymbol{T}^{\text{T}}$

$$\begin{aligned}\text{curl}\,\boldsymbol{T} &= \boldsymbol{\epsilon}^{\alpha st}T_{\beta t,s}\,\mathbf{g}_\alpha\otimes\mathbf{g}^\beta \\ &= S^{\alpha}_{.\beta}\mathbf{g}_\alpha\otimes\mathbf{g}^\beta\end{aligned}$$

$$\text{curl}\,\boldsymbol{S} = \epsilon^{ijk}S^{\alpha}_{.k,j}\,\mathbf{g}_i\otimes\mathbf{g}_\alpha$$

so that

$$\text{curl}\,\boldsymbol{S} = \text{curl}(\text{curl}\,\boldsymbol{T}) = \epsilon^{ijk}\epsilon^{\alpha st}T_{kt,sj}\,\mathbf{g}_i\otimes\mathbf{g}_\alpha$$

$$= \begin{vmatrix} g^{i\alpha} & g^{is} & g^{it} \\ g^{j\alpha} & g^{js} & g^{jt} \\ g^{k\alpha} & g^{ks} & g^{kt} \end{vmatrix} T_{kt,sj}\, \mathbf{g}_i \otimes \mathbf{g}_\alpha$$

$$= \begin{bmatrix} g^{i\alpha}\left(g^{js}g^{kt} - g^{jt}g^{ks}\right) + g^{is}\left(g^{jt}g^{k\alpha} - g^{j\alpha}g^{kt}\right) \\ +g^{it}\left(g^{j\alpha}g^{ks} - g^{js}g^{k\alpha}\right) \end{bmatrix} T_{kt,sj}\, \mathbf{g}_i \otimes \mathbf{g}_\alpha$$

$$= \left[g^{js}T^{t}_{.t,sj} - T^{sj}_{..},_{sj}\right](\mathbf{g}^\alpha \otimes \mathbf{g}_\alpha) + \left[T^{\alpha j}_{..},_{sj} - g^{j\alpha}T^{t}_{.t,sj}\right](\mathbf{g}^s \otimes \mathbf{g}_\alpha)$$
$$+ \left[g^{j\alpha}T^{s.}_{.t,sj} - g^{js}T^{\alpha.}_{.t,sj}\right](\mathbf{g}^t \otimes \mathbf{g}_\alpha)$$

$$= [\nabla^2(\mathrm{tr}\,\boldsymbol{T}) - \mathrm{div}(\mathrm{div}\,\boldsymbol{T})]\boldsymbol{I} + \left(\mathrm{grad}(\mathrm{div}\,\boldsymbol{T})\right)^{\mathrm{T}} - \mathrm{grad}\left(\mathrm{grad}\,(\mathrm{tr}\mathbf{T})\right)$$
$$+ \left(\mathrm{grad}(\mathrm{div}\,\mathbf{T})\right) - \nabla^2\mathbf{T}^{\mathrm{T}}$$

32. When $\mathbf{T}$ is symmetric, show that $\mathrm{tr}(\mathrm{curl}\,\mathbf{T})$ vanishes.

$$\mathrm{curl}\,\mathbf{T} = \boldsymbol{\epsilon}^{ijk}T_{\beta k,j}\,\mathbf{g}_i \otimes \mathbf{g}^\beta$$
$$\mathrm{tr}(\mathrm{curl}\,\mathbf{T}) = \epsilon^{ijk}T_{\beta k,j}\,\mathbf{g}_i \cdot \mathbf{g}^\beta$$
$$= \epsilon^{ijk}T_{\beta k,j}\,\delta_i^\beta = \epsilon^{ijk}T_{ik,j}$$

which obviously vanishes on account of the symmetry and antisymmetry in i and k. In this case,

$$\begin{aligned}\operatorname{curl}(\operatorname{curl}\mathbf{T}) &= [\nabla^2(\operatorname{tr}\mathbf{T}) - \operatorname{div}(\operatorname{div}\mathbf{T})]\mathbf{1} - \operatorname{grad}\big(\operatorname{grad}(\operatorname{tr}\boldsymbol{T})\big) + 2\big(\operatorname{grad}(\operatorname{div}\boldsymbol{T})\big) \\ &\quad - \nabla^2\mathbf{T}\end{aligned}$$

as $\big(\operatorname{grad}(\operatorname{div}\mathbf{T})\big)^{\mathrm{T}} = \operatorname{grad}(\operatorname{div}\mathbf{T})$ if the order of differentiation is immaterial and $\mathbf{T}$ is symmetric.

33. For a scalar function Φ and a vector $\boldsymbol{v}^{\boldsymbol{i}}$ show that the divergence of the vector $\boldsymbol{v}^{\boldsymbol{i}}\boldsymbol{\Phi}$ is equal to, $\mathbf{v}\cdot\boldsymbol{\nabla}\Phi + \Phi\, div\,\mathbf{v}$

$$\left(v^i\Phi\right)_{,i} = \Phi v^i{}_{,i} + v^i\Phi_{,\mathrm{i}}$$

Hence the result.

34. Show that $\operatorname{curl}\mathbf{u}\times\mathbf{v} = (\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u}) + (\mathbf{u}\cdot\operatorname{div}\mathbf{v}) - (\mathbf{v}\cdot\operatorname{div}\mathbf{u}) - (\mathbf{u}\cdot\boldsymbol{\nabla}\,\mathbf{v})$

Taking the associated (covariant) vector of the expression for the cross product in the last example, it is straightforward to see that the LHS in indicial notation is,

$$\epsilon^{lmi}\left(\epsilon_{ijk}u^j v^k\right)_{,m}$$

Expanding in the usual way, noting the relation between the alternating tensors and the Kronecker deltas,

$$\epsilon^{lmi}\left(\varepsilon_{ijk}u^j v^k\right)_{,m} = \delta^{lmi}_{jki}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right)$$

$$= \delta^{lm}_{jk}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right) = \begin{vmatrix} \delta^l_j & \delta^m_j \\ \delta^l_k & \delta^m_k \end{vmatrix}\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right)$$

$$= \left(\delta^l_j\delta^m_k - \delta^l_k\delta^m_j\right)\left(u^j{}_{,m}v^k - u^j v^k{}_{,m}\right)$$

$$= \delta^l_j\delta^m_k u^j{}_{,m}v^k - \delta^l_j\delta^m_k u^j v^k{}_{,m} + \delta^l_k\delta^m_j u^j{}_{,m}v^k - \delta^l_k\delta^m_j u^j v^k{}_{,m}$$

$$= u^l{}_{,m}v^m - u^m{}_{,m}v^l + u^l v^m{}_{,m} - u^m v^l{}_{,m}$$

Which is the result we seek in indicial notation.

35. . In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i\mathbf{e}_i$. Show that (a) $x_{,j} = \frac{x_j}{x}$, (b) $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$, (c) $x_{,ii} = \frac{2}{x}$, (d) If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ $U_{,ii} = \mathbf{0}$ and $\operatorname{div}\left(\frac{\mathbf{r}}{x}\right) = \frac{2}{x}$.

$$(a)\ \ x = \sqrt{x_i x_i}$$

$$x_{,j} = \frac{\partial\sqrt{x_i x_i}}{\partial x_j} = \frac{\partial\sqrt{x_i x_i}}{\partial(x_i x_i)} \times \frac{\partial(x_i x_i)}{\partial x_j} = \frac{1}{2\sqrt{x_i x_i}}\left[x_i\delta_{ij} + x_i\delta_{ij}\right] = \frac{x_j}{x}.$$

$$(b)\ x_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial\sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right) = \frac{x\frac{\partial x_i}{\partial x_j} - x_i\frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2}$$

$$= \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$$

$$(c)\ x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3} = \frac{3}{x} - \frac{(x)^2}{(x)^3} = \frac{2}{x}.$$

$(d)\ U = \frac{1}{x}$ so that

$$U_{,j} = \frac{\partial\frac{1}{x}}{\partial x_j} = \frac{\partial\frac{1}{x}}{\partial x} \times \frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$

Consequently,

$$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right) = \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i\frac{\partial}{\partial x_j}(x^3)}{x^6}$$

$$= \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial(x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6} = \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$

$$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5} = \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$

$$\text{div}\left(\frac{\mathbf{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x}x_{j,j} + \left(\frac{1}{x}\right)_{,j} = \frac{3}{x} + x_j\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\frac{dx}{dx_j}\right)$$

$$= \frac{3}{x} + x_j\left[-\left(\frac{1}{x^2}\right)\frac{x_j}{x}\right] = \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$

36. For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times$.

The tensor $(\mathbf{u}\times) = -\epsilon_{lmn}u^n\mathbf{g}^l \otimes \mathbf{g}^m$ similarly, $(\mathbf{v}\times) = -\epsilon^{\alpha\beta\gamma}v_\gamma\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$ and $(\mathbf{w}\times) = -\epsilon^{ijk}w_k\mathbf{g}_i \otimes \mathbf{g}_j$. Clearly,

$$
\begin{aligned}
(\mathbf{u}\,\times)(\mathbf{v}\,\times)(\mathbf{w}\,\times) &= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}^l\otimes\mathbf{g}^m)(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= -\epsilon^{\alpha\beta\gamma}\epsilon_{lmn}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\delta^l_\beta\delta^m_i\\
&= -\epsilon^{\alpha l\gamma}\epsilon_{lin}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&= -\epsilon^{l\alpha\gamma}\epsilon_{lni}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&= -(\delta^\alpha_n\delta^\gamma_i-\delta^\alpha_i\delta^\gamma_n)\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\\
&= -\epsilon^{ijk}u^\alpha v_i w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)+\epsilon^{ijk}u^\gamma v_\gamma w_k(\mathbf{g}_i\otimes\mathbf{g}_j)\\
&= [\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w})-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\,\times]
\end{aligned}
$$

37. Show that $[\mathbf{u},\mathbf{v},\mathbf{w}]=\mathrm{tr}[(\mathbf{u}\,\times)(\mathbf{v}\,\times)(\mathbf{w}\,\times)]$

In the above we have shown that $(\mathbf{u}\,\times)(\mathbf{v}\,\times)(\mathbf{w}\,\times)=[\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w})-(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\,\times]$

Because the vector cross is traceless, the trace of $[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\,\times]=0$. The trace of the first term, $\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w})$ is obviously the same as $[\mathbf{u},\mathbf{v},\mathbf{w}]$ which completes the proof.

38. Show that $(\mathbf{u}\times)(\mathbf{v}\times) = (\mathbf{u}\cdot\mathbf{v})\mathbf{1} - \mathbf{u}\otimes\mathbf{v}$ and that $\mathrm{tr}[(\mathbf{u}\times)(\mathbf{v}\times)] = 2(\mathbf{u}\cdot\mathbf{v})$

$$(\mathbf{u}\times)(\mathbf{v}\times) = -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}^l\otimes\mathbf{g}^m)$$
$$= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)\delta^l_\beta = -\epsilon_{\beta mn}\epsilon^{\beta\gamma\alpha}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)$$
$$= [\delta^\gamma_n\delta^\alpha_m - \delta^\gamma_m\delta^\alpha_n]u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)$$
$$= u^n v_n(\mathbf{g}_\alpha\otimes\mathbf{g}^\alpha) - u^n v_m(\mathbf{g}_n\otimes\mathbf{g}^m) = (\mathbf{u}\cdot\mathbf{v})\mathbf{1} - \mathbf{u}\otimes\mathbf{v}$$

Obviously, the trace of this tensor is $2(\mathbf{u}\cdot\mathbf{v})$

39. The position vector in the above example $\boldsymbol{r} = x_i\boldsymbol{e}_i$. Show that (a) div $\boldsymbol{r} = \mathbf{3}$, **(b)** div $(\boldsymbol{r}\otimes\boldsymbol{r}) = \mathbf{4}\boldsymbol{r}$, **(c)** div $\boldsymbol{r} = 3$, and (d) grad $\boldsymbol{r} = \mathbf{1}$ and (e) curl $(\boldsymbol{r}\otimes\boldsymbol{r}) = -\boldsymbol{r}\times$

$$\mathrm{grad}\,\boldsymbol{r} = x_{i,\boldsymbol{j}}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_{\boldsymbol{j}}$$
$$= \delta_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_{\boldsymbol{j}} = \mathbf{1}$$
$$\mathrm{div}\,\boldsymbol{r} = x_{i,\boldsymbol{j}}\,\boldsymbol{e}_i\cdot\boldsymbol{e}_{\boldsymbol{j}}$$
$$= \delta_{ij}\delta_{ij} = \delta_{jj} = 3.\,\boldsymbol{r}\otimes\boldsymbol{r} = x_i\boldsymbol{e}_i\otimes x_j\boldsymbol{e}_j = x_i x_j\boldsymbol{e}_i\otimes\boldsymbol{e}_{\boldsymbol{j}}\mathrm{grad}(\boldsymbol{r}\otimes\boldsymbol{r})$$
$$= (x_i x_j)_{,k}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_{\boldsymbol{j}}\otimes\boldsymbol{e}_{\boldsymbol{k}} = (x_{i,k}\,x_j + x_i x_{j,k})\boldsymbol{e}_i\otimes\boldsymbol{e}_{\boldsymbol{j}}\cdot\boldsymbol{e}_{\boldsymbol{k}}$$
$$= (\delta_{ik}\boldsymbol{x}_{\boldsymbol{j}} + \boldsymbol{x}_{\boldsymbol{i}}\delta_{jk})\delta_{jk}\boldsymbol{e}_i = (\delta_{ik}\boldsymbol{x}_{\boldsymbol{k}} + \boldsymbol{x}_{\boldsymbol{i}}\delta_{jj})\boldsymbol{e}_i$$

$$= 4x_i \boldsymbol{e}_i = 4\boldsymbol{r}$$

$$\text{curl}(\boldsymbol{r} \otimes \boldsymbol{r}) = \epsilon_{\alpha\beta\gamma}\left(x_i x_\gamma\right)_{,\beta} \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i$$

$$= \epsilon_{\alpha\beta\gamma}\left(x_{i,\beta}\, x_\gamma + x_i x_{\gamma,\beta}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i$$

$$= \epsilon_{\alpha\beta\gamma}\left(\delta_{i\beta} x_\gamma + x_i \delta_{\gamma\beta}\right)\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i$$

$$= \epsilon_{\alpha i\gamma} x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i + \epsilon_{\alpha\beta\beta} x_i \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i = -\epsilon_{\alpha\gamma i} x_\gamma \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_i = -\boldsymbol{r} \times$$

40. Define the magnitude of tensor $\mathbf{A}$ as, $|\mathbf{A}| = \sqrt{\text{tr}(\mathbf{A}\mathbf{A}^{\text{T}})}$ Show that $\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$

By definition, given a scalar α, the derivative of a scalar function of a tensor $f(\mathbf{A})$ is

$$\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} f(\mathbf{A} + \alpha\mathbf{B})$$

for any arbitrary tensor **B.**

In the case of $f(\mathbf{A}) = |\mathbf{A}|$,

$$\frac{\partial |\mathbf{A}|}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} |\mathbf{A} + \alpha\mathbf{B}|$$

$$|\mathbf{A} + \alpha\mathbf{B}| = \sqrt{\text{tr}(\mathbf{A} + \alpha\mathbf{B})(\mathbf{A} + \alpha\mathbf{B})^{\text{T}}} = \sqrt{\text{tr}(\mathbf{A}\mathbf{A}^{\text{T}} + \alpha\mathbf{B}\mathbf{A}^{\text{T}} + \alpha\mathbf{A}\mathbf{B}^{\text{T}} + \alpha^2\mathbf{B}\mathbf{B}^{\text{T}})}$$

Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write,

$$\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}|\mathbf{A}+\alpha\mathbf{B}| = \lim_{\alpha\to 0}\frac{\text{tr}\,(\mathbf{B}\mathbf{A}^{\text{T}})+\text{tr}\,(\mathbf{A}\mathbf{B}^{\text{T}})+2\alpha\text{tr}\,(\mathbf{B}\mathbf{B}^{\text{T}})}{2\sqrt{\text{tr}(\mathbf{A}\mathbf{A}^{\text{T}}+\alpha\mathbf{B}\mathbf{A}^{\text{T}}+\alpha\mathbf{A}\mathbf{B}^{\text{T}}+\alpha^2\mathbf{B}\mathbf{B}^{\text{T}})}} = \frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}}$$

$$= \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

So that,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}}:\mathbf{B} = \frac{\mathbf{A}}{|\mathbf{A}|}:\mathbf{B}$$

or,

$$\frac{\partial|\mathbf{A}|}{\partial\mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

as required since $\mathbf{B}$ is arbitrary.

41. Show that $\frac{\partial I_3(\boldsymbol{S})}{\partial\boldsymbol{S}} = \frac{\partial\det(\boldsymbol{S})}{\partial\boldsymbol{S}} = \boldsymbol{S}^c$ the cofactor of $\boldsymbol{S}$.

Clearly $\boldsymbol{S}^c = \det(\boldsymbol{S})\,\boldsymbol{S}^{-\text{T}} = \boldsymbol{I}_3(\boldsymbol{S})\,\boldsymbol{S}^{-\text{T}}$. Details of this for the contravariant components of a tensor is presented below. Let

$$\det(\boldsymbol{S}) \equiv |\boldsymbol{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha\delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha\delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha\delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}\left[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\boldsymbol{S}$

42. For a scalar variable α, if the tensor $\mathbf{T} = \mathbf{T}(\alpha)$ and $\dot{\mathbf{T}} \equiv \frac{d\mathbf{T}}{d\alpha}$, Show that

$$\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\mathrm{tr}(\dot{\mathbf{T}}\mathbf{T}^{-1})$$

Let $\boldsymbol{A} \equiv \dot{\boldsymbol{T}}\boldsymbol{T}^{-1}$ so that, $\dot{\boldsymbol{T}} = \boldsymbol{A}\boldsymbol{T}$. In component form, we have $\dot{T}_j^i = A_m^i T_j^m$.

Therefore,

$$
\begin{aligned}
\frac{d}{d\alpha}\det(\boldsymbol{T}) &= \frac{d}{d\alpha}\left(\epsilon^{ijk}T_i^1T_j^2T_k^3\right) = \epsilon^{ijk}\left(\dot{T}_i^1T_j^2T_k^3 + T_i^1\dot{T}_j^2T_k^3 + T_i^1T_j^2\dot{T}_k^3\right) \\
&= \epsilon^{ijk}\left(A_l^1T_i^lT_j^2T_k^3 + T_i^1A_m^2T_j^mT_k^3 + T_i^1T_j^2A_n^3T_k^n\right) \\
&= \epsilon^{ijk}\left[\left(A_1^1T_i^1 + \boxed{A_2^1T_i^2} + \boxed{A_3^1T_i^3}\right)T_j^2T_k^3 + T_i^1\left(\boxed{A_1^2T_j^1} + A_2^2T_j^2\right.\right. \\
&\quad \left.\left. + \boxed{A_3^2T_j^3}\right)T_k^3 + T_i^1T_j^2\left(\boxed{A_1^3T_k^1} + \boxed{A_2^3T_k^2} + A_3^3T_k^3\right)\right]
\end{aligned}
$$

All the boxed terms in the above equation vanish on account of the contraction of a symmetric tensor with an antisymmetric one.

(For example, the first boxed term yields, $\epsilon^{ijk}A_2^1T_i^2T_j^2T_k^3$

Which is symmetric as well as antisymmetric in i and j. It therefore vanishes. The same is true for all other such terms.)

$$
\begin{aligned}
\frac{d}{d\alpha}\det(\mathbf{T}) &= \epsilon^{ijk}\left[\left(A_1^1T_i^1\right)T_j^2T_k^3 + T_i^1\left(A_2^2T_j^2\right)T_k^3 + T_i^1T_j^2\left(A_3^3T_k^3\right)\right] \\
&= A_m^m\epsilon^{ijk}T_i^1T_j^2T_k^3 = \mathrm{tr}\left(\dot{\mathbf{T}}\mathbf{T}^{-1}\right)\det(\mathbf{T})
\end{aligned}
$$

as required.

43. Without breaking down into components, establish the fact that $\frac{\partial \det(\mathbf{T})}{\partial \mathbf{T}} = \mathbf{T}^c$

Start from Liouville's Theorem, given a scalar parameter such that $\mathbf{T} = \mathbf{T}(\alpha)$,

$$\frac{\partial}{\partial \alpha}\left(\det(\mathbf{T})\right) = \det(\mathbf{T}) \operatorname{tr}\left[\left(\frac{\partial \mathbf{T}}{\partial \alpha}\right) \mathbf{T}^{-1}\right] = [\det(\mathbf{T})\, \mathbf{T}^{-\mathrm{T}}] : \left(\frac{\partial \mathbf{T}}{\partial \alpha}\right)$$

By the simple rules of multiple derivative,

$$\frac{\partial}{\partial \alpha}\left(\det(\mathbf{T})\right) = \left[\frac{\partial}{\partial \mathbf{T}}\left(\det(\mathbf{T})\right)\right] : \left(\frac{\partial \mathbf{T}}{\partial \alpha}\right)$$

It therefore follows that,

$$\left[\frac{\partial}{\partial \mathbf{T}}\left(\det(\mathbf{T})\right) - [\det(\mathbf{T})\, \mathbf{T}^{-\mathrm{T}}]\right] : \left(\frac{\partial \mathbf{T}}{\partial \alpha}\right) = 0$$

Hence

$$\frac{\partial}{\partial \mathbf{T}}\left(\det(\mathbf{T})\right) = [\det(\mathbf{T})\, \mathbf{T}^{-\mathrm{T}}] = \mathbf{T}^{\mathrm{c}}$$

44. **[Gurtin 3.4.2a]** If T is invertible, show that $\frac{\partial}{\partial \mathbf{T}}(\log \det(\mathbf{T})) = \mathbf{T}^{-\mathrm{T}}$

$$\frac{\partial}{\partial \mathbf{T}}(\log \det(\mathbf{T})) = \frac{\partial(\log \det(\mathbf{T}))}{\partial \det(\mathbf{T})} \frac{\partial \det(\mathbf{T})}{\partial \mathbf{T}}$$

$$= \frac{1}{\det(\mathbf{T})}\mathbf{T}^{\mathbf{c}} = \frac{1}{\det(\mathbf{T})}\det(\mathbf{T})\,\mathbf{T}^{-\mathbf{T}}$$
$$= \mathbf{T}^{-\mathbf{T}}$$

45. **[Gurtin 3.4.2a]** If $\mathbf{T}$ is invertible, show that $\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathbf{T}}$

$$\frac{\partial}{\partial \mathbf{T}}(\log\det(\mathbf{T}^{-1})) = \frac{\partial(\log\det(\mathbf{T}^{-1}))}{\partial\det(\mathbf{T}^{-1})}\frac{\partial\det(\mathbf{T}^{-1})}{\partial \mathbf{T}^{-1}}\frac{\partial \mathbf{T}^{-1}}{\partial \mathbf{T}}$$
$$= \frac{1}{\det(\mathbf{T}^{-1})}\mathbf{T}^{-\mathbf{c}}(-\mathbf{T}^{-2})$$
$$= \frac{1}{\det(\mathbf{T}^{-1})}\det(\mathbf{T}^{-1})\,\mathbf{T}^{\mathbf{T}}(-\mathbf{T}^{-2})$$
$$= -\mathbf{T}^{-\mathbf{T}}$$

46. Given that $\mathbf{A}$ is a constant tensor, Show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \mathbf{A}^{\mathrm{T}}$

In invariant components terms, let $\mathbf{A} = \mathrm{A}^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$ and let $\mathbf{S} = \mathrm{S}_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta}$.

$$\mathbf{AS} = \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}(\mathbf{g}_i \otimes \mathbf{g}_j)(\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta})$$

$$= \mathrm{A}^{ij}\mathrm{S}_{\alpha\beta}\left(\mathbf{g}_i \otimes \mathbf{g}^\beta\right)\delta_j^\alpha$$
$$= \mathrm{A}^{ij}\mathrm{S}_{j\beta}\left(\mathbf{g}_i \otimes \mathbf{g}^\beta\right)$$
$$\mathrm{tr}(\mathbf{AS}) = \mathrm{A}^{ij}\mathrm{S}_{j\beta}\left(\mathbf{g}_i \cdot \mathbf{g}^\beta\right)$$
$$= \mathrm{A}^{ij}\mathrm{S}_{j\beta}\delta_i^\beta = \mathrm{A}^{ij}\mathrm{S}_{ji}$$
$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}) = \frac{\partial}{\partial \mathrm{S}_{\alpha\beta}}\mathrm{tr}(\mathbf{AS})\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= \frac{\partial \mathrm{A}^{ij}\mathrm{S}_{ji}}{\partial \mathrm{S}_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$
$$= \mathrm{A}^{ij}\delta_j^\alpha\delta_i^\beta\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \mathrm{A}^{ij}\mathbf{g}_j \otimes \mathbf{g}_i = \mathbf{A}^\mathrm{T} = \frac{\partial}{\partial \mathbf{S}}(\mathbf{A}^\mathrm{T}:\mathbf{S})$$

as required.

47. Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{ASB}^\mathrm{T}) = \mathbf{A}^\mathrm{T}\mathbf{B}$

First observe that $\mathrm{tr}(\mathbf{ASB}^\mathrm{T}) = \mathrm{tr}(\mathbf{B}^\mathrm{T}\mathbf{AS})$. If we write, $\mathbf{C} \equiv \mathbf{B}^\mathrm{T}\mathbf{A}$, it is obvious from the above that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{CS}) = \mathbf{C}^\mathrm{T}$. Therefore,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathbf{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$$

48. Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathbf{B}^{\mathrm{T}}\mathbf{A}$

Observe that $\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS}^{\mathrm{T}}) = \mathrm{tr}[\mathbf{S}(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}] = \mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}]$

[The transposition does not alter trace; neither does a cyclic permutation. Ensure you understand why each equality here is true.] Consequently,

$$\frac{\partial}{\partial \mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \frac{\partial}{\partial \mathbf{S}}\mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}] = [(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}]^{\mathbf{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$$

49. Let $\boldsymbol{S}$ be a symmetric and positive definite tensor and let $I_1(\boldsymbol{S}), I_2(\boldsymbol{S})\text{and}I_3(\boldsymbol{S})$ be the three principal invariants of $\boldsymbol{S}$ show that *(a)* $\frac{\partial \boldsymbol{I}_1(\boldsymbol{S})}{\partial \boldsymbol{S}} = \mathbf{1}$ the identity tensor, *(b)* $\frac{\partial \boldsymbol{I}_2(\boldsymbol{S})}{\partial \boldsymbol{S}} = I_1(\boldsymbol{S})\mathbf{1} - \boldsymbol{S}$ and (c) $\frac{\partial I_3(\boldsymbol{S})}{\partial \boldsymbol{S}} = I_3(\boldsymbol{S})\,\boldsymbol{S}^{-1}$

$\frac{\partial \boldsymbol{I}_1(\boldsymbol{S})}{\partial \boldsymbol{S}}$ can be written in the invariant component form as,

$$\frac{\partial I_1(\boldsymbol{S})}{\partial \boldsymbol{S}} = \frac{\partial I_1(\boldsymbol{S})}{\partial S_i^j}\mathbf{g}_i \otimes \mathbf{g}^j$$

Recall that $I_1(\mathbf{S}) = \mathrm{tr}(\mathbf{S}) = S_\alpha^\alpha$ hence

$$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_i^j} \mathbf{g}_i \otimes \mathbf{g}^j = \frac{\partial S_\alpha^\alpha}{\partial S_i^j} \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \delta_\alpha^i \delta_j^\alpha \mathbf{g}_i \otimes \mathbf{g}^j = \delta_j^i \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \mathbf{1}$$

which is the identity tensor as expected.

$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}}$ in a similar way can be written in the invariant component form as,

$$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2} \frac{\partial I_1(\mathbf{S})}{\partial S_i^j} \left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta \right] \mathbf{g}_i \otimes \mathbf{g}^j$$

where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)]$. Consequently,

$$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2} \frac{\partial}{\partial S_i^j} \left[S_\alpha^\alpha S_\beta^\beta - S_\beta^\alpha S_\alpha^\beta \right] \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2} \left[\delta_\alpha^i \delta_j^\alpha S_\beta^\beta + \delta_\beta^i \delta_j^\beta S_\alpha^\alpha - \delta_\beta^i \delta_j^\alpha S_\alpha^\beta - \delta_\alpha^i \delta_j^\beta S_\beta^\alpha \right] \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= \frac{1}{2} \left[\delta_j^i S_\beta^\beta + \delta_j^i S_\alpha^\alpha - S_i^j - S_i^j \right] \mathbf{g}_i \otimes \mathbf{g}^j = \left(\delta_j^i S_\alpha^\alpha - S_i^j \right) \mathbf{g}_i \otimes \mathbf{g}^j$$

$$= I_1(\mathbf{S}) \mathbf{1} - \mathbf{S}$$

$$\det(\boldsymbol{S}) \equiv |\boldsymbol{S}| \equiv S = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}S_{ir}S_{js}S_{kt}$$

Differentiating wrt $S_{\alpha\beta}$, we obtain,

$$\frac{\partial S}{\partial S_{\alpha\beta}}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}}S_{js}S_{kt} + S_{ir}\frac{\partial S_{js}}{\partial S_{\alpha\beta}}S_{kt} + S_{ir}S_{js}\frac{\partial S_{kt}}{\partial S_{\alpha\beta}}\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{ijk}\epsilon^{rst}\left[\delta_i^\alpha\delta_r^\beta S_{js}S_{kt} + S_{ir}\delta_j^\alpha\delta_s^\beta S_{kt} + S_{ir}S_{js}\delta_k^\alpha\delta_t^\beta\right]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{3!}\epsilon^{\alpha jk}\epsilon^{\beta st}[S_{js}S_{kt} + S_{js}S_{kt} + S_{js}S_{kt}]\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

$$= \frac{1}{2!}\epsilon^{\alpha jk}\epsilon^{\beta st}S_{js}S_{kt}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta \equiv [S^{\mathrm{c}}]^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\boldsymbol{S}$

50. For a tensor field $\boldsymbol{\Xi}$, The volume integral in the region $\Omega \subset \mathcal{E}$, $\int_{\Omega} (\text{grad}\, \boldsymbol{\Xi})\, dv = \int_{\partial\Omega} \boldsymbol{\Xi} \otimes \boldsymbol{n}\, ds$ where $\boldsymbol{n}$ is the outward drawn normal to $\partial\Omega$ – the boundary of Ω. Show that for a vector field $\boldsymbol{f}$

$$\int_{\Omega} (\text{div}\, \boldsymbol{f})\, dv = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\, ds$$

Replace $\boldsymbol{\Xi}$ by **the** vector field $\boldsymbol{f}$ we have,

$$\int_{\Omega} (\text{grad}\, \boldsymbol{f})\, dv = \int_{\partial\Omega} \boldsymbol{f} \otimes \boldsymbol{n}\, ds$$

Taking the trace of both sides and noting that both trace and the integral are linear operations, therefore we have,

$$\begin{aligned} \int_{\Omega} (\text{div}\, \boldsymbol{f})\, dv &= \int_{\Omega} \text{tr}(\text{grad}\, \boldsymbol{f})\, dv \\ &= \int_{\partial\Omega} \text{tr}(\boldsymbol{f} \otimes \boldsymbol{n})\, ds \\ &= \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n}\, ds \end{aligned}$$

51. Show that for a scalar function Hence the divergence theorem becomes, $\int_{\Omega} (\text{grad } \phi) \, dv = \int_{\partial\Omega} \phi \boldsymbol{n} \, ds$

Recall that for a vector field, that for a vector field $\boldsymbol{f}$

$$\int_{\Omega} (\text{div } \boldsymbol{f}) \, dv = \int_{\partial\Omega} \boldsymbol{f} \cdot \boldsymbol{n} \, ds$$

if we write, $\mathbf{f} = \phi \boldsymbol{a}$ where $\boldsymbol{a}$ is an arbitrary constant vector, we have,

$$\int_{\Omega} (\text{div}[\phi \boldsymbol{a}]) \, dv = \int_{\partial\Omega} \phi \boldsymbol{a} \cdot \mathbf{n} \, ds = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n} \, ds$$

For the LHS, note that, $\text{div}[\phi \boldsymbol{a}] = \text{tr}(\text{grad}[\phi \boldsymbol{a}])$

$$\text{grad}[\phi \boldsymbol{a}] = (\phi a^i)_{,j} \, \mathbf{g}_i \otimes \mathbf{g}^j = a^i \phi_{,j} \, \mathbf{g}_i \otimes \mathbf{g}^j$$

The trace of which is,

$$a^i \phi_{,j} \, \mathbf{g}_i \cdot \mathbf{g}^j = a^i \phi_{,j} \, \delta_i^j = a^i \phi_{,i} = \boldsymbol{a} \cdot \text{grad } \phi$$

For the arbitrary constant vector $\boldsymbol{a}$, we therefore have that,

$$\int_{\Omega} (\text{div}[\phi \boldsymbol{a}]) \, dv = \boldsymbol{a} \cdot \int_{\Omega} \text{grad } \phi \; dv = \boldsymbol{a} \cdot \int_{\partial\Omega} \phi \mathbf{n} \, ds$$

$$\int_{\Omega} \operatorname{grad} \phi \ dv = \int_{\partial\Omega} \phi \mathbf{n} \, ds$$