Homework 2.1

1. The easiest proof is to observe that $\boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha$ is the identity tensor.

Consequently, $(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}_\alpha = \boldsymbol{S}(\boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha) = \boldsymbol{S}\mathbf{1} = \boldsymbol{S}.$

Alternatively, a longer route is to obtain the components of this result on the $\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ base. Recall that this is simply the inner product of this with the dual $\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ *of* the base:

$$[(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}_\alpha]:(\boldsymbol{g}^i \otimes \boldsymbol{g}^j) = \mathrm{tr}\{[(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}_\alpha](\boldsymbol{g}^j \otimes \boldsymbol{g}^i)\}$$
$$= \mathrm{tr}\{[(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}^i]\delta_\alpha^j\} = \mathrm{tr}[(\boldsymbol{S}\boldsymbol{g}^j) \otimes \boldsymbol{g}^i] = \boldsymbol{g}^i \cdot \boldsymbol{S}\boldsymbol{g}^j = S^{ij}$$

Consequently, we may express $(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}_\alpha$ in component form as,

$$(\boldsymbol{S}\boldsymbol{g}^\alpha) \otimes \boldsymbol{g}_\alpha = S^{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \boldsymbol{S}$$

2. The tensor $(\boldsymbol{u}\,\times) = -\epsilon_{lmn}u^n\boldsymbol{g}^l \otimes \boldsymbol{g}^m$

Similarly, $(\boldsymbol{v}\,\times) = -\epsilon^{\alpha\beta\gamma}v_\gamma\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta$ and $(\boldsymbol{w}\,\times) = -\epsilon^{ijk}w_k\boldsymbol{g}_i \otimes \boldsymbol{g}_j$.

Clearly,

$$\begin{aligned}(\boldsymbol{u}\,\times)(\boldsymbol{v}\,\times)(\boldsymbol{w}\,\times) &= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}u^n v_\gamma w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_\beta)(\boldsymbol{g}^l \otimes \boldsymbol{g}^m)(\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\\
&= -\epsilon^{\alpha\beta\gamma}\epsilon_{lmn}\epsilon^{ijk}u^n v_\gamma w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_j)\delta_\beta^l\delta_i^m\\
&= -\epsilon^{\alpha l\gamma}\epsilon_{lin}\epsilon^{ijk}u^n v_\gamma w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_j)\\
&= -\epsilon^{l\alpha\gamma}\epsilon_{lni}\epsilon^{ijk}u^n v_\gamma w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_j)\\
&= -(\delta_n^\alpha\delta_i^\gamma - \delta_i^\alpha\delta_n^\gamma)\epsilon^{ijk}u^n v_\gamma w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_j)\\
&= -\epsilon^{ijk}u^\alpha v_i w_k(\boldsymbol{g}_\alpha \otimes \boldsymbol{g}_j)\\
&\quad + \epsilon^{ijk}u^\gamma v_\gamma w_k(\boldsymbol{g}_i \otimes \boldsymbol{g}_j) = [\boldsymbol{u} \otimes (\boldsymbol{v} \times \boldsymbol{w}) - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}\,\times]\end{aligned}$$

3. First note that if the tensor $\mathbf{T}$ is invertible, for any vector $\mathbf{k}$,

$$\mathbf{Tk} = \mathbf{o}$$

automatically means that $\mathbf{k} = \mathbf{o}.$ The proof is easy and all we need to do is to contract with a tensor inverse of $\mathbf{T}$:

$$\mathbf{T}^{-1}\mathbf{Tk} = \mathbf{k} = \mathbf{o}$$

as required.

4 Next we can show that if the vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are independent vectors and $\mathbf{T}$ is invertible, then the vectors $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also independent. We provide a reduction ad absurdum proof of this assertion:

Imagine that our conditions are satisfied, and yet $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are not independent. This would mean that $\exists \alpha, \beta$ and γ not all of which are zero such that,

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \boldsymbol{o}.$$

This means that,

$$\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = \mathbf{Tk} = \mathbf{o}$$

where, in this case, $\mathbf{k} \equiv \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ must now necessarily vanish since $\mathbf{T}$ is invertible. Obviously, this shows that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are not independent. This contradicts our premise!

In a similar way, we may prove that the independence of the vectors $\mathbf{u}$ and $\mathbf{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$.

5. We show immediately that $(\mathbf{w}\times)(\mathbf{w}\times) = \mathbf{w}\otimes\mathbf{w} - \|\mathbf{w}\|^2\mathbf{1}$ by writing one covariantly and the other in contravariant components:

$$\begin{aligned}(\mathbf{w}\times)(\mathbf{w}\times) &= \left(\epsilon_{ijk}w^j\mathbf{g}^i\otimes\mathbf{g}^k\right)(\epsilon^{lmn}w_m\mathbf{g}_l\otimes\mathbf{g}_n) = \epsilon_{ijk}\epsilon^{lmn}w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n\delta_l^k \\ &= \epsilon_{ijl}\epsilon^{lmn}w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n = \left(\delta_i^m\delta_j^n - \delta_j^m\delta_i^n\right)w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n \\ &= (w^nw_i - w^mw_m\delta_i^n)\mathbf{g}^i\otimes\mathbf{g}_n \\ &= \mathbf{w}\otimes\mathbf{w} - \|\mathbf{w}\|^2\mathbf{1}\end{aligned}$$

We show that $\mathbf{w}\times(\mathbf{w}\otimes\mathbf{w}) = \mathbf{0}$

$$\begin{aligned}\mathbf{w}\times(\mathbf{w}\otimes\mathbf{w}) &= \left(\epsilon_{\alpha\beta\gamma}w^\beta\mathbf{g}^\alpha\otimes\mathbf{g}^\gamma\right)\left(w_iw_j\mathbf{g}^i\otimes\mathbf{g}^j\right) \\ &= \epsilon_{\alpha\beta\gamma}w^\beta w_iw_jg^{\gamma i}\mathbf{g}^\alpha\otimes\mathbf{g}^j \\ &= \epsilon_{\alpha\beta\gamma}w^\beta w^\gamma w_j\mathbf{g}^\alpha\otimes\mathbf{g}^j \\ &= \mathbf{0}\end{aligned}$$

On account of the symmetry and antisymmetry in β and γ

6. Multiplying the two tensors, we find,

$$
\begin{aligned}
&(\mathbf{1}+\mathbf{u}\otimes\mathbf{v})[\mathbf{1}-(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}\mathbf{u}\otimes\mathbf{v}] = \\
&\quad=\mathbf{1}-(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}\,(\mathbf{u}\otimes\mathbf{v})(\mathbf{u}\otimes\mathbf{v})-(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}(\mathbf{u}\otimes\mathbf{v})+(\mathbf{u}\otimes\mathbf{v})\\
&\quad=\mathbf{1}-(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}(\mathbf{u}\otimes\mathbf{v})\,\mathbf{u}\cdot\mathbf{v}-(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}(\mathbf{u}\otimes\mathbf{v})+(\mathbf{u}\otimes\mathbf{v})\\
&\quad=\mathbf{1}-(\mathbf{u}\otimes\mathbf{v})\{(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}\,\mathbf{u}\cdot\mathbf{v}+(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}\}+(\mathbf{u}\otimes\mathbf{v})\\
&\quad=\mathbf{1}-(\mathbf{u}\otimes\mathbf{v})(1+\mathbf{u}\cdot\ \mathbf{v})^{-1}(\mathbf{u}\cdot\mathbf{v}+1)+(\mathbf{u}\otimes\mathbf{v})\\
&\quad=\mathbf{1}
\end{aligned}
$$

And since this product gives the identity, the two tensors are inverses of each other.

7. As before, the product,

$$
\begin{aligned}
(\mathbf{1}+\mathbf{w}\times)&[\mathbf{1}-(1+\|\mathbf{w}\|^2)^{-1}(\|\mathbf{w}\|^2\mathbf{1}-\mathbf{w}\otimes\mathbf{w}+\mathbf{w}\times)]\\
&=\mathbf{1}+(1+\|\mathbf{w}\|^2)^{-1}(\|\mathbf{w}\|^2\mathbf{1}-\mathbf{w}\otimes\mathbf{w}+\mathbf{w}\times)+\ \mathbf{w}\times\mathbf{1}\\
&\qquad-(1+\|\mathbf{w}\|^2)^{-1}[\|\mathbf{w}\|^2\mathbf{w}\times\mathbf{1}-\mathbf{w}\times(\mathbf{w}\otimes\mathbf{w})+(\mathbf{w}\times)(\mathbf{w}\times)]
\end{aligned}
$$

We show immediately that $(\mathbf{w}\times)(\mathbf{w}\times)=\ \mathbf{w}\otimes\mathbf{w}-\|\mathbf{w}\|^2\mathbf{1}$ by writing one covariantly and the other in contravariant components:

$$
\begin{aligned}
(\mathbf{w}\times)(\mathbf{w}\times)&=\left(\epsilon_{ijk}w^j\mathbf{g}^i\otimes\mathbf{g}^k\right)(\epsilon^{lmn}w_m\mathbf{g}_l\otimes\mathbf{g}_n)=\epsilon_{ijk}\epsilon^{lmn}w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n\delta_l^k\\
&=\epsilon_{ijl}\epsilon^{lmn}w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n=\left(\delta_i^m\delta_j^n-\delta_j^m\delta_i^n\right)w^jw_m\mathbf{g}^i\otimes\mathbf{g}_n\\
&=(w^nw_i-w^mw_m\delta_i^n)\mathbf{g}^i\otimes\mathbf{g}_n\\
&=\ \mathbf{w}\otimes\mathbf{w}-\|\mathbf{w}\|^2\mathbf{1}
\end{aligned}
$$

We show that $\mathbf{w}\times(\mathbf{w}\otimes\mathbf{w})=\mathbf{0}$

$$
\begin{aligned}
\mathbf{w}\times(\mathbf{w}\otimes\mathbf{w})&=\left(\epsilon_{\alpha\beta\gamma}w^\beta\mathbf{g}^\alpha\otimes\mathbf{g}^\gamma\right)\left(w_iw_j\mathbf{g}^i\otimes\mathbf{g}^j\right)\\
&=\epsilon_{\alpha\beta\gamma}w^\beta w_iw_jg^{\gamma i}\mathbf{g}^\alpha\otimes\mathbf{g}^j
\end{aligned}
$$

$$=\epsilon_{\alpha\beta\gamma}w^\beta w^\gamma w_j\mathbf{g}^\alpha\otimes\mathbf{g}^j$$

$$= \mathbf{0}$$

On account of the symmetry and antisymmetry in β and γ

We also note that the above expression is the same as $(\mathbf{w} \times \mathbf{w}) \otimes \mathbf{w}$ which is obviously zero.

An easier proof comes from allowing the tensor $\mathbf{T} \equiv \mathbf{w} \times$. Observe that,

$$\begin{aligned} \mathbf{w} \times (\mathbf{w} \otimes \mathbf{w}) &= \mathbf{T}(\mathbf{w} \otimes \mathbf{w}) \\ &= \mathbf{T}\mathbf{w} \otimes \mathbf{w} \\ &= (\mathbf{w} \times \mathbf{w}) \otimes \mathbf{w} = \mathbf{0} \end{aligned}$$

We therefore have that,

$$\begin{aligned} &(\mathbf{1} + \mathbf{w} \times)[\mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}(\|\mathbf{w}\|^2\mathbf{1} - \mathbf{w} \otimes \mathbf{w} + \mathbf{w} \times)] \\ &\quad = \mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}(\|\mathbf{w}\|^2\mathbf{1} - \mathbf{w} \otimes \mathbf{w} + \mathbf{w} \times) + \mathbf{w} \times \mathbf{1} \\ &\qquad - (1 + \|\mathbf{w}\|^2)^{-1}[\|\mathbf{w}\|^2\mathbf{w} \times \mathbf{1} - \mathbf{w} \times (\mathbf{w} \otimes \mathbf{w}) + (\mathbf{w} \times)(\mathbf{w} \times)] \\ &\quad = \mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}[\|\mathbf{w}\|^2\mathbf{1} - \mathbf{w} \otimes \mathbf{w} + \mathbf{w} \times] + (\mathbf{w} \times) \\ &\qquad - (1 + \|\mathbf{w}\|^2)^{-1}[\|\mathbf{w}\|^2(\mathbf{w} \times) + \mathbf{w} \otimes \mathbf{w} - \|\mathbf{w}\|^2\mathbf{1}] \\ &\quad = \mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}[\|\mathbf{w}\|^2\mathbf{1} - \mathbf{w} \otimes \mathbf{w} + (\mathbf{w} \times) + \|\mathbf{w}\|^2(\mathbf{w} \times) + \mathbf{w} \\ &\qquad \otimes \mathbf{w} - \|\mathbf{w}\|^2\mathbf{1}] + (\mathbf{w} \times) \\ &\quad = \mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}[(\mathbf{w} \times) + \|\mathbf{w}\|^2(\mathbf{w} \times)] + (\mathbf{w} \times) \\ &\quad = \mathbf{1} - (1 + \|\mathbf{w}\|^2)^{-1}(\mathbf{w} \times)[1 + \|\mathbf{w}\|^2] + (\mathbf{w} \times) = \mathbf{1} \end{aligned}$$

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$$(\mathbf{TST}^{-1})_0 \equiv \operatorname{dev}(\mathbf{TST}^{-1}) = \mathbf{TST}^{-1} - \frac{1}{3}\operatorname{tr}(\mathbf{TST}^{-1})\mathbf{1}$$

But $\operatorname{tr}(\mathbf{TST}^{-1}) = \operatorname{tr}(\mathbf{T}^{-1}\mathbf{TS}) = \operatorname{tr}\mathbf{S}$. We may therefore write that,

$$\begin{aligned} &(\mathbf{TST}^{-1})_0 = \mathbf{TST}^{-1} - \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{1} = \mathbf{TST}^{-1} - \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{T1T}^{-1} \\ &= \mathbf{T}\left(\mathbf{S} - \left[\frac{(\operatorname{tr}\mathbf{S})}{3}\mathbf{1}\right]\right)\mathbf{T}^{-1} = \mathbf{TS_0T}^{-1} \end{aligned}$$