

# SSG 805 Mechanics of Continua

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# Purpose of the Course

- \* Available to beginning graduate students in Engineering
- \* Provides a background to several other courses such as Elasticity, Plasticity, Fluid Mechanics, Heat Transfer, Fracture Mechanics, Rheology, Dynamics, Acoustics, etc. These courses are taught in several of our departments. This present course may be viewed as an advanced introduction to the modern approach to these courses
- \* It is taught so that related graduate courses can build on this modern approach.

# What you will need

- \* The slides here are quite extensive. They are meant to assist the serious learner. They are **NO SUBSTITUTES** for the course text which must be read and followed concurrently.
- \* Preparation by reading ahead is **ABSOLUTELY** necessary to follow this course
- \* Assignments are given at the end of each class and they are due (No excuses) **exactly** five days later.
- \* Late submission carry zero grade.

# Scope of Instructional Material

## Course Schedule:

Slide Title	Slides	Weeks	Text	Read Pages
Vectors and Linear Spaces	80	2	Gurtin	1-8
Tensor Algebra	110	3	Gurtin	9-37
Tensor Calculus	119	3	Gurtin	39-57
Kinematics: Deformation & Motion	106	3	Gurtin	59-123
Theory of Stress & Heat Flux	43	1	Holz	109-129
Balance Laws	58	2	Gurtin	125-205

The read-ahead materials are from Gurtin except the part marked red. There please read Holzapfel. Home work assignments will be drawn from the range of pages in the respective books.

# Examination

The only remedy for late submission is that you fight for the rest of your grade in the final exam if your excuse is considered to be genuine. Ordinarily, the following will hold:

Evaluation	Obtainable
Quiz	10
Homework	50
Midterm	20
Exam	20
Total	100

# Course Texts

- \* This course was prepared with several textbooks and papers. They will be listed below. However, the main course text is: Gurtin ME, Fried E & Anand L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) 2010
- \* The course will cover pp1-240 of the book. You can view the course as a way to assist your reading and understanding of this book
- \* The specific pages to be read each week are given ahead of time. **It is a waste of time to come to class without the preparation of reading ahead.**
- \* This preparation requires going through the slides and the area in the course text that will be covered.

# Software

- \* The software for the Course is Mathematica 9 by Wolfram Research. Each student is entitled to a licensed copy. Find out from the LG Laboratory
- \* It your duty to learn to use it. Students will find some examples too laborious to execute by manual computation. It is a good idea to start learning Mathematica ahead of your need of it.
- \* For later courses, commercial FEA Simulations package such as ANSYS, COMSOL or NASTRAN will be needed. Student editions of some of these are available. We have COMSOL in the LG Laboratory

# Texts

- \* Gurtin, ME, Fried, E & Anand, L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) 2010
- \* Bertram, A, **Elasticity and Plasticity of Large Deformations**, Springer-Verlag Berlin Heidelberg, 2008
- \* Tadmor, E, Miller, R & Elliott, R, **Continuum Mechanics and Thermodynamics From Fundamental Concepts to Governing Equations**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) , 2012
- \* Nagahban, M, **The Mechanical and Thermodynamical Theory of Plasticity**, CRC Press, Taylor and Francis Group, June 2012
- \* Heinbockel, JH, **Introduction to Tensor Calculus and Continuum Mechanics**, Trafford, 2003



# Texts

- \* Bower, AF, **Applied Mechanics of Solids**, CRC Press, 2010
- \* Taber, LA, **Nonlinear Theory of Elasticity**, World Scientific, 2008
- \* Ogden, RW, **Nonlinear Elastic Deformations**, Dover Publications, Inc. NY, 1997
- \* Humphrey, JD, **Cardiovascular Solid Mechanics: Cells, Tissues and Organs**, Springer-Verlag, NY, 2002
- \* Holzapfel, GA, **Nonlinear Solid Mechanics**, Wiley NY, 2007
- \* McConnell, AJ, **Applications of Tensor Analysis**, Dover Publications, NY 1951
- \* Gibbs, JW “**A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces**,” Transactions of the Connecticut Academy of Arts and Sciences 2, Dec. 1873, pp. 382-404.

# Texts

- \* Romano, A, Lancellotta, R, & Marasco A, **Continuum Mechanics using Mathematica, Fundamentals, Applications and Scientific Computing**, Modeling and Simulation in Science and Technology, Birkhauser, Boston 2006
- \* Reddy, JN, **Principles of Continuum Mechanics**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) 2012
- \* Brannon, RM, **Functional and Structured Tensor Analysis for Engineers**, UNM BOOK DRAFT, 2006, pp 177-184.
- \* Atluri, SN, **Alternative Stress and Conjugate Strain Measures, and Mixed Variational Formulations Involving Rigid Rotations, for Computational Analysis of Finitely Deformed Solids with Application to Plates and Shells**, Computers and Structures, Vol. 18, No 1, 1984, pp 93-116

# Texts

- \* Wang, CC, **A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of my Papers on Representations for Isotropic Functions Part 1. Scalar-Valued Isotropic Functions**, Archives of Rational Mechanics, 1969 pp
- \* Dill, EH, **Continuum Mechanics, Elasticity, Plasticity, Viscoelasticity**, CRC Press, 2007
- \* Bonet J & Wood, RD, **Nonlinear Mechanics for Finite Element Analysis**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) 2008
- \* Wenger, J & Haddow, JB, **Introduction to Continuum Mechanics & Thermodynamics**, Cambridge University Press, [www.cambridge.org](http://www.cambridge.org) 2010

# Texts

- \* Li, S & Wang G, **Introduction to Micromechanics and Nanomechanics**, World Scientific, 2008
- \* Wolfram, S **The Mathematica Book**, 5<sup>th</sup> Edition  
Wolfram Media 2003
- \* Trott, M, **The Mathematica Guidebook, 4 volumes: Symbolics, Numerics, Graphics & Programming**, Springer 2000
- \* Sokolnikoff, IS, **Tensor Analysis, Theory and Applications to Geometry and Mechanics of Continua**, John Wiley, 1964

# Quiz of the day

1. If  $\forall \mathbf{v} \in \mathcal{V}, \mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$ , Show that  $\mathbf{a} = \mathbf{b}$
2. If  $\forall \mathbf{v} \in \mathcal{V}, \mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$ , Show that  $\mathbf{a} = \mathbf{b}$

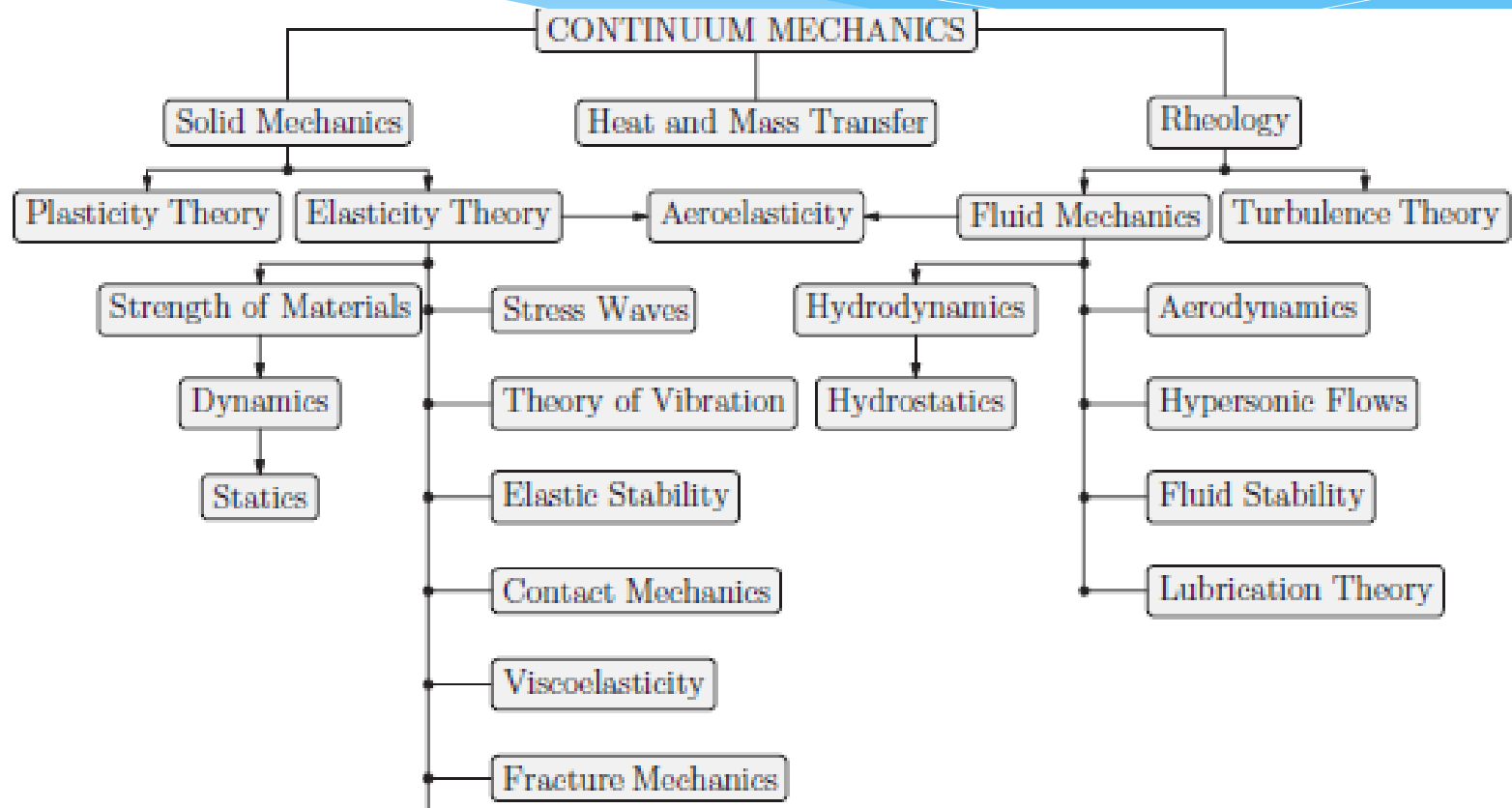
# Linear Spaces

Introduction

# Unified Theory

- \* Continuum Mechanics can be thought of as the grand unifying theory of engineering science.
- \* Many of the courses taught in an engineering curriculum are closely related and can be obtained as special cases of the general framework of continuum mechanics.
- \* This fact is easily lost on most undergraduate and even some graduate students.

# Continuum Mechanics





# Physical Quantities

Continuum Mechanics views matter as continuously distributed in space. The physical quantities we are interested in can be

- **Scalars** or **reals**, such as time, energy, power,
- **Vectors**, for example, position vectors, velocities, or forces,
- **Tensors**: deformation gradient, strain and stress measures.

# Physical Quantities

- \* Since we can also interpret scalars as zeroth-order tensors, and vectors as 1st-order tensors, all continuum mechanical quantities can generally be considered as tensors of different orders.
- \* It is therefore clear that a thorough understanding of Tensors is essential to continuum mechanics. This is NOT always an easy requirement;
- \* The notational representation of tensors is often inconsistent as different authors take the liberty to express themselves in several ways.

# Physical Quantities

- \* There are two major divisions common in the literature: Invariant or direct notation and the component notation.
- \* Each has its own advantages and shortcomings. It is possible for a reader that is versatile in one to be handicapped in reading literature written from the other viewpoint. In fact, it has been alleged that
- \* *“Continuum Mechanics may appear as a fortress surrounded by the walls of tensor notation”* It is our hope that the course helps every serious learner overcome these difficulties

# Real Numbers & Tuples

- \* The set of real numbers is denoted by  $\mathcal{R}$
- \* Let  $\mathcal{R}^n$  be the set of  $n$ -tuples so that when  $n = 2$ ,  $\mathcal{R}^2$  we have the set of pairs of real numbers. For example, such can be used for the  $x$  and  $y$  coordinates of points on a Cartesian coordinate system.

# Vector Space

A real vector space  $\mathcal{V}$  is a set of elements (called vectors) such that,

- 1. Addition operation** is **defined** and it is **commutative** and **associative** under  $\mathcal{V}$ : that is,  $\mathbf{u} + \mathbf{v} \in \mathcal{V}$ ,  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ ,  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ ,  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ .  
Furthermore,  $\mathcal{V}$  is **closed** under addition: That is, given that  $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ , then  $\mathbf{w} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ ,  $\Rightarrow \mathbf{w} \in \mathcal{V}$ .
- 2.  $\mathcal{V}$  contains a zero element  $\mathbf{o}$**  such that  $\mathbf{u} + \mathbf{o} = \mathbf{u} \forall \mathbf{u} \in \mathcal{V}$ . For every  $\mathbf{u} \in \mathcal{V}$ ,  $\exists -\mathbf{u}: \mathbf{u} + (-\mathbf{u}) = \mathbf{o}$ .
- 3. Multiplication by a scalar.** For  $\alpha, \beta \in \mathcal{R}$  and  $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ ,  $\alpha\mathbf{u} \in \mathcal{V}$ ,  $1\mathbf{u} = \mathbf{u}$ ,  $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$ ,  $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ ,  $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ .

# Euclidean Vector Space

- \* An **Inner-Product** (also called a **Euclidean Vector Space**)  $\mathcal{E}$  is a real vector space that defines the scalar product: for each pair  $\mathbf{u}, \mathbf{v} \in \mathcal{E}$ ,  $\exists l \in \mathcal{R}$  such that,  $l = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ . Further,  $\mathbf{u} \cdot \mathbf{u} \geq 0$ , the zero value occurring only when  $\mathbf{u} = \mathbf{0}$ . It is called “Euclidean” because the laws of Euclidean geometry hold in such a space.
- \* The inner product also called a dot product, is the mapping

$$" \cdot " : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$$

from the product space to the real space.

# Magnitude & Direction

- \* We were previously told that a vector is something that has magnitude and direction. We often represent such objects as directed lines. Do such objects conform to our present definition?
- \* To answer, we only need to see if the three conditions we previously stipulated are met:

# Dimensionality

From our definition of the Euclidean space, it is easy to see that,

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$$

such that,  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{R}$  and  $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n \in \mathcal{E}$ , is also a vector. The subset

$$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \subset \mathcal{E}$$

is said to be **linearly independent** or **free** if for any choice of the subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathcal{R}$  other than  $\{0, 0, \dots, 0\}$ ,  $\mathbf{v} \neq 0$ . If it is possible to find linearly independent vector systems of order  $n$  where  $n$  is a finite integer, but there is no free system of order  $n + 1$ , then the dimension of  $\mathcal{E}$  is  $n$ . In other words, the dimension of a space is the highest number of linearly independent members it can have.



# Basis

- \* Any linearly independent subset of  $\mathcal{E}$  is said to form a **basis** for  $\mathcal{E}$  in the sense that any other vector in  $\mathcal{E}$  can be expressed as a linear combination of members of that subset. In particular our familiar Cartesian vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  is a famous such subset in three dimensional Euclidean space.
- \* From the above definition, it is clear that a basis is not necessarily unique.

# Directed Line

- 1. Addition operation** for a directed line segment is defined by the parallelogram law for addition.
- 2.  $\mathcal{V}$  contains a zero element  $\mathbf{o}$**  in such a case is simply a point with zero length..
- 3. Multiplication by a scalar  $\alpha$ .** Has the meaning that  
 $0 < \alpha \leq 1 \rightarrow$  Line is shrunk along the same direction by  $\alpha$   
 $\alpha > 1 \rightarrow$  Elongation by  $\alpha$   
Negative value is same as the above with a change of direction.

# Other Vectors

- \* Now we have confirmed that our original notion of a vector is accommodated. It is not all that possess magnitude and direction that can be members of a vector space.
- \* A book has a size and a direction but because we cannot define addition, multiplication by a scalar as we have done for the directed line segment, it is not a vector.

# Other Vectors

**Complex Numbers.** The set  $\mathcal{C}$  of complex numbers is a real vector space or equivalently, a vector space over  $\mathcal{R}$ .

**2-D Coordinate Space.** Another real vector space is the set of all pairs of  $x_i \in \mathcal{R}$  forms a 2-dimensional vector space over  $\mathcal{R}$  is the two dimensional coordinate space you have been graphing on! It satisfies each of the requirements:

# Set of Pairs

$$\mathbf{x} = \{x_1, x_2\}, x_1, x_2 \in \mathcal{R}, \mathbf{y} = \{y_1, y_2\}, y_1, y_2 \in \mathcal{R}.$$

- \* **Addition** is easily defined as  $\mathbf{x} + \mathbf{y} = \{x_1 + y_1, x_2 + y_2\}$  clearly  $\mathbf{x} + \mathbf{y} \in \mathcal{R}^2$  since  $x_1 + y_1, x_2 + y_2 \in \mathcal{R}$ .

Addition operation creates other members for the vector space – Hence closure exists for the operation.

- \* **Multiplication by a scalar:**  $\alpha \mathbf{x} = \{\alpha x_1, \alpha x_2\}, \forall \alpha \in \mathcal{R}$ .

- \* **Zero element:**  $\mathbf{o} = \{0, 0\}$ . Additive Inverse:  $-\mathbf{x} = \{-x_1, -x_2\}, x_1, x_2 \in \mathcal{R}$

Type equation here. A standard basis for this :

$$\mathbf{e}_1 = \{1, 0\}, \mathbf{e}_2 = \{0, 1\}$$

Type equation here. Any other member can be expressed in terms of this basis.

# N-tuples

**$n$  –D Coordinate Space.** For any positive number  $n$ , we may create  $n$  –tuples such that,  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$  where  $x_1, x_2, \dots, x_n \in \mathcal{R}$  are members of  $\mathcal{R}^n$  – a real vector space over the  $\mathcal{R}$ .  $\mathbf{y} = \{y_1, y_2, \dots, y_n\}$ ,  $y_1, y_2, \dots, y_n \in \mathcal{R}$ .  $\mathbf{x} + \mathbf{y} = \{x_1 + y_1, x_2 + y_2, \dots, x_n + y_n\}$ ,  $\mathbf{x} + \mathbf{y} \in \mathcal{R}^n$  since  $x_1 + y_1, x_2 + y_2, \dots, x_n + y_n \in \mathcal{R}$

# N-tuples

- \* **Addition operation** creates other members for the vector space – Hence closure exists for the operation.
- \* **Multiplication by a scalar:**  $\alpha \mathbf{x} = \{\alpha x_1, \alpha x_2, \dots, \alpha x_n\}$ ,  $\forall \alpha \in R$ .
- \* **Zero element**  $\mathbf{o} = \{0, 0, \dots, 0\}$ . **Additive Inverse**  
–  $-\mathbf{x} = \{-x_1, -x_2, \dots, -x_n\}$ ,  $x_1, x_2, \dots, x_n \in \mathcal{R}$   
There is also a standard basis which is easily proved to be linearly independent:  $\mathbf{e}_1 = \{1, 0, \dots, 0\}$ ,  $\mathbf{e}_2 = \{0, 1, \dots, 0\}$ , ...,  $\mathbf{e}_n = \{0, 0, \dots, 0\}$

# Matrices

Let  $\mathcal{R}^{m \times n}$  denote the set of matrices with entries that are real numbers (same thing as saying members of the real space  $\mathcal{R}$ ). Then,  $\mathcal{R}^{m \times n}$  is a real vector space. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry by the same scalar). The zero vector here is just the zero matrix. The dimension of this space is  $mn$ . For example, in  $\mathcal{R}^{3 \times 3}$  we can choose basis in the form,

$$* \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



# The Polynomials

Forming polynomials with a single variable  $x$  to order  $n$  when  $n$  is a real number creates a vector space. It is left as an exercise to demonstrate that this satisfies all the three rules of what a vector space is.

# Euclidean Vector Space

- \* An **Inner-Product** (also called a **Euclidean Vector Space**)  $\mathcal{E}$  is a real vector space that defines the scalar product: for each pair  $\mathbf{u}, \mathbf{v} \in \mathcal{E}$ ,  $\exists l \in \mathcal{R}$  such that,  $l = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ . Further,  $\mathbf{u} \cdot \mathbf{u} \geq 0$ , the zero value occurring only when  $\mathbf{u} = \mathbf{0}$ . It is called “Euclidean” because the laws of Euclidean geometry hold in such a space.
- \* The inner product also called a dot product, is the mapping

$$" \cdot " : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$$

from the product space to the real space.

# Co-vectors

- \* A mapping from a vector space is also called a functional; a term that is more appropriate when we are looking at a function space.
- \* A linear functional  $\mathbf{v}^* : \mathcal{V} \rightarrow \mathcal{R}$  on the vector space  $\mathcal{V}$  is called a covector or a dual vector. For a finite dimensional vector space, the set of all covectors forms the dual space  $\mathcal{V}^*$  of  $\mathcal{V}$ . If  $\mathcal{V}$  is an Inner Product Space, then there is no distinction between the vector space and its dual.

# Magnitude & Direction Again

**Magnitude** The norm, length or magnitude of  $\mathbf{u}$ , denoted  $\|\mathbf{u}\|$  is defined as the positive square root of  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ . When  $\|\mathbf{u}\| = 1$ ,  $\mathbf{u}$  is said to be a unit vector. When  $\mathbf{u} \cdot \mathbf{v} = 0$ ,  $\mathbf{u}$  and  $\mathbf{v}$  are said to be orthogonal.

**Direction** Furthermore, for any two vectors  $\mathbf{u}$  and  $\mathbf{v}$ , the angle between them is defined as,

$$\cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} \right)$$

The scalar **distance**  $d$  between two vectors  $\mathbf{u}$  and  $\mathbf{v}$

$$d = \|\mathbf{u} - \mathbf{v}\|$$

# 3-D Euclidean Space

- \* A 3-D Euclidean space is a Normed space because the inner product induces a norm on every member.
- \* It is also a metric space because we can find distances and angles and therefore measure areas and volumes
- \* Furthermore, in this space, we can define the cross product, a mapping from the product space

$$" \times " : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V}$$

Which takes two vectors and produces a vector.

# Cross Product

Without any further ado, our definition of cross product is exactly the same as what you already know from elementary texts. We simply repeat a few of these for emphasis:

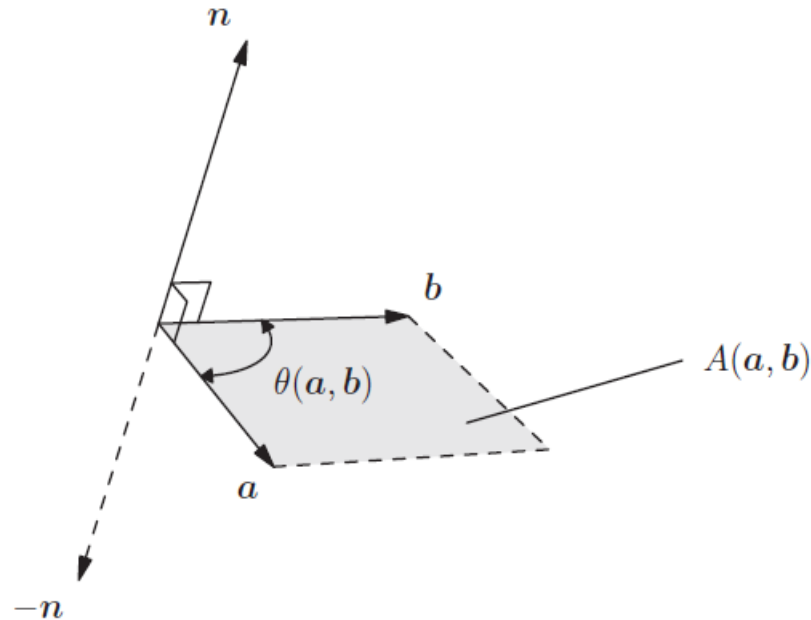
1. The magnitude

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \quad (0 \leq \theta \leq \pi)$$

of the cross product  $\mathbf{a} \times \mathbf{b}$  is the area  $A(\mathbf{a}, \mathbf{b})$  spanned by the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . This is the area of the parallelogram defined by these vectors. This area is non-zero only when the two vectors are linearly independent.

2.  $\theta$  is the angle between the two vectors.
3. The direction of  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$

# Cross Product



The area  $A(\mathbf{a}, \mathbf{b})$  spanned by the vectors  $\mathbf{a}$  and  $\mathbf{b}$ . The unit vector  $\mathbf{n}$  in the direction of the cross product can be obtained from the quotient,  $\frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|}$ .

# Cross Product

The cross product is bilinear and anti-commutative:

Given  $\alpha \in \mathcal{R}, \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{V}$ ,

$$(\alpha \mathbf{a} + \mathbf{b}) \times \mathbf{c} = \alpha(\mathbf{a} \times \mathbf{c}) + \mathbf{b} \times \mathbf{c}$$

$$\mathbf{a} \times (\alpha \mathbf{b} + \mathbf{c}) = \alpha(\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{c}$$

So that there is linearity in both arguments.

Furthermore,  $\forall \mathbf{a}, \mathbf{b} \in \mathcal{V}$

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$



# Tripple Products

The trilinear mapping,

$$[ \cdot, \cdot ] : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{R}$$

From the product set  $\mathcal{V} \times \mathcal{V} \times \mathcal{V}$  to real space is defined by:

$$[ \mathbf{u}, \mathbf{v}, \mathbf{w} ] \equiv \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

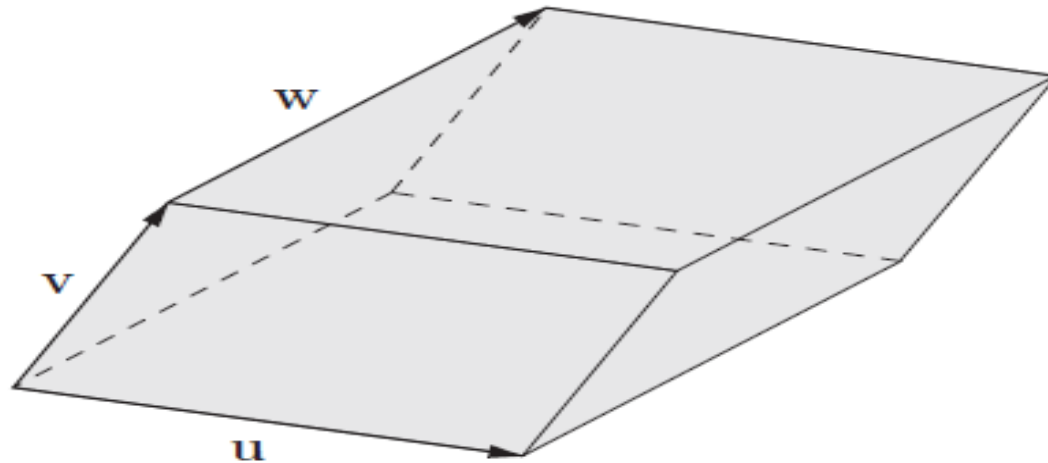
Has the following properties:

1.  $[ \mathbf{a}, \mathbf{b}, \mathbf{c} ] = [ \mathbf{b}, \mathbf{c}, \mathbf{a} ] = [ \mathbf{c}, \mathbf{a}, \mathbf{b} ] = - [ \mathbf{b}, \mathbf{a}, \mathbf{c} ] = - [ \mathbf{c}, \mathbf{b}, \mathbf{a} ] = - [ \mathbf{a}, \mathbf{c}, \mathbf{b} ]$

**HW: Prove this**

1. Vanishes when  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are linearly dependent.
2. It is the volume of the parallelepiped defined by  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$

# Tripple product



Parallelepiped defined by  $u$ ,  $v$  and  $w$

# Summation Convention

- \* We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we deal with higher ranked objects, such notational conveniences become even more important. We shall often deal with coordinate transformations.

# Summation Convention

- \* When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

# Summation Convention

- \* Consider transformation equations such as,

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

- \* We may write these equations using the summation symbols as:

$$y_1 = \sum_{j=1}^n a_{1j}x_j$$

$$y_2 = \sum_{j=1}^n a_{2j}x_j$$

$$y_3 = \sum_{j=1}^n a_{3j}x_j$$

# Summation Convention

- \* In each of these, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j$$

$$y_2 = a_{2j}x_j$$

$$y_3 = a_{3j}x_j$$

- \* Finally, we observe that  $y_1$ ,  $y_2$ , and  $y_3$  can be represented as we have been doing by  $y_i$ ,  $i = 1,2,3$  so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \quad i = 1,2,3$$

# Summation Convention

Please note here that while  $j$  in each equation is a dummy index,  $i$  is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of  $i$ , we may leave it out completely and write,

$$y_i = a_{ij}x_j$$

to represent compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

# Summation Convention

If  $a_{ij}$  represents the components of a  $3 \times 3$  matrix  $\mathbf{A}$ , we can show that,

$$a_{ij}a_{jk} = b_{ik}$$

Where  $\mathbf{B}$  is the product matrix  $\mathbf{AA}$ .

To show this, apply summation convention and see that,

$$\text{for } i = 1, k = 1, a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31} = b_{11}$$

$$\text{for } i = 1, k = 2, a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32} = b_{12}$$

$$\text{for } i = 1, k = 3, a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33} = b_{13}$$

$$\text{for } i = 2, k = 1, a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31} = b_{21}$$

$$\text{for } i = 2, k = 2, a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32} = b_{22}$$

$$\text{for } i = 2, k = 3, a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33} = b_{23}$$

$$\text{for } i = 3, k = 1, a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31} = b_{31}$$

$$\text{for } i = 3, k = 2, a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32} = b_{32}$$

$$\text{for } i = 3, k = 3, a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33} = b_{33}$$



# Summation Convention

The above can easily be verified in matrix notation as,

$$\begin{aligned}\mathbf{AA} &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B}\end{aligned}$$

In this same way, we could have also proved that,

$$a_{ij}a_{kj} = b_{ik}$$

- \* Where  $\mathbf{B}$  is the product matrix  $\mathbf{AA}^T$ . Note the arrangements could sometimes be counter intuitive.

# Vector Components

Suppose our basis vectors  $\mathbf{g}_i, i = 1,2,3$  are not only not unit in magnitude, but in addition are NOT orthogonal. The only assumption we are making is that  $\mathbf{g}_i \in \mathcal{V}, i = 1,2,3$  are linearly independent vectors.

With respect to this basis, we can express vectors  $\mathbf{v}, \mathbf{w} \in \mathcal{V}$  in terms of the basis as,

$$\mathbf{v} = v^i \mathbf{g}_i, \mathbf{w} = w^i \mathbf{g}_i$$

Where each  $v^i$  is called the contravariant component of  $\mathbf{v}$

# Vector Components

Clearly, addition and linearity of the vector space  $\Rightarrow$

$$\mathbf{v} + \mathbf{w} = (v^i + w^i)\mathbf{g}_i$$

Multiplication by scalar rule implies that if  $\alpha \in \mathcal{R}, \forall \mathbf{v} \in \mathcal{V}$ ,

$$\alpha \mathbf{v} = (\alpha v^i)\mathbf{g}_i$$

# Reciprocal Basis

For any basis vectors  $\mathbf{g}_i \in \mathcal{V}, i = 1,2,3$  there is a dual (or reciprocal) basis defined by the reciprocity relationship:

$$\mathbf{g}^i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}^i = \delta_j^i$$

Where  $\delta_j^i$  is the Kronecker delta

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

**Let the fact that the above equations are actually nine equations each sink. Consider the full meaning:**

# Kronecker Delta

Kronecker Delta:  $\delta_{ij}$ ,  $\delta^{ij}$  or  $\delta_j^i$  has the following properties:

$$\delta_{11} = 1, \delta_{12} = 0, \delta_{13} = 0$$

$$\delta_{21} = 0, \delta_{22} = 1, \delta_{23} = 0$$

$$\delta_{31} = 0, \delta_{32} = 0, \delta_{33} = 1$$

As is obvious, these are obtained by allowing the indices to attain all possible values in the range. The Kronecker delta is defined by the fact that when the indices explicit values are equal, it has the value of unity. Otherwise, it is zero. The above nine equations can be written more compactly as,

$$\delta_j^i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

# Covariant components

\* For any  $\forall \mathbf{v} \in \mathcal{V}$ ,

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i$$

Are two related representations in the reciprocal bases. Taking the inner product of the above equation with the basis vector  $\mathbf{g}_j$ , we have

$$\mathbf{v} \cdot \mathbf{g}_j = v^i \mathbf{g}_i \cdot \mathbf{g}_j = v_i \mathbf{g}^i \cdot \mathbf{g}_j$$

Which gives us the *covariant* component,

$$\mathbf{v} \cdot \mathbf{g}_j = v^i g_{ij} = v_i \delta_j^i = v_j$$

The last equality earns the Kronecker delta the epithet of “Substitution symbol”. **Work it out**

# Contravariant Components

In the same easy manner, we may evaluate the contravariant components of the same vector by taking the dot product of the same equation with the contravariant base vector  $\mathbf{g}^j$ :

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \mathbf{g}_i \cdot \mathbf{g}^j = v_i \mathbf{g}^i \cdot \mathbf{g}^j$$

So that,

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \delta_i^j = v_i g^{ij} = v^j$$

# Metric coefficients

The nine scalar quantities,  $g^{ij}$  as well as the nine related quantities  $g_{ij}$  play important roles in the coordinate system spanned by these arbitrary reciprocal set of basis vectors as we shall see.

They are called metric coefficients because they **metrize** the space defined by these bases.



# Levi Civita Symbol

\* The Levi-Civita Symbol:  $e_{ijk}$

\*  $e_{111} = 0, e_{112} = 0, e_{113} = 0, e_{121} = 0, e_{122} = 0, e_{123} = 1, e_{131} = 0, e_{132} = -1, e_{133} = 0$

$e_{211} = 0, e_{212} = 0, e_{213} = -1, e_{221} = 0, e_{222} = 0, e_{223} = 0, e_{231} = 1, e_{232} = 0, e_{233} = 0$

$e_{311} = 0, e_{312} = 1, e_{313} = 0, e_{321} = -1, e_{322} = 0, e_{323} = 0, e_{331} = 0, e_{332} = 0, e_{333} = 0$

# Levi Civita Symbol

- \* While the above equations might look arbitrary at first, a closer look shows there is a simple logic to it all. In fact, note that whenever the value of an index is repeated, the symbol has a value of zero. Furthermore, we can see that once the indices are an even arrangement (permutation) of 1,2, and 3, the symbols have the value of 1, When we have an odd arrangement, the value is -1. Again, we desire to avoid writing twenty seven equations to express this simple fact. Hence we use the index notation to define the Levi-Civita symbol as follows:

$$* e_{ijk} = \begin{cases} 1 & \text{if } i, j \text{ and } k \text{ are an even permutation of } 1,2 \text{ and } 3 \\ -1 & \text{if } i, j \text{ and } k \text{ are an odd permutation of } 1,2 \text{ and } 3 \\ 0 & \text{In all other cases} \end{cases}$$

# Cross Product of Basis Vectors

Given that  $g = \det g_{ij}$  of the covariant metric coefficients, It is not difficult to prove that

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} \equiv \sqrt{g} e_{ijk}$$

- \* This relationship immediately implies that,

$$\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k.$$

- \* The dual of the expression, the equivalent contravariant equivalent also follows from the fact that,

$$\mathbf{g}^1 \times \mathbf{g}^2 \cdot \mathbf{g}^3 = 1/\sqrt{g}$$

# Cross Product of Basis Vectors

This leads in a similar way to the expression,

$$\mathbf{g}^i \times \mathbf{g}^j \cdot \mathbf{g}^k = \frac{e^{ijk}}{\sqrt{g}} = \epsilon^{ijk}$$

It follows immediately from this that,

$$\mathbf{g}^i \times \mathbf{g}^j = \epsilon^{ijk} \mathbf{g}_k$$

# Exercises

- \* Given that,  $\mathbf{g}_1$ ,  $\mathbf{g}_2$  and  $\mathbf{g}_3$  are three linearly independent vectors and satisfy  $\mathbf{g}^i \cdot \mathbf{g}_j = \delta_j^i$ , show that  $\mathbf{g}^1 = \frac{1}{V} \mathbf{g}_2 \times \mathbf{g}_3$ ,  $\mathbf{g}^2 = \frac{1}{V} \mathbf{g}_3 \times \mathbf{g}_1$ , and  $\mathbf{g}^3 = \frac{1}{V} \mathbf{g}_1 \times \mathbf{g}_2$ , where  $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$

It is clear, for example, that  $\mathbf{g}^1$  is perpendicular to  $\mathbf{g}_2$  as well as to  $\mathbf{g}_3$  (an obvious fact because  $\mathbf{g}^1 \cdot \mathbf{g}_2 = 0$  and  $\mathbf{g}^1 \cdot \mathbf{g}_3 = 0$ ), we can say that the vector  $\mathbf{g}^1$  must necessarily lie on the cross product  $\mathbf{g}_2 \times \mathbf{g}_3$  of  $\mathbf{g}_2$  and  $\mathbf{g}_3$ . It is therefore correct to write,

$$\mathbf{g}^1 = \frac{1}{V} \mathbf{g}_2 \times \mathbf{g}_3$$

Where  $V^{-1}$  is a constant we will now determine. We can do this right away by taking the dot product of both sides of the equation (5) with  $\mathbf{g}_1$  we immediately obtain,

$$\mathbf{g}_1 \cdot \mathbf{g}^1 = V^{-1} \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = 1$$

So that,  $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$

the volume of the parallelepiped formed by the three vectors  $\mathbf{g}_1$ ,  $\mathbf{g}_2$ , and  $\mathbf{g}_3$  when their origins are made to coincide.

# Jacobian of Transformation

Suppose you have a function  $f(x, y, z)$  of variables  $x, y$  and  $z$ . Let us assume there are some variables  $r, \phi$  and  $Z$  such that, the original variables are themselves functions  $x = x(r, \phi, Z)$ ,  $y = y(r, \phi, Z)$ , and  $z = z(r, \phi, Z)$ . A simple example is the polar coordinate transformation:  $x = r \cos \phi$ ,  $y = r \sin \phi$ ,  $z = Z$ . We can always get a new function  $F(r, \phi, Z) = f(x, y, z)$  by doing a coordinate transformation using these equations. It is a well known fact that the transformation equations are invertible provided that the Jacobian of the transformation,

$$\frac{\partial(x, y, z)}{\partial(r, \phi, Z)} \equiv \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{vmatrix} \neq 0$$

We prefer to use indexed variables. Hence instead of  $x, y$  and  $z$ , we prefer  $x^i = x^i(u^1, u^2, u^3)$  where  $i = 1, 2, 3$  as you can obviously see that instead of  $r, \phi, Z$ , we are now talking about  $u^1, u^2, u^3$ . As before, we can say that the transformation will have an inverse provided the

Jacobian,  $\left| \frac{\partial x^k}{\partial u^i} \right|$  does not vanish. Therefore to say that

the transformation is invertible ensures that  $\left| \frac{\partial x^k}{\partial u^i} \right| \neq 0$ .

Recall that in Cartesian coordinates, the vector connecting an arbitrary point to the origin, also called a position vector can be written as

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x^i \mathbf{e}_i$$

Or, in order to emphasize the functional dependencies,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3) \mathbf{e}_i$$



# Basis Vectors

First notice that once you have a correct expression for your position vector for an arbitrary location, you can, by partial differentiation obtain an alternative representation for your basis vectors. It is elementary, for example to see clearly that,

$$\mathbf{i} = \frac{\partial \mathbf{r}}{\partial x}$$

And in general Cartesian coordinates, using index notation,

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}, \quad i = 1, 2, 3$$

# Natural Bases

We generalize the result now in terms of natural bases that arise in coordinate transformations from the Cartesian:

In the curvilinear system  $(u^1, u^2, u^3)$  obtained from the transformation  $x^i = x^i(u^1, u^2, u^3)$  from Cartesian

coordinates, let  $\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i}$  and let  $\mathbf{g}^j$  be the corresponding dual

basis. Show that  $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j}$ . If  $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3$

and  $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$ , show that  $v V = 1$ . Show also that

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g} e_{ijk}.$$

The position vector  $\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3) \mathbf{e}_i$  where  $\mathbf{e}_i, i = 1, 2, 3$  are unit vectors that are orthonormal in the Euclidean space.

# Natural Bases

Changing variables, we can write that,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i = \mathbf{r}(u^1, u^2, u^3)$$

So that we have new coordinates  $u^k, k = 1, 2, 3$ . In this new system, the differential of the position vector  $\mathbf{r}$  is,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i \equiv \mathbf{g}_i du^i$$

the above equation, as we shall soon show, defines the natural basis vectors in the new coordinate system. The vectors  $\mathbf{g}_1, \mathbf{g}_2$  and  $\mathbf{g}_3$  are not necessarily unit vectors but they form a basis of the new system provided,

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \neq 0$$

# Dual Bases

Clearly, the reciprocal basis vectors are

$$\mathbf{g}^1 = V^{-1} \mathbf{g}_2 \times \mathbf{g}_3$$

$$\mathbf{g}^2 = V^{-1} \mathbf{g}_3 \times \mathbf{g}_1$$

$$\mathbf{g}^3 = V^{-1} \mathbf{g}_1 \times \mathbf{g}_2$$

(dot the first with  $\mathbf{g}_1$  to see) Now we are given that

$v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$ . Using the above relations, we can write,

$$\begin{aligned} \mathbf{g}^2 \times \mathbf{g}^3 &= (V^{-1} \mathbf{g}_3 \times \mathbf{g}_1) \times (V^{-1} \mathbf{g}_1 \times \mathbf{g}_2) \\ &= V^{-2} [(\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2) \mathbf{g}_1 - (\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_1) \mathbf{g}_2] \\ &= V^{-2} (\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3) \mathbf{g}_1 = V^{-1} \mathbf{g}_1 \end{aligned}$$

We can now write,

$$v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3 = \mathbf{g}^1 \cdot V^{-1} \mathbf{g}_1 = V^{-1} \mathbf{g}^1 \cdot \mathbf{g}_1 = V^{-1}$$

Showing that,  $v V = 1$  as required.

We now show that if the Jacobian of the transformation

$\left| \frac{\partial x^k}{\partial u^i} \right|$  does not vanish, then the  $\mathbf{g}_i$  are independent:

Now,

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial x^k}{\partial u^i} \mathbf{e}_k$$

$$\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \\ \frac{\partial x^1}{\partial u^3} & \frac{\partial x^2}{\partial u^3} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \left| \frac{\partial x^k}{\partial u^i} \right| \neq 0.$$

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} = \left( \frac{\partial x^k}{\partial u^i} \mathbf{e}_k \right) \cdot \left( \frac{\partial x^l}{\partial u^j} \mathbf{e}_l \right) \\ &= \frac{\partial x^k}{\partial u^i} \frac{\partial x^l}{\partial u^j} \mathbf{e}_k \cdot \mathbf{e}_l = \frac{\partial x^k}{\partial u^i} \frac{\partial x^l}{\partial u^j} \delta_{kl} = \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \end{aligned}$$

Clearly, the determinant of  $g_{ij}$  (we shall prove later that the determinant of a product of matrices is the product of the determinants)

$$g \equiv |g_{ij}| = \left| \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \right| = \left| \frac{\partial x^k}{\partial u^i} \right|^2 = V^2$$

This means,  $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \left| \frac{\partial x^i}{\partial u^j} \right| = \sqrt{g}$ . We can therefore write,

$$\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = e_{123} \sqrt{g}$$

Swapping indices 2 and 3, we have,

$$\mathbf{g}_1 \cdot \mathbf{g}_3 \times \mathbf{g}_2 = -\sqrt{g} = e_{132} \sqrt{g} = \mathbf{g}_1 \times \mathbf{g}_3 \cdot \mathbf{g}_2$$

The second equality coming from the fact that swapping the cross with the dot changes nothing. Lastly, swapping 1 and 3 in the last equation shows that,

$\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2 = -(-\sqrt{g}) = e_{312} \sqrt{g}$ . These three expressions together imply that,

$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g} e_{ijk}$  as required.

This relationship immediately implies that,

$\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k$  as a dot product of this with  $\mathbf{g}_\alpha$  recovers the previous. The dual of the expression, the equivalent contravariant equivalent also follows from the fact that .

$$\mathbf{g}^1 \times \mathbf{g}^2 \cdot \mathbf{g}^3 = 1/\sqrt{g}$$

as it must be since we proved that the two volumes must be inverses. This leads in a similar way to the expression,

$$\mathbf{g}^i \times \mathbf{g}^j \cdot \mathbf{g}^k = \frac{e^{ijk}}{\sqrt{g}} = \epsilon^{ijk}$$

It follows immediately from this that,

$$\mathbf{g}^i \times \mathbf{g}^j = \epsilon^{ijk} \mathbf{g}_k$$

**Show that  $\mathbf{g}^j = g^{ij} \mathbf{g}_i = g^{ji} \mathbf{g}_i$  and establish the relation,  $g_{ij} g^{jk} = \delta_i^k$**

First expand  $\mathbf{g}^j$  in terms of the  $\mathbf{g}_i$ s:

$$\mathbf{g}^j = \alpha \mathbf{g}_1 + \beta \mathbf{g}_2 + \gamma \mathbf{g}_3$$

Dotting with  $\mathbf{g}^1 \Rightarrow \mathbf{g}^j \cdot \mathbf{g}^1 = \alpha \mathbf{g}_1 \cdot \mathbf{g}^1 + \beta \mathbf{g}_2 \cdot \mathbf{g}^1 + \gamma \mathbf{g}_3 \cdot \mathbf{g}^1 = g^{j1} = \alpha$ . In the same way we find that  $\beta = g^{j2}$  and  $\gamma = g^{j3}$  so that,

$$\mathbf{g}^j = g^{j1} \mathbf{g}_1 + g^{j2} \mathbf{g}_2 + g^{j3} \mathbf{g}_3 = g^{ji} \mathbf{g}_i.$$

Similarly,  $\mathbf{g}_i = g_{i\alpha} \mathbf{g}^\alpha$ .

Recall the reciprocity relationship:  $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k$ . Using the above, we can write

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}^k &= (g_{i\alpha} \mathbf{g}^\alpha) \cdot (g^{k\beta} \mathbf{g}_\beta) \\ &= g_{i\alpha} g^{k\beta} \mathbf{g}^\alpha \cdot \mathbf{g}_\beta \\ &= g_{i\alpha} g^{k\beta} \delta_\beta^\alpha = \delta_i^k \end{aligned}$$

which shows that

$$g_{i\alpha} g^{k\alpha} = g_{ij} g^{jk} = \delta_i^k$$

As required. This shows that the tensor  $g_{ij}$  and  $g^{ij}$  are inverses of each other.



**Show that the cross product of vectors  $\mathbf{a}$  and  $\mathbf{b}$  in general coordinates is  $a^i b^j \epsilon_{ijk} \mathbf{g}^k$  or  $\epsilon^{ijk} a_i b_j \mathbf{g}_k$  where  $a^i, b^j$  are the respective contravariant components and  $a_i, b_j$  the covariant.**

Express vectors  $\mathbf{a}$  and  $\mathbf{b}$  as contravariant components:  $\mathbf{a} = a^i \mathbf{g}_i$ , and  $\mathbf{b} = b^j \mathbf{g}_j$ . Using the above result, we can write that,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a^i \mathbf{g}_i) \times (b^j \mathbf{g}_j) \\ &= a^i b^j \mathbf{g}_i \times \mathbf{g}_j = a^i b^j \epsilon_{ijk} \mathbf{g}^k.\end{aligned}$$

Express vectors  $\mathbf{a}$  and  $\mathbf{b}$  as covariant components:  $\mathbf{a} = a_i \mathbf{g}^i$  and  $\mathbf{b} = b_j \mathbf{g}^j$ . Again, proceeding as before, we can write,

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_i \mathbf{g}^i) \times (b_j \mathbf{g}^j) \\ &= \epsilon^{ijk} a_i b_j \mathbf{g}_k\end{aligned}$$

Express vectors  $\mathbf{a}$  as a contravariant components:  $\mathbf{a} = a^i \mathbf{g}_i$  and  $\mathbf{b}$  as covariant components:  $\mathbf{b} = b_j \mathbf{g}^j$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a^i \mathbf{g}_i) \times (b_j \mathbf{g}^j) \\ &= a^i b_j (\mathbf{g}_i \times \mathbf{g}^j)\end{aligned}$$

# Answer to Quiz 1.2

- \* The answer to Quiz 1.1 is on page 5 of Gurtin. Once a piece of information has been made available to you in some form, it is your responsibility to take note.

In the Q1.2, we are given that  $\forall \mathbf{v} \in \mathcal{V}, \mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v}$ ,

Now take a dot product with  $\mathbf{a}$ , we have that,

$$\mathbf{a} \cdot \mathbf{b} \times \mathbf{v} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v} = 0 = \mathbf{o} \cdot \mathbf{v}$$

for all  $\mathbf{v}$  proving from Quiz 1.1 that  $\mathbf{a} \times \mathbf{b} = \mathbf{o}$ . This shows that  $\mathbf{a} \times \mathbf{b}$  are collinear. We can therefore write that  $\mathbf{b} = \alpha \mathbf{a}$

Hence,  $\mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} = \alpha \mathbf{a} \times \mathbf{v}$  where  $\alpha$  is a scalar. So that

$$(\mathbf{a} \times \mathbf{v})(1 - \alpha) = 0 \Rightarrow 1 = \alpha$$

showing that  $\mathbf{a} = \mathbf{b}$  as was required.

Given that,

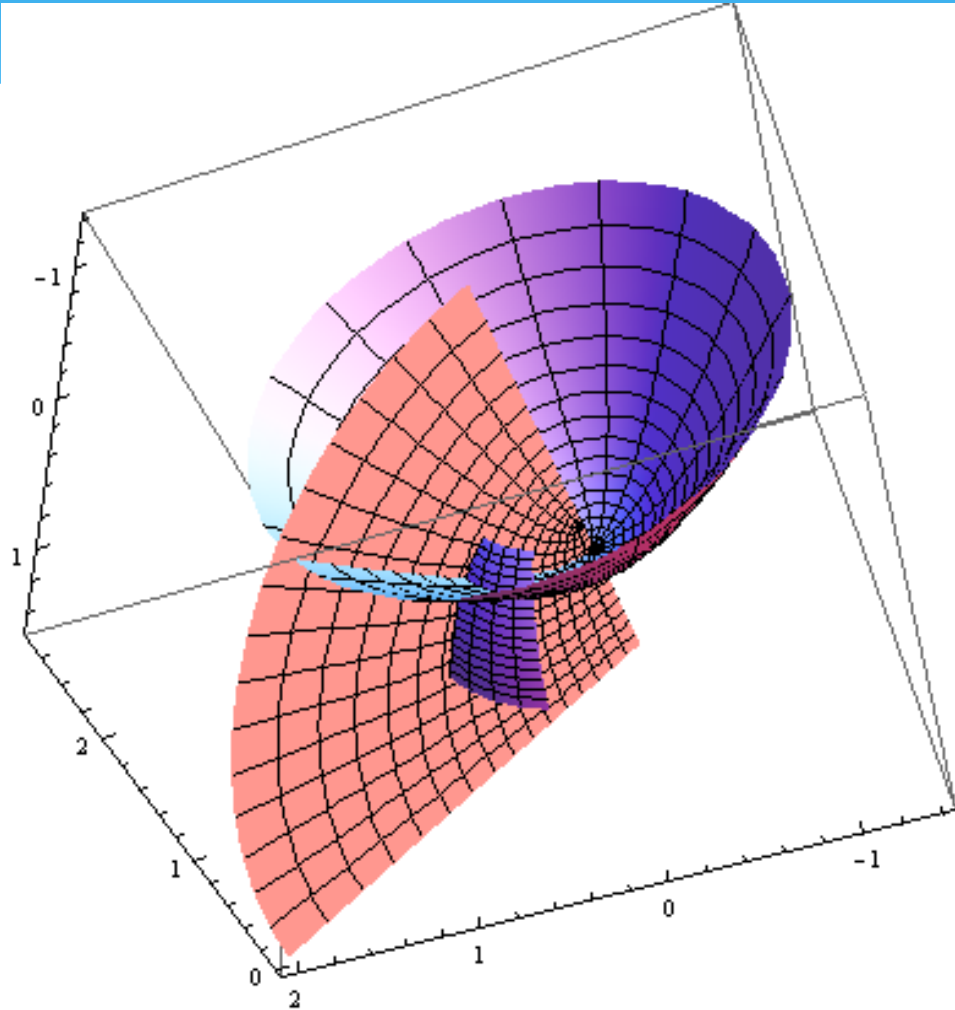
$$\delta_{ijk}^{rst} \equiv e^{rst} e_{ijk} = \begin{vmatrix} \delta_i^r & \delta_j^r & \delta_k^r \\ \delta_i^s & \delta_j^s & \delta_k^s \\ \delta_i^t & \delta_j^t & \delta_k^t \end{vmatrix}$$

Show that  $\delta_{ijk}^{rsk} = \delta_i^r \delta_j^s - \delta_i^s \delta_j^r$

Expanding the equation, we have:

$$\begin{aligned} e_{ijk} e^{rsk} &= \delta_{ijk}^{rsk} = \delta_i^k \begin{vmatrix} \delta_j^r & \delta_k^r \\ \delta_j^s & \delta_k^s \end{vmatrix} - \delta_j^k \begin{vmatrix} \delta_i^r & \delta_k^r \\ \delta_i^s & \delta_k^s \end{vmatrix} + 3 \begin{vmatrix} \delta_i^r & \delta_j^r \\ \delta_i^s & \delta_j^s \end{vmatrix} \\ &= \delta_i^k (\delta_j^r \delta_k^s - \delta_j^s \delta_k^r) - \delta_j^k (\delta_i^r \delta_k^s - \delta_i^s \delta_k^r) \\ &\quad + 3(\delta_i^r \delta_j^s - \delta_i^s \delta_j^r) = \delta_j^r \delta_i^s - \delta_j^s \delta_i^r - \delta_i^r \delta_j^s \\ &\quad + \delta_i^s \delta_j^r + 3(\delta_i^r \delta_j^s \\ &\quad - \delta_i^s \delta_j^r) = -2(\delta_i^r \delta_j^s - \delta_i^s \delta_j^r) + 3(\delta_i^r \delta_j^s - \delta_i^s \delta_j^r) \\ &= \delta_i^r \delta_j^s - \delta_i^s \delta_j^r \end{aligned}$$

# Coordinate Surfaces



**Show that**  $\delta_{ijk}^{rjk} = 2\delta_i^r$

Contracting one more index, we have:

$$e_{ijk}e^{rjk} = \delta_{ijk}^{rjk} = \delta_i^r \delta_j^j - \delta_i^j \delta_j^r = 3\delta_i^r - \delta_i^r = 2\delta_i^r$$

These results are useful in several situations.

**Show that  $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$**

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \epsilon^{ijk} u_i v_j \mathbf{g}_k \\ \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= u_\alpha \mathbf{g}^\alpha \cdot (\epsilon^{ijk} u_i v_j \mathbf{g}_k) \\ &= u_\alpha (\epsilon^{ijk} u_i v_j) \delta_k^\alpha = \epsilon^{ijk} u_i v_j u_k \\ &= 0\end{aligned}$$

On account of the symmetry and antisymmetry in  $i$  and  $k$ .

**Show that  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$**

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \epsilon^{ijk} u_i v_j \mathbf{g}_k \\ &= -\epsilon^{jik} u_i v_j \mathbf{g}_k = -\epsilon^{ijk} v_i u_j \mathbf{g}_k \\ &= -\mathbf{v} \times \mathbf{u}\end{aligned}$$

**Show that  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ .**

Let  $\mathbf{z} = \mathbf{v} \times \mathbf{w} = \epsilon^{ijk} v_i w_j \mathbf{g}_k$

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \mathbf{u} \times \mathbf{z} = \epsilon_{\alpha\beta\gamma} u^\alpha z^\beta \mathbf{g}^\gamma \\ &= \epsilon_{\alpha\beta\gamma} u^\alpha z^\beta \mathbf{g}^\gamma = \epsilon_{\alpha\beta\gamma} u^\alpha \epsilon^{ij\beta} v_i w_j \mathbf{g}^\gamma \\ &= \epsilon^{ij\beta} \epsilon_{\gamma\alpha\beta} u^\alpha v_i w_j \mathbf{g}^\gamma \\ &= (\delta_\gamma^i \delta_\alpha^j - \delta_\alpha^i \delta_\gamma^j) u^\alpha v_i w_j \mathbf{g}^\gamma \\ &= u^j v_\gamma w_j \mathbf{g}^\gamma - u^i v_i w_\gamma \mathbf{g}^\gamma \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\end{aligned}$$



# Exercises

1. In the transformation from the  $(x, y, z)$  system to the  $(r, \phi, Z)$  coordinate system, the position vector changed from  $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  to  $\mathbf{R} = r\mathbf{e}_r(\phi) + Z\mathbf{e}_z$ . Show by partial differentiation only, that the basis vectors in respective coordinates are  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  and  $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$  respectively,  $\mathbf{e}_\phi(\phi) = r \frac{\partial \mathbf{e}_r(\phi)}{\partial \phi}$ .
2. If the position vector in another system with coordinate variables  $(\rho, \phi, \theta)$  is  $\mathbf{R} = \rho\mathbf{e}_\rho(\phi, \theta)$ , use the same method to find the basis vectors in this system also.

3. In Problem 1 above, if the transformation from Cartesian to the other system is given explicitly as  $x(r, \phi, Z) = r \cos \phi$ ,  $y(r, \phi, Z) = r \sin \phi$  and  $z(r, \phi, Z) = Z$ , find explicit expression for the basis vectors  $\mathbf{g}_i, i = 1, 2, 3$ . Also find the reciprocal basis vectors  $\mathbf{g}^j, j = 1, 2, 3$ . [Hint:  $2\mathbf{g}^i = \epsilon^{ijk} \mathbf{g}_j \times \mathbf{g}_k$ ]
4. Are these basis vectors orthogonal? Are they normalized?
5. Find the dual bases for the Cartesian system.
6. Find the reciprocal bases for the spherical coordinate systems. Are they orthogonal? Are they normalized?

7. Find the metric tensor for each of the above systems.
8. Find the determinant of the metric tensor and confirm in these cases that  $\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g} e_{ijk}$  and that  $\mathbf{g}^i \times \mathbf{g}^j \cdot \mathbf{g}^k = \frac{e^{ijk}}{\sqrt{g}} = \epsilon^{ijk}$ .
9. Beginning with the equation,  $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k$ , take a contraction with  $\epsilon^{ija}$  and find the expression for  $\mathbf{g}^k$

10. Elliptical Cylindrical Coordinates is defined by the position vector,

$$\begin{aligned}\mathbf{R}(x, y, z) &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= \mathbf{R}(\xi, \eta, w) \\ &= \mathbf{i} \cos \eta \cosh \xi + \mathbf{j} \sin \eta \sinh \xi + \mathbf{k}w\end{aligned}$$

Use **Mathematica** and show that this system of coordinates is orthogonal. Hint:

$$\begin{aligned}R[\xi_, \eta_, w_] &= R(\xi, \eta, w) := i\text{Cos}[\eta]\text{Cosh}[\xi] + j\text{Sin}[\eta]\text{Sinh}[\xi] + kw \\ &= i \cos \eta \cosh \xi + j \sin \eta \sinh \xi + kw\end{aligned}$$

$$\begin{aligned}g &= D[R[\xi, \eta, w], \{\{\xi, \eta, w\}\}] = \frac{\partial R(\xi, \eta, w)}{\partial \{\xi, \eta, w\}} \\ &= \{\text{icos}(\eta) \sinh(\xi) + \text{jsin}(\eta) \cosh(\xi), \text{jcos}(\eta) \sinh(\xi) \\ &\quad - \text{isin}(\eta) \cosh(\xi), k\}\end{aligned}$$

**KroneckerProduct[g,g]//MatrixForm**

11. Compare the results of  
`KroneckerProduct[g,g]//MatrixForm`

In Q10 to

```
D[{Cos[η]Cosh[ξ], Sin[η]Sinh[ξ], kw}, {{ξ, η, w}}]  
Transpose[%]. %
```

Explain what the two commands are doing differently.

12. Repeat Q10, 11 for Spherical coordinates
13. Plot Coordinate surfaces for Elliptical Cylindrical coordinates using Mathematica.

14. Simplify the following by employing the substitution properties of the Kronecker delta (a)  $e_{ijk}\delta_{kn}$ , (b)  $e_{ijk}\delta_{is}\delta_{jm}$  (c)  $e_{ijk}\delta_{is}\delta_{jm}$  (d)  $a_{ij}\delta_{in}$  (e)  $\delta_{ij}\delta_{jn}$  (f)  $\delta_{ij}\delta_{jn}\delta_{ni}$

15. Show that the moments of inertia  $I_{ij}$  defined by

$$I_{11} = \iiint_V (y^2 + z^2) \rho(x, y, z) dx dy dz, \quad I_{21} = I_{12} = \iiint_V xy \rho(x, y, z) dx dy dz,$$

$$I_{22} = \iiint_V (z^2 + x^2) \rho(x, y, z) dx dy dz, \quad I_{32} = I_{23} = \iiint_V yz \rho(x, y, z) dx dy dz,$$

$$I_{31} = I_{13} = \iiint_V xy \rho(x, y, z) dx dy dz,$$

$$I_{33} = \iiint_V (x^2 + y^2) \rho(x, y, z) dx dy dz$$

can be represented in the index notation as

$$I_{ij} = \iiint_V (x^m x^m \delta_{ij} - x^i x^j) \rho(x^1, x^2, x^3) dx^1 dx^2 dx^3$$

where  $x = x^1, y = x^2, z = x^3$  and  $\rho(x^1, x^2, x^3)$  is scalar density of the material

16. Show that the Cylindrical Polar basis vectors,

$$\begin{aligned} \mathbf{e}_r(r, \phi, z) &= \cos \phi \mathbf{i} + \sin \phi \mathbf{j} \\ \mathbf{e}_\phi(r, \phi, z) &= -\sin \phi \mathbf{i} + \cos \phi \mathbf{j} \\ \mathbf{e}_z(r, \phi, z) &= \mathbf{k} \end{aligned}$$

Constitute an orthonormal system. [Hint: Show their magnitudes are unity and they are pairwise orthogonal].

17. Show that the contraction of a symmetric object with an antisymmetric object equals zero. For example given that  $a_{mn}$ ,  $m, n = 1, 2, 3$  is antisymmetric, Show that  $a_{mn}x^m x^n = \mathbf{0}$
18. Noting that  $e_{ijk}\sigma_{jk} = 0$  observe that  $e_{ijk}$  is perfectly antisymmetric. What does this tell about  $\sigma_{jk}$ ?
19. For any tensor  $\mathbf{A}$ , define  $(\text{Sym}(\mathbf{A}))_{ij} = \frac{1}{2}(\mathbf{A}_{ij} + \mathbf{A}_{ji})$ . Show that  $\text{Sym}(\mathbf{A}^T \mathbf{S} \mathbf{A}) = \mathbf{A}^T \text{Sym}(\mathbf{S}) \mathbf{A}$

20. Given that,  $\epsilon_{rst}\epsilon_{ijk} = \begin{vmatrix} g_{ri} & g_{rj} & g_{rk} \\ g_{si} & g_{sj} & g_{sk} \\ g_{ti} & g_{tj} & g_{tk} \end{vmatrix}$

Find  $\epsilon_{ist}\epsilon_{ijk}$  and  $\epsilon_{ijt}\epsilon_{ijk}$ .