Boundary Conditions

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Basic Equations

* From the foregoing, we can conclude that the basic equations of linear elasticity are:
  1. Geometry of Deformation
  2. Balance of Momentum: Cauchy's laws of motion or equilibrium.
  3. Isotrotropic, Homogeneous elastic law

Geometry of Deformation

* Simplifying the general Lagrangian or Eulerian deformation relations upon the assumption of small displacements we arrived at:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$$

Given any three continuously differentiable functions, $u_i, i = 1,2,3$, we may derive all the components of strain in any system of coordinates from this set of equations.

(How many are they)?

Balance of Momentum

Assuming equilibrium conditions, we note that linear momentum balance implies that,

$$\text{div}\,\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{o}$$

Where $\boldsymbol{b}$ is the body force per unit volume. The second Cauchy law maintains the symmetry of the spatial stress tensor $\boldsymbol{\sigma}$. In Cartesian systems, this simply implies,

$$\left(\frac{\partial \sigma^{11}}{\partial x^1} + \frac{\partial \sigma^{12}}{\partial x^2} + \frac{\partial \sigma^{13}}{\partial x^3}\right) + b^1 = 0$$

$$\left(\frac{\partial \sigma^{21}}{\partial x^1} + \frac{\partial \sigma^{22}}{\partial x^2} + \frac{\partial \sigma^{23}}{\partial x^3}\right) + b^1 = 0$$

$$\left(\frac{\partial \sigma^{21}}{\partial x^1} + \frac{\partial \sigma^{22}}{\partial x^2} + \frac{\partial \sigma^{23}}{\partial x^3}\right) + b^1 = 0$$

Indeterminacy

- By virtue of symmetry, we have six strain components, six stress components and three displacement components.
- Fifteen variables in all. It is obvious that the above 9 equations are insufficient.

The constitutive equations, as we have seen, gives the remainder of the puzzle:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\mathrm{tr}\boldsymbol{\varepsilon}$$

where μ and λ are called Lame's material constants.

Constitutive Equations

A more familiar expression is,

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}$$

With the inverse relation,

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

So that we now have 15 equations and 15 unknowns

Solution Strategies

- In principle, the equations can be solved. In practice, they are extremely difficult equations to solve. These have attracted the best mathematical minds in history.
- Our main task in linear elasticity is to understand the classical solutions that are well preserved in the literature and have created the built environment in the world today.
- On further reason to study these is that forays into new solutions of new problems depend on the language and terminology of the known solutions.

Boundary Conditions

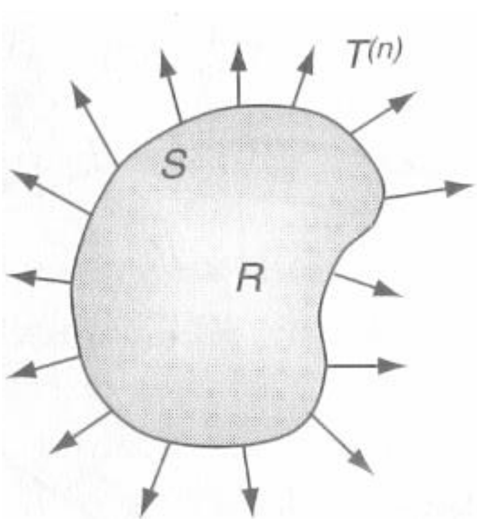
* The 15 equations we have are differential equations and algebraic. We know that we will need to specify boundary conditions to obtain useful solutions.
* Historically this gives rise to three different ways of formulating elasticity problems

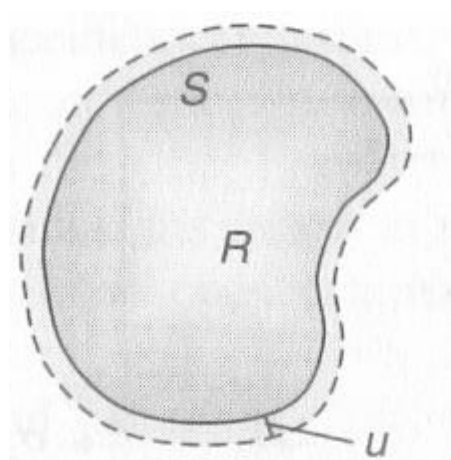
Three ways of specifying Boundary Conditions

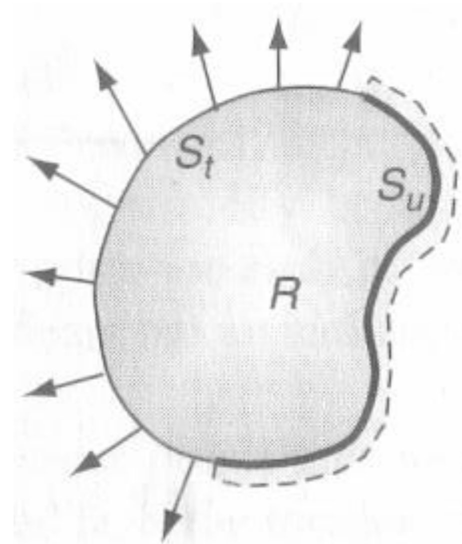
1. Specify the surface tractions on the boundary and subject the material to the governing equations so that these conditions are met.
2. Specify the displacements on the boundary and subject the material to the governing equations so that these conditions are met.
3. Mixture of 1 & 2

Boundary Conditions

The boundary conditions are mainly specifications of how the body is supported and/or loaded. This is mathematically equivalent to the specification of the *displacements* or *tractions*.







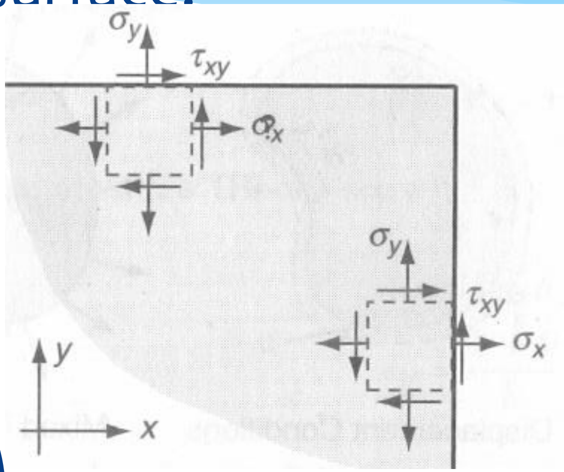
Traction Conditions

- Whether you solve the problems directly or use a commercial package, it will be often necessary to one specification or another.
- It is easy to confuse the traction condition with the specification of stresses. The traction is the stress vector: *Provided the stress vector* $\mathbf{T}^{(\mathbf{n})}$ *acting on a surface with outwardly drawn unit normal* $\mathbf{n}$ *is a continuous function of the coordinate variables, there exists a second-order tensor field* $\boldsymbol{\sigma}(\mathbf{x})$*, independent of* $\mathbf{n}$*, such that* $\mathbf{T}^{(\mathbf{n})}$ *is a linear function of* $\mathbf{n}$ *such that:*

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}$$

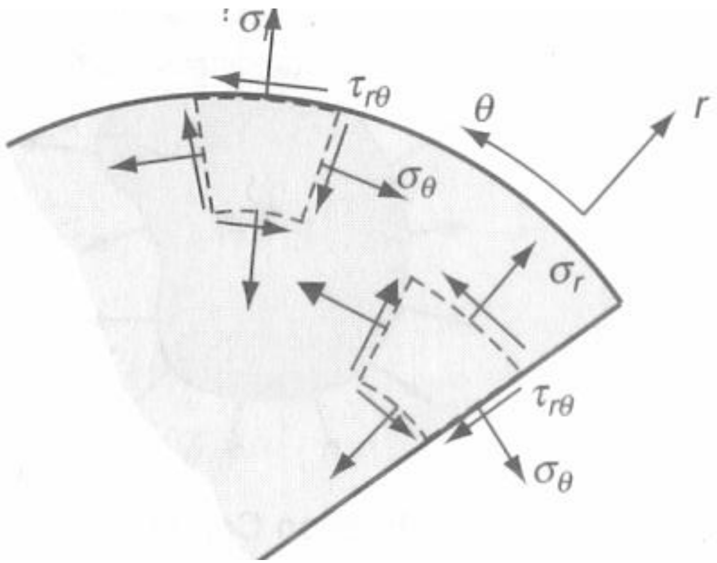
Boundary Examples

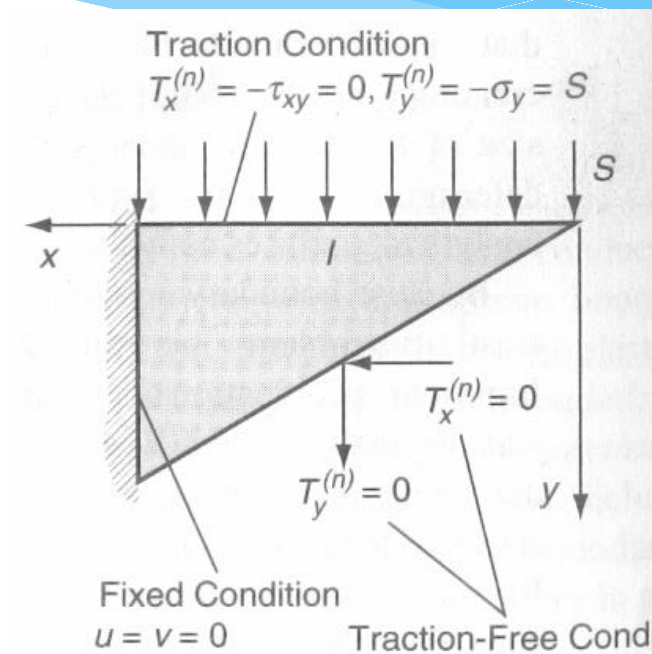
- It is very convenient, when possible, to align the boundary with the coordinate surface:
- On the horizontal plane here, we see that the unit normal is simply the y-axis. The traction vector normal



- $$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau_{xy} \\ \sigma_y \\ \tau_{yz} \end{pmatrix}$$
- From which we can see here that the traction coincides with the stresses as shown.

Boundary Conditions





Boundary Conditions

* $$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \tau_{xy} \\ \sigma_y \\ \tau_{yz} \end{pmatrix} = \begin{pmatrix} \tau_{xy} \\ -S \\ \tau_{yz} \end{pmatrix}$$
* $$\begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
* Or, $T_x^{(n)} = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = 0$ etc. for the free boundary conditions. Note that this follows directly from Cauchy's theorem.

Displacement Formulation

- If the boundary conditions are all given in terms of displacements, we might as well formulate the governing equations entirely in these terms. We employ the stress-strain relations, $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\text{tr}\boldsymbol{\varepsilon}$
- Or in covariant component form,

$$\sigma_{ij} = \mu\left(u_{i,j} + u_{j,i}\right) + \lambda\delta_{ij}u_{k,k}$$

So that the equilibrium equations become,

$$\sigma_{ij,j} + b_i = \mu\left(u_{i,jj} + u_{j,ij}\right) + \lambda\delta_{ij}u_{k,kj} + b_i = 0$$

Displacement form of Governing Equations

$$\mu\left(u_{i,jj} + u_{j,ij}\right) + \lambda u_{k,ki} + b_i = 0$$

Or if the functions are so differentiable continuously that the order of differentiation is immaterial,

$$(\lambda + \mu)u_{j,ij} + \mu u_{i,jj} + b_i = 0$$

$$(\lambda + \mu)\left(u_{j,j}\right)_{,i} + \mu u_{i,jj} + b_i = 0$$

Or, in terms of vector operators,

$$(\lambda + \mu)\mathrm{grad}(\mathrm{div}\,\mathbf{u}) + \mu\mathrm{div}(\mathrm{grad}\mathbf{u}) + \mathbf{b} = \mathbf{o}$$

Which now combines the balance and constitutive equations into the requirement of finding three functions.

Navier's Equations

* We have thus obtained the famous Navier's equations of Elastodynamics:

$$(\lambda + \mu)\text{grad}(\text{div}\,\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{b} = \varrho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

Which comes from simply inserting the constitutive conditions into the equations from the balance of linear momentum in an elastic body. Note that this is the base of the Navier Stokes Equations of Fluid Flow.

Navier's Equations: General Solutions

We begin to find the classical solutions to Navier's Equations. Before we proceed further, we must first consider the well-known Helmholtz Theorem for vector functions.

Helmholtz theorem suggests a decomposition of the unknown field $\mathbf{u} = \mathbf{u}^1 + \mathbf{u}^2$ where these components have some special properties.

Helmholz Theorem

Any continuous vector field can be represented as the sum of an irrotational field and a solenoidal field.

* It is necessary to define these terms:
  1. **IRROTATIONAL:** A vector point function (or a field) $\mathbf{u}^1(x, y, z)$is irrotational if a scalar field $\phi(x, y, z)$ can be found such that $\mathbf{u}^1 = \nabla\phi$
  2. **SOLENOIDAL:** A vector point function (or a field) $\mathbf{u}^2(x, y, z)$is solenoidal if a vector field $\boldsymbol{\varphi}(x, y, z)$ can be found such that $\mathbf{u}^2 = \nabla \times \boldsymbol{\varphi}$

Helmholtz Theorem

Our unknown displacement function can thus be written as,

$$\mathbf{u} = \nabla\phi + \nabla \times \boldsymbol{\varphi}$$

In formulating the general solution of Navier's Equations, we will use the special characteristics of the irrotational and solenoidal fields. Recall that it can easily be proved that the divergence of curl as well as the curl of grad vanishes. (This proof is a simple example of the vector identities we established last term).

Lame’s Potentials

- The first set of classical solutions we will look at are associated with the name of Lame. To obtain these, we make the observation that:
  - Helmholz theorem creates a redundancy in the sense that we are looking for three displacements from four functions, and
  - We shall assume that the body forces vanish.

Lame's Potentials

* Recall (Solution 19 of 50 problems) that,

$$\nabla^2 \mathbf{u} = \text{grad}\,(\text{div}\,\mathbf{u}) - \text{curl}\,(\text{curl}\,\mathbf{u})$$

We can therefore write Navier's equation with identically zero body forces as:

$$(\lambda + 2\mu)\text{grad}(\text{div}\,\mathbf{u}) - \mu\text{curl}\,(\text{curl}\,\mathbf{u}) = \varrho \frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \boldsymbol{\varphi})$$

Or in terms of the Helmholz decomposition,

$$(\lambda + 2\mu)\text{grad}(\text{div}(\nabla\phi)) - \mu\text{curl}\,(\text{curl}\nabla \times \boldsymbol{\varphi}) = \varrho \frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \boldsymbol{\varphi})$$

Lame's Solution

* Again applying solution 19, we can see clearly that,

$$\text{curl}(\text{curl}\ (\text{curl}\boldsymbol{\varphi})) = \text{grad}\ (\text{div}\ (\text{curl}\boldsymbol{\varphi})) - \nabla^2(\text{curl}\ \boldsymbol{\varphi})$$
$$= -\nabla^2(\text{curl}\ \boldsymbol{\varphi})$$

So that Navier's equations become,

$$(\lambda + 2\mu)\text{grad}(\text{div}(\nabla\phi)) + \mu\nabla^2(\text{curl}\ \boldsymbol{\varphi})$$
$$= \varrho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla\times\boldsymbol{\varphi})$$

Lame's Solution

* Interchanging the Laplacian and the curl, we have that for equilibrium, Navier's equations become,

$$(\lambda + 2\mu)\mathrm{grad}(\mathrm{div}(\nabla\phi)) + \mu\mathrm{curl}(\nabla^2\boldsymbol{\varphi}) = \boldsymbol{o}$$

Or,

$$(\lambda + 2\mu)\mathrm{grad}(\nabla^2\phi) + \mu\mathrm{curl}(\nabla^2\boldsymbol{\varphi}) = \boldsymbol{o}$$

Upon noting that the divergence of grad is the Laplacian.

These equations are satisfied identically if ϕ and $\boldsymbol{\varphi}$ are harmonic.

Lame's solution

* $\nabla^2 \phi = 0$ and $\boldsymbol{\varphi} = \boldsymbol{o}$ are special cases and these are called Lame's solutions. In the Literature, Lame's Solution is written with a constant factor such that,

$$2\mu\mathbf{u} = \text{grad}\ \phi$$

Dynamic Solutions

$$(\lambda + 2\mu)\mathrm{grad}(\nabla^2\phi) + \mu\mathrm{curl}(\nabla^2\boldsymbol{\varphi}) = \varrho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla\times\boldsymbol{\varphi})$$

Which simplifies to,

$$\mathrm{grad}\left[(\lambda + 2\mu)\nabla^2\phi - \varrho\frac{\partial^2\phi}{\partial t^2}\right] + \mathrm{curl}\left[\mu\nabla^2\boldsymbol{\varphi} - \varrho\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}\right]$$

This equations are satisfied once the wave equations in the brackets are satisfied:

Wave Equations

$$(\lambda + 2\mu)\nabla^2\phi = \varrho\frac{\partial^2\phi}{\partial t^2}$$

$$\mu\nabla^2\boldsymbol{\varphi} = \varrho\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2}$$

The two functions in the dynamic solution to the Navier equations are therefore wave functions as shown.

Cartesian Coordinates

* For the equilibrumsolutions, the Lame's functions can be used to derive the strains and consequently, the constitutive equations derive the stresses:

$$2\mu\mathbf{u} = \text{grad }\phi$$

In tensor form,

$$2\mu u_i = \phi_{,i}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) = \frac{1}{2\mu}\phi_{,ij}$$

Homework

* Write out these functions in Cartesian, Cylindrical & Spherical Systems