Linear Elasticity

Constitutive Laws for Linear Elasticity

Scope for Today

* Generalization of Uniaxial Hooke's Law.
* Tensor Transformation Laws. Covariant, Contravariant
* Material Symmetries: Anisotropy, Isotropy, Aelotropy, Orthotropy and Transverse Isotropy
* Effects of Symmetry on Material Constants. Isotropic Tensor Functions
* Elastic Constants and their implications

Hooke's Law

* Our first encounter with constitutive relationships start as always with Hooke's famous law.
* The theory that stress is proportional to strain is a naturally appealing theory. This comes first from our familiarity with the theory; and secondly by the fact that it is easily demonstrated in simple laboratory experiments.
* The proportionality of the stress to the strain is the linearity.
* The ability of the material to regain its original state after removal of any loading is elasticity. An elastic material has no memory. Its state is dependent only on its load; not on the rate at which the load is applied nor on the history of loading.

One Dimensional Strain

Hooke's Law, as we know states that the load is proportional to the displacement. Or, equivalently, stress is proportional to strain. The theory assumes we are talking of a single scalar stress and a single scalar strain.

Recall that the Lagrangian Strain Tensor, in its spectral form can be written as,

The Lagrangian Strain Tensor,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \mathbf{1}) = \frac{1}{2}\Sigma_{i=1}^{3}\left(\lambda_i^2 - 1\right)\mathbf{u}_i \otimes \mathbf{u}_i$$

Where the eigenvalues, $\lambda_i, i = 1,2,3$ are the principal stretches.

In one dimension this degenerates into a scalar value – the stretch, $\lambda = \frac{l}{l_0}$ that is, ratio of lengths in the spatial state and the reference state. Therefore the uniaxial Lagrangian strain,

$$E = \frac{1}{2}(\lambda^2 - 1) = \frac{1}{2}\left(\left(\frac{l}{l_0}\right)^2 - 1\right) =$$

Small Strain

$$E = \frac{1}{2}(\lambda^2 - 1) = \frac{1}{2}\left(\left(\frac{l}{l_0}\right)^2 - 1\right)$$

$$= \frac{l^2 - l_0^2}{2l_0^2} = \frac{(l - l_0)(l + l_0)}{2l_0^2}$$

$$\approx \frac{2l_0(l - l_0)}{2l_0^2} \approx \frac{l - l_0}{l_0} \approx \frac{\text{Increase in length}}{\text{Original length}}$$

* We see clearly that in one dimension, the Lagrangian strain tensor degenerates to our definition of strain only if we make one crucial assumption: Second order quantities are insignificant. What does this mean? When is original length indistinguishable from final length? Small strain?

Generalization to Multi-Axial States

* The above manipulations show clearly that the definition of strain we are used to is only valid in small strain.
* Defining strain as increase in length over original length is not the same as our definition of strain when the deformations are sufficiently large such that second order quantities become significant!
* We have also shown in previous coverage from Cauchy stress principle that we are dealing not with one scalar when we talk about stress but with nine: the components of the stress tensor!

Extension of Elementary Notions

- Of course, the stress is symmetrical (Cauchy's second law) and the small strain, as we have seen, is also symmetrical.
- Still we are dealing with six stress components and six strain components.
- It is clear that the simple formulation of Hookean constitutive relations are not very helpful in the general stress state.

Generalized Hooke's Law

* The generalized Hooke's Law states that for an isotropic, homogeneous material, the stress is a linear function of the strain. That is,

$$\sigma_{11} = \alpha_1\varepsilon_{11} + \alpha_2\varepsilon_{12} + \cdots + \alpha_9\varepsilon_{33}$$
$$\sigma_{12} = \beta_1\varepsilon_{11} + \beta_2\varepsilon_{12} + \cdots + \beta_9\varepsilon_{33}$$
$$= \cdots$$
$$= \cdots$$
$$\sigma_{33} = \gamma_1\varepsilon_{11} + \gamma_2\varepsilon_{12} + \cdots + \gamma_9\varepsilon_{33}$$

Where the 81 scalar quantities, α_1, α_2,..., γ_9 are the constants of proportionality.

Symmetry of Stress & Strain

We can write this equations as a tensor equation,

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$$

where $\mathbb{C}$ is the fourth-order tensor in,

$$\sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = C^{ij}_{kl}\varepsilon^{kl}\mathbf{g}_i \otimes \mathbf{g}_j$$

whose 81 components are the values

$$\alpha_1 = C^{11}_{11}, \alpha_2 = C^{11}_{12}, \dots, \gamma_9 = C^{33}_{33}$$

Note that the tensor equation is simply a convenient form of expressing the linearity in the context here. ***It has no other meaning.***

Leaving out the coordinate vectors, we have,

$$\sigma^{ij} = C^{ij}_{kl}\varepsilon^{kl}$$

Symmetry of Stress & Strain

- By Cauchy's first law of motion, we know that the stress tensor is symmetric. We also know that the small strain tensor is a symmetric quantity. On account of these, it is clear that we have a linear relationship among a set of six stress components and six strain components rather than nine each.
- This obviously reduces our constants from 81 to 36

Simplification

It is a bit more difficult to visualize a tensor of a higher order than 2. We therefore adopt the following simplification, taking advantage of the symmetry of the stress and strain tensors with a revised material tensor that is two dimensional. This time however, the tensors are in six dimensional space. Let,

$$E_1 = \varepsilon_{11}, E_2 = \varepsilon_{22}, E_3 = \varepsilon_{33}, E_4 = \varepsilon_{12}, E_5 = \varepsilon_{13}, E_6 = \varepsilon_{23}$$

$$\Sigma_1 = \sigma_{11}, \Sigma_2 = \sigma_{22}, \Sigma_3 = \sigma_{33}, \Sigma_4 = \sigma_{12}, \Sigma_5 = \sigma_{13}, \Sigma_6 = \sigma_{23}$$

Similarly, we introduce the material constants,

$c_1^m = C_{11}^{ij}, c_1^m = C_{12}^{ij}, c_{m3} = C_{13}^{ij}, c_{m4} = C_{12}^{ij}, c_{m5} = C_{13}^{ij}$ and $c_{m6} = C_{23}^{ij}$

Simplification

The tensor equation, $\sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = C^{ij}_{kl}\varepsilon^{kl}\mathbf{g}_i \otimes \mathbf{g}_j$ can now be written in a 2-D matrix form as,

$$\begin{pmatrix} \Sigma_1 \\ \Sigma_2 \\ \Sigma_3 \\ \Sigma_4 \\ \Sigma_5 \\ \Sigma_6 \end{pmatrix} = \begin{bmatrix} c_1^1 & c_1^2 & c_1^3 & c_1^4 & c_1^5 & c_1^6 \\ c_2^1 & c_2^2 & c_2^3 & c_2^4 & c_2^5 & c_2^6 \\ c_3^1 & c_3^2 & c_3^3 & c_3^4 & c_3^5 & c_3^6 \\ c_4^1 & c_4^2 & c_4^3 & c_4^4 & c_4^5 & c_4^6 \\ c_5^1 & c_5^2 & c_5^3 & c_5^4 & c_5^5 & c_5^6 \\ c_6^1 & c_6^2 & c_6^3 & c_6^4 & c_6^5 & c_6^6 \end{bmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{pmatrix}$$

The strains now appear as vectors in a six-D space as,

$$\Sigma_i = c_i^j E_j$$

Compliance

In a linear equation, the inverse can always be found if the coefficient matrix in non-singular:

The strains now appear as vectors in a six-D space as,

$$\Sigma_i = c_i^j E_j$$

Material symmetries

* An Isotropic material is symmetrical over all planes cutting at a point.
* A material symmetric about one plane is said to be aelotropic
* material (crystal), with one plane of symmetry, the 36 constants c_i^j reduce to 20 constants.
* Heinbockel as well as several authors

Component Transformation

In *n*-dimensional space V_n consider a curve C defined by the set of parametric equations

$$C: \quad X^i = X^i(t), \qquad i = 1, \dots, n$$

which is the parametric representation of a curve in *n*-dimensions. The tangent vector is defined as,

$$T^i = \frac{dX^i}{dt}, \qquad i = 1, \dots, n$$

Again consider the transformation, $X^i = X^i(x^1, x^2, \dots, x^n), i = 1,2, \dots, n$ which is simply a way of replacing the old variables X^i by new ones $x^i, i = 1, \dots, n$. As usual, if the Jacobian does not vanish, we may also have the inverse transformation, $x^i = x^i(X^1, X^2, \dots, X^n),\ i = 1,2, \dots, n$

Contravariant Vectors

In the new coordinates, the tangent vector can be defined τ^i which satisfies,

$$\tau^i = \frac{dx^i}{dt} = \frac{\partial x^i}{\partial X^j}\frac{dX^j}{dt} \equiv T^j \frac{\partial x^i}{\partial X^j}$$

This defines the transformation law of an absolute contravariant tensor of rank one. We can proceed to define a general tensor of rank one as follows:

Contravariant Vectors

Whenever n quantities α^i in a coordinate system $(x^1, x^2, \ldots, x^n)$ are related to n quantities a^j in a coordinate system $(X^1, X^2, \ldots, X^n)$ such that the Jacobian J is different from zero, then if the transformation law,

$$\alpha^i(x^1, x^2, \ldots, x^n) = J^W a^j(X^1, X^2, \ldots, X^n)\frac{\partial x^i}{\partial X^j}$$

is satisfied, these quantities are called the components of a relative contravariant tensor of rank or order one with weight W. Whenever $W = 0$ these quantities are called the components of an absolute contravariant tensor of rank or order one.

Tensor Definition

To prove that the above definition is consistent with the definition of a vector, consider n basis vectors $\mathbf{g}_i$ with their reciprocal bases $\mathbf{g}^i$. We are interested in how components will transform in the event of a change of basis to the set $\boldsymbol{\gamma}_i$ and its reciprocal basis $\boldsymbol{\gamma}^i$ such that the second-order tensor $\boldsymbol{A}$ is the transformation tensor from one natural basis to another so that,

$$\boldsymbol{\gamma}_i = \boldsymbol{A}\mathbf{g}_i$$

for the natural bases. And, for the reciprocal bases, the tensor $\boldsymbol{B}$ is given

$$\boldsymbol{\gamma}^i = \boldsymbol{B}\mathbf{g}^i$$

Tensor Definition

We can represent the transformation tensors in terms of the product basis themselves and write, $\boldsymbol{A} = A^{i}_{.j}\mathbf{g}_i \otimes \mathbf{g}^j$. We can see immediately that, in terms of the components of $\boldsymbol{A}$,

$$\boldsymbol{\gamma}_i = \boldsymbol{A}\mathbf{g}_i = A^{k}_{.j}\left(\mathbf{g}_k \otimes \mathbf{g}^j\right)\mathbf{g}_i = A^{k}_{.j}\mathbf{g}_k\delta^j_i = A^{k}_{.i}\mathbf{g}_k$$

Similarly, $\boldsymbol{\gamma}^i = \boldsymbol{B}\mathbf{g}^i = B^{.i}_{j}\mathbf{g}^j$.

From these expressions, we can easily conclude that $\boldsymbol{A} = \boldsymbol{\gamma}_i \otimes \mathbf{g}^i$ and $\boldsymbol{B} = \boldsymbol{\gamma}^i \otimes \mathbf{g}_i$ [Test and see they have the same effects on vectors; for example, $\boldsymbol{A}\mathbf{g}_i = \left(\boldsymbol{\gamma}_j \otimes \mathbf{g}^j\right)\mathbf{g}_i = \boldsymbol{\gamma}_j\delta^j_i = \boldsymbol{\gamma}_i$ as expected.] Furthermore, we can find expressions for the components of the transformation tensors. Contracting $\boldsymbol{\gamma}_i = A^{k}_{.i}\mathbf{g}_k$ with $\mathbf{g}^j$, we see that $A^{j}_{.i} = \boldsymbol{\gamma}_i \cdot \mathbf{g}^j$, and from its reciprocal equivalent, we see that $B^{.i}_{j} = \boldsymbol{\gamma}^i \cdot \mathbf{g}_j$.

Reciprocity

- If the two bases $\boldsymbol{\gamma}_i, and\ \mathbf{g}_i$ are the same, then the transformation is an identity transformation. In such a case, $\boldsymbol{A} = \boldsymbol{B} = \mathbf{1}$ the identity tensor. In the more general case, we recall that reciprocity implies,

$$\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}^j = (\boldsymbol{A}\mathbf{g}_i) \cdot (\boldsymbol{B}\mathbf{g}^j) = \mathbf{g}_i(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B})\mathbf{g}^i = \delta_i^j$$

which is true only if

$$\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \mathbf{1}$$

Or, $\boldsymbol{B} = \boldsymbol{A}^{-\mathrm{T}}$. Writing the transformation tensors in component form, we could also state the reciprocity relationship as,

$$\begin{aligned}\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}^j &= (\boldsymbol{A}\mathbf{g}_i) \cdot (\boldsymbol{B}\mathbf{g}^j) \\ &= (A_{.i}^{k}\mathbf{g}_k) \cdot (B_l^{\cdot j}\mathbf{g}^l) \\ &= A_{.i}^{k}\, B_l^{\cdot j}\mathbf{g}_k \cdot \mathbf{g}^l = A_{.i}^{k}\, B_l^{\cdot j}\delta_k^l = A_{.i}^{k}\, B_k^{\cdot j} = \delta_i^j\end{aligned}$$

Reciprocity Components

In transforming from $X^1, X^2, \dots, X^n$ to $x^1, x^2, \dots, x^n$, the unit differential element,

$$d\boldsymbol{r} = \mathbf{g}_j dX^j = \boldsymbol{\gamma}_i dx^i$$
$$= \mathbf{g}_j \frac{\partial X^j}{\partial x^i} dx^i$$

from which we see that $\boldsymbol{\gamma}_i = \frac{\partial X^j}{\partial x^i}\mathbf{g}_j = A^j_{.i}\mathbf{g}_j$. Clearly,

$A^j_{.i} = \frac{\partial X^j}{\partial x^i}$

since we know that $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B} = \mathbf{1}$. And, $\frac{\partial X^j}{\partial x^i}\frac{\partial x^i}{\partial X^k} = \delta^j_k$

Comparing to the reciprocity relationship (in terms of components), we see immediately that $B^i_k = \frac{\partial x^i}{\partial X^k}$

Reciprocity Components

In the two basis discussed here, we can write a vector quantity $\boldsymbol{a}$ as,

$$\boldsymbol{a} = a^i \mathbf{g}_i = \alpha^i \boldsymbol{\gamma}_i$$

so that a^i and α^i are the contravariant components of the vectors as we refer to the chosen (natural bases). We can now observe the coordinates transform from one natural base to another:

$$\alpha^i = \boldsymbol{a} \cdot \boldsymbol{\gamma}^i = a^j \mathbf{g}_j \cdot \boldsymbol{\gamma}^i = a^j B_j^{.i} = a^j \frac{\partial x^i}{\partial X^j}$$

Which is the expected relationship between absolute tensors of order or rank one.

Covariant Vectors

We introduce covariant tensor laws by way of another example: Consider an absolute scalar whose component in one coordinate system is, $f(X^1, X^2, \ldots, X^n)$ and that this is equal to the components $h(x^1, x^2, \ldots, x^n)$ after a coordinate transformation with a non-vanishing Jacobian as before,

$$h(x^1, x^2, \ldots, x^n) = f(X^1, X^2, \ldots, X^n)$$

Consider the gradient of this scalar in the new system of coordinates:

$$\frac{\partial h(x^1, x^2, \ldots, x^n)}{\partial x^i} = \frac{\partial f(X^1, X^2, \ldots, X^n)}{\partial x^i}$$

$$= \frac{\partial f(X^1, X^2, \ldots, X^n)}{\partial X^j} \frac{\partial X^j}{\partial x^i}$$

Covariant Vectors

If we define,

$$h_i(x^1, x^2, \dots, x^n) \equiv \frac{\partial h(x^1, x^2, \dots, x^n)}{\partial x^i}, \text{and}$$

$$f_j(X^1, X^2, \dots, X^n) \equiv \frac{\partial f(X^1, X^2, \dots, X^n)}{\partial X^j}$$

Then we may write, ignoring the obvious functional dependencies,

$$h_i = f_j \frac{\partial X^j}{\partial x^i}$$

This is the transformation law for an absolute covariant tensor of rank one. In general therefore, we may write that,

Definition

Whenever n *quantities* α_i *in a coordinate system* $(x^1, x^2, \dots, x^n)$ *are related to* n *quantities* a_j *in a coordinate system* $(X^1, X^2, \dots, X^n)$ *such that the Jacobian* J *is different from zero, then if the transformation law,*

$$\alpha_i(x^1, x^2, \dots, x^n) = J^W a_j(X^1, X^2, \dots, X^n) \frac{\partial X^j}{\partial x^i}$$

is satisfied, these quantities are called the components of a relative covariant tensor of rank or order one with weight W. Whenever $\mathrm{W} = 0$ these quantities are called the components of an absolute covariant tensor of rank or order one.

Again we show that a vector, expressed in its reciprocal basis, has the components of a rank one covariant tensor:

Expressing the same vector quantity $\boldsymbol{a}$ as,

$$\boldsymbol{a} = a_i \mathbf{g}^i = \alpha_i \boldsymbol{\gamma}^i$$

in terms of bases reciprocal to the earlier chosen (natural) bases. We seek the relationship between the components in one basis to another:

$$\alpha_i = \boldsymbol{a} \cdot \boldsymbol{\gamma}_i = a_j \mathbf{g}^j \cdot \boldsymbol{\gamma}_i$$

$$= a_j A^j_{.i} = a_j \frac{\partial X^j}{\partial x^i}$$

which is the transformation relation between the components of two absolute order one tensors from one coordinate basis to another. This proves that a vector of rank one, referred to basis reciprocal to the originally chosen (natural) system is an absolute covariant tensor of rank one.

Higher-Order Tensors

In the case of tensors of orders higher than one, we may define relative tensors of these orders that are contravariant, covariant or mixed. The rank two tensors have most of the attributes expected in higher ranked quantities. We will therefore define contravariant, covariant and mixed tensors of rank two as follows:

Contravariance Definition

Whenever n^2 *quantities* τ^{ij} *in a coordinate system* $(x^1, x^2, \dots, x^n)$ *are related to* n^2 *quantities* $T^{\alpha\beta}$ *in a coordinate system* $(X^1, X^2, \dots, X^n)$ *such that the Jacobian* J *of transformation is different from zero, then if the transformation law,*

$$\tau^{ij}(x^1, x^2, \dots, x^n) = J^W T^{\alpha\beta}(X^1, X^2, \dots, X^n) \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta}$$

is satisfied, these quantities are called the components of a relative contravariant tensor of rank or order two with weight W*. Whenever* $W = 0$ *these quantities are called the components of an absolute contravariant tensor of rank or order two. The tensor itself is the totality of all such components.*

A second-order tensor is, as we have seen, a mapping from one vector space unto itself. Under an assumed product basis, $\mathbf{g}_i \otimes \mathbf{g}_j$, we can write the tensor in component form so that,

$$\boldsymbol{T} = T^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$$

But this same tensor could be expressed along product bases that are related to the natural bases as before so that,

$$\boldsymbol{T} = T^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta = \tau^{ij}\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}_j$$

These components can be obtained from the dot product of the tensors themselves with the respective reciprocal product bases. In particular,

$$\begin{aligned}\tau^{ij} &= \boldsymbol{T}\left(\boldsymbol{\gamma}^i \otimes \boldsymbol{\gamma}^j\right) = \boldsymbol{\gamma}^i \cdot \left(\boldsymbol{T}\boldsymbol{\gamma}^j\right) \\ &= \boldsymbol{\gamma}^i \cdot \left[\left(T^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\boldsymbol{\gamma}^j\right] = T^{\alpha\beta}\left(\boldsymbol{\gamma}^i \cdot \mathbf{g}_\alpha\right)\left(\mathbf{g}_\beta \cdot \boldsymbol{\gamma}^j\right) \\ &= T^{\alpha\beta}B_{\alpha}^{.i}B_{\beta}^{.j} = T^{\alpha\beta}\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^j}{\partial X^\beta}\end{aligned}$$

which are the transformation equations of the components of a rank two contravariant tensor.

Covariance Definition

Whenever n^2 *quantities* g_{ij} *in a coordinate system* $(x^1, x^2, \dots, x^n)$ *are related to* n^2 *quantities* $f_{\alpha\beta}$ *in a coordinate system* $(X^1, X^2, \dots, X^n)$ *such that the Jacobian J is different from zero, then if the transformation law,*

$$\tau_{ij}(x^1, x^2, \dots, x^n) = J^W T_{\alpha\beta}(X^1, X^2, \dots, X^n) \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j}$$

is satisfied, these quantities are called the components of a relative covariant tensor of rank or order two with weight W*. Whenever* $W = 0$ *these quantities are called the components of an absolute covariant tensor of rank or order two.*

Expressing the same tensor above in terms of its reciprocal product bases we can write,

$$\boldsymbol{T} = T_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta} = \tau_{ij}\boldsymbol{\gamma}^{i} \otimes \boldsymbol{\gamma}^{j}$$

And, as before, we find the relationship between the two sets of components

$$\begin{aligned}\tau_{ij} &= \boldsymbol{T}(\boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}_j) = \boldsymbol{\gamma}_i \cdot (\boldsymbol{T}\boldsymbol{\gamma}_j) \\ &= \boldsymbol{\gamma}_i \cdot (T_{\alpha\beta}\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta})\boldsymbol{\gamma}_j = T_{\alpha\beta}\boldsymbol{\gamma}_i \cdot (\mathbf{g}^{\alpha} \otimes \mathbf{g}^{\beta})\boldsymbol{\gamma}_j \\ &= T_{\alpha\beta}(\boldsymbol{\gamma}_i \cdot \mathbf{g}^{\alpha})(\mathbf{g}^{\beta} \cdot \boldsymbol{\gamma}_j) = T_{\alpha\beta}A^{\alpha}_{.i}A^{\beta}_{.j} \\ &= T_{\alpha\beta}\frac{\partial X^{\alpha}}{\partial x^{i}}\frac{\partial X^{\beta}}{\partial x^{j}}\end{aligned}$$

so that this tensor, expressed in terms of its covariant components is an absolute covariant second-order tensor.

Aelotropic Materials

* A material with a plane of symmetry is said to be aelotropic.
* We will now show that for an aelotropic material, the number of necessary constants reduces from 36 to 24.
* In order to do this, we transform the constitutive relations about the plane of symmetry. We will then see that the constants must reduce in number accordingly for the material to possess this symmetry.
* We elect to do this transformation using contravariant tensors. For the same proof, using covariant components, see Heinbockel.

Aelotropic Transformation

From the above definition of contravariance, transforming from the Reference system to the Spatial,

$$\sigma^{ij}(\mathbf{x}) = \Sigma^{\alpha\beta}(\mathbf{X}) \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta}$$

Where we are looking at a stress tensor $\boldsymbol{\Sigma}$ in one system and its image $\boldsymbol{\sigma}$ in a transformed system. We are utilizing the known relationships between the transformed components

Aelotropy

For an aelotropic material, let the plane of symmetry be the $X^1 - X^2$ and imagine that we reverse the X^3 axis. (For a visual on planes of symmetry, se the Youtube video http://www.youtube.com/watch?v=nmr46D5Cy9E)

The transformation equation is,

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$$

Where we are simply restating the fact that

$$x^1 = X^1, x^2 = X^2$$

So that the $X^1 - X^2$ plane remains unchanged while

$$x^3 = -X^3$$

indicates that the transformed axes reverses the third coordinate.

Transformed Stress & Strain

The above transformation implies,

$$\frac{\partial x^1}{\partial X^1} = \frac{\partial x^2}{\partial X^2} = -\frac{\partial x^3}{\partial X^3} = 1$$

All other terms in the Jacobian of transformation vanish so that,

$$\frac{\partial x^i}{\partial X^j} = 0 \ \forall i \neq j$$

Under these circumstances, the following relationship holds between the reference stress/strain components and and the transformed (spatial)

Transformed Relations

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \\ \sigma^4 \\ \sigma^5 \\ \sigma^6 \end{pmatrix} = \begin{pmatrix} \Sigma^1 \\ \Sigma^2 \\ \Sigma^3 \\ \Sigma^4 \\ -\Sigma^5 \\ -\Sigma^6 \end{pmatrix}, \text{ and } \begin{pmatrix} \varepsilon^1 \\ \varepsilon^2 \\ \varepsilon^3 \\ \varepsilon^4 \\ \varepsilon^5 \\ \varepsilon^6 \end{pmatrix} = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \\ E^4 \\ -E^5 \\ -E^6 \end{pmatrix}$$

This constitutive relation, that holds in the reference system is a material relationship and must survive this transformation. We must therefore have,

$\Sigma^i = c^i_j E^j$ as well as $\sigma^i = c^i_j \varepsilon^j$ with the same material constants c^i_j. The only way this can happen is that

Transformed Relations

$$\begin{bmatrix} c_1^1 & c_1^2 & c_1^3 & c_1^4 & c_1^5 & c_1^6 \\ c_2^1 & c_2^2 & c_2^3 & c_2^4 & c_2^5 & c_2^6 \\ c_3^1 & c_3^2 & c_3^3 & c_3^4 & c_3^5 & c_3^6 \\ c_4^1 & c_4^2 & c_4^3 & c_4^4 & c_4^5 & c_4^6 \\ c_5^1 & c_5^2 & c_5^3 & c_5^4 & c_5^5 & c_5^6 \\ c_6^1 & c_6^2 & c_6^3 & c_6^4 & c_6^5 & c_6^6 \end{bmatrix} = \begin{bmatrix} c_1^1 & c_1^2 & c_1^3 & c_1^4 & 0 & 0 \\ c_2^1 & c_2^2 & c_2^3 & c_2^4 & 0 & 0 \\ c_3^1 & c_3^2 & c_3^3 & c_3^4 & 0 & 0 \\ c_4^1 & c_4^2 & c_4^3 & c_4^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5^5 & c_5^6 \\ 0 & 0 & 0 & 0 & c_6^5 & c_6^6 \end{bmatrix}$$

So that aelotropy reduces the material constants from 36 to 20

Orthotropy

Some engineering materials, including certain piezoelectric materials (e.g. Rochelle salt) and 2-ply fiber-reinforced composites, are **orthotropic**. By definition, an orthotropic material has at least 2 orthogonal planes of symmetry, where material properties are independent of direction within each plane. Such materials require 9 independent variables (i.e. elastic constants) in their constitutive matrices.

http://www.efunda.com/formulae/solid_mechanics/mat_mechanics/hooke_orthotropic.cfm

Orthotropic transformation

* Inclusion of another plane of symmetry and proceeding as we have just done, we can easily see that orthotropy reduces the number of constants to 12. See page 247 of Heinbockel to see how such a transformation is accomplished.

Isotropy

If a material is symmetric about all planes – that is in every direction, the constants reduce to three.

Such a material is said to be isotropic.

It can be further shown that the three constants in an isotropic material are not all independent. The number reduces to 2.

Hence, when a material is homogeneous and isotropic, only two constants are needed to describe its constitution. This is the generalized Hooke's law.

Representation Theorem

Thus far we have followed a usual engineering approach to the constitutive formulation for linear elasticity.

The mathematical approach is via the representation theorem for isotropic functions.

Assume now that the body is isotropic. Then $\boldsymbol{\sigma}$ is an isotropic function of $\boldsymbol{\varepsilon}$ and, by the Representation Theorem for Isotropic Linear Tensor Functions the only possible linear relationship is that

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{1}\mathrm{tr}\boldsymbol{\varepsilon}$$

where μ and λ are called Lame's material constants.

Elastic Modulus

A more familiar expression is,

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}$$

With the inverse relation,

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

On page 74 of Bower, you may find the relationships among different pairs of constants that are in use.

Elastic Modulus

Nd, if we include temperature-induced strains, expression becomes,

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\} - \frac{E\alpha\Delta T}{1-2\nu}\delta_{ij}$$

With the inverse relation,

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \alpha\Delta T\delta_{ij}$$

Physical Interpretation (Bower 75)

Young's modulus E is the slope of the stress-strain curve in uniaxial tension. It has dimensions of stress N/m^2 and is usually large for steel, $E = 210 \times 10^9 N/m^2$. You can think of E as a measure of the stiffness of the solid. The larger the value of E, the stiffer the solid. For a stable material, $E > 0$.

Poisson's ratio ν is the ratio of lateral to longitudinal strain in uniaxial tensile stress. It is dimensionless and typically ranges from $0.2 - 0.49$, and is around 0.3 for most metals. For a stable material, $-1 \leq \nu \leq 0.5$. It is a measure of the compressibility of the solid. If $\nu = 0.5$, the solid is incompressible its volume remains constant, no matter how it is deformed. If $\nu = 0$, then stretching a specimen causes no lateral contraction. Some bizarre materials have $\nu < 0$ if you stretch a round bar of such a material, the bar increases in diameter!!

Thermal expansion coefficient quantifies the change in volume of a material if it is heated in the absence of stress. It has dimensions of inverse degrees

Kelvin.$°K^{-1}$ and is usually very small. For steel, $\alpha \approx 6 - 10 \times 10^{-1} K^{-1}$
The **bulk modulus** quantifies the resistance of the solid to volume changes. It has a large value (usually bigger than *E*).
The **shear modulus** quantifies its resistance to volume preserving shear deformations. Its value is usually somewhat smaller than *E*.

Strain Energy Density for Isotropic Solids

Work Done

Note the following observations

* If you deform a block of material, you do work on it (or, in some cases, it may do work on you…)
* In an elastic material, the work done during loading is stored as recoverable strain energy in the solid. If you unload the material, the specimen does work on you, and when it reaches its initial configuration you come out even.
* The work done to deform a specimen depends only on the state of strain at the end of the test. It is independent of the history of loading.