SSG 806 Linear & Non Linear Elasticity

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Purpose of the Course

* Elasticity, in its linear form, is the simplest model applied to continuum systems
* Elasticity is the most successful constitutive model and is responsible for the built environment: Eiffel Tower, Skyscrapers, Bridges, etc
* Many, if not most of the problems arising from the linear theory can be solved in closed form or numerically using the modern commercial software such as FEA packages.

What you will need

* The slides here are quite extensive. They are meant to assist the serious learner. They are NO SUBSTITUTES for the course text which must be read and followed concurrently.
* Preparation by reading ahead is **ABSOLUTELY** necessary to follow this course
* Assignments are given at the end of each class and they are due (No excuses) **exactly** five days later.
* Late submission carry zero grade.

Scope of Instructional Material

Course Schedule:

Slide Title	Slides	Weeks	Text	Read Pages
Governing Equations and Global Charts	80	2	Heinfokel	1-8
Constitutive Models (Linear Elasticity)	110	3	Gurtin	9-37
Nonlinear Models	119	3	Gurtin	39-57
Solutions for Linear Elastic Solids	106	3	Gurtin	59-123
Solutions for Hyper-Elasticity	43	1	Holz	109-129

The read-ahead materials are from Gurtin except the part marked red. There please read Holzapfel. Home work assignments will be drawn from the range of pages in the respective books.

Examination

The only remedy for late submission is that you fight for the rest of your grade in the final exam if your excuse is considered to be genuine. Ordinarily, the following will hold:

Evaluation	Obtainable
Quiz	10
Homework	50
Midterm	20
Exam	20
Total	100

Course Texts

* This course was prepared with several textbooks and papers. They will be listed below. However, the main course text is: Gurtin ME, Fried E & Anand L, **The Mechanics and Thermodynamics of Continua,** Cambridge University Press, www.cambridge.org 2010
* The course will cover pp1-240 of the book. You can view the course as a way to assist your reading and understanding of this book
* The specific pages to be read each week are given ahead of time. It is a waste of time to come to class without the preparation of reading ahead.
* This preparation requires going through the slides and the area in the course text that will be covered.

Software

* The software for the Course is Mathematica 9 by Wolfram Research. Each student is entitled to a licensed copy. Find out from the LG Laboratory
* It your duty to learn to use it. Students will find some examples too laborious to execute by manual computation. It is a good idea to start learning Mathematica ahead of your need of it.
* For later courses, commercial FEA Simulations package such as ANSYS, COMSOL or NASTRAN will be needed. Student editions of some of these are available. We have COMSOL in the LG Laboratory

Texts

* Gurtin, ME, Fried, E & Anand, L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, www.cambridge.org 2010
* Bertram, A, **Elasticity and Plasticity of Large Deformations**, Springer-Verlag Berlin Heidelberg, 2008
* Tadmore, E, Miller, R & Elliott, R, **Continuum Mechanics and Thermodynamics From Fundamental Concepts to Governing Equations**, Cambridge University Press, www.cambridge.org , 2012
* Nagahban, M, **The Mechanical and Thermodynamical Theory of Plasticity**, CRC Press, Taylor and Francis Group, June 2012
* Heinbockel, JH, **Introduction to Tensor Calculus and Continuum Mechanics**, Trafford, 2003

Texts

* Bower, AF, **Applied Mechanics of Solids**, CRC Press, 2010
* Taber, LA, **Nonlinear Theory of Elasticity**, World Scientific, 2008
* Ogden, RW, **Nonlinear Elastic Deformations**, Dover Publications, Inc. NY, 1997
* Humphrey, JD, **Cadiovascular Solid Mechanics: Cells, Tissues and Organs**, Springer-Verlag, NY, 2002
* Holzapfel, GA, **Nonlinear Solid Mechanics**, Wiley NY, 2007
* McConnell, AJ, **Applications of Tensor Analysis**, Dover Publications, NY 1951
* Gibbs, JW "**A Method of Geometrical Representation of the Thermodynamic Properties of Substances by Means of Surfaces,**" Transactions of the Connecticut Academy of Arts and Sciences 2, Dec. 1873, pp. 382-404.

Texts

* Romano, A, Lancellotta, R, & Marasco A, **Continuum Mechanics using Mathematica, Fundamentals, Applications and Scientific Computing**, Modeling and Simulation in Science and Technology, Birkhauser, Boston 2006
* Reddy, JN, **Principles of Continuum Mechanics**, Cambridge University Press, www.cambridge.org 2012
* Brannon, RM, **Functional and Structured Tensor Analysis for Engineers**, UNM BOOK DRAFT, 2006, pp 177-184.
* Atluri, SN, **Alternative Stress and Conjugate Strain Measures, and Mixed Variational Formulations Involving Rigid Rotations, for Computational Analysis of Finitely Deformed Solids with Application to Plates and Shells**, Computers and Structures, Vol. 18, No 1, 1984, pp 93-116

Texts

- Wang, CC, **A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of my Papers on Representations for Isotropic Functions Part 1. Scalar-Valued Isotropic Functions**, Archives of Rational Mechanics, 1969 pp
- Dill, EH, **Continuum Mechanics, Elasticity, Plasticity, Viscoelasticity,** CRC Press, 2007
- Bonet J & Wood, RD, **Nonlinear Mechanics for Finite Element Analysis**, Cambridge University Press, www.cambridge.org 2008
- Wenger, J & Haddow, JB, **Introduction to Continuum Mechanics & Thermodynamics**, Cambridge University Press, www.cambridge.org 2010

Texts

- Li, S & Wang G, **Introduction to Micromechanics and Nanomechanics**, World Scientific, 2008
- Wolfram, S **The Mathematica Book,** 5th Edition Wolgram Media 2003
- Trott, M, **The Mathematica Guidebook, 4 volumes: Symbolics, Numerics, Graphics &Programming,** Springer 2000
- Sokolnikoff, IS, **Tensor Analysis, Theory and Applications to Geometry and Mechanics of Continua**, John Wiley, 1964

Linear Theory of Elasticity

Introduction

Reference & Spatial Placement

* In our previous work, we looked at a continuously distributed material occupying a region of space. When such a body is subjected to deformations or motion, we call the original placement in the Euclidean Point Space the "Reference State"
* At any point in time, we observe placements that are generally functions of time in the sense that a new placement can be physically observed at a particular time. These are, as we know, called Spatial Placements.

Small Deformations

- The spatial placement can be any arbitrary deformation of the Reference Placement. In general, the Deformation Gradient defines the change between the reference placement and the spatial.
- When deformations are such that the shape changes between the spatial and reference placements are not large, it is convenient to introduce another variable called the displacement.

Displacement Gradient

The vector quantity,

$$\mathbf{u}(\mathbf{X}, t) = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}$$

which is a function of the reference point $\mathbf{X}$ is called the displacement. It is the displacement of the material point $\mathbf{X}$ at time t.

Recall that the Green-St Venant strain tensor is defined as,

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})$$
$$= \frac{1}{2}\left(\mathbf{F}^{\mathbf{T}}\mathbf{F} - \mathbf{1}\right)$$

Now

$$\mathbf{F} = \text{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t)$$

and taking the reference gradient of the relation,

$$\mathbf{u}(\mathbf{X}, t) = \boldsymbol{\chi}(\mathbf{X}, t) - \mathbf{X}$$

Small Deformations

we can write,

$$\begin{aligned}\text{Grad}\,\mathbf{u}(\mathbf{X},t) &= \text{Grad}\,\boldsymbol{\chi}(\mathbf{X},t) - \text{Grad}\,\mathbf{X} \\ &= \text{Grad}\,\boldsymbol{\chi}(\mathbf{X},t) - \mathbf{1} \\ &= \mathbf{F} - \mathbf{1}\end{aligned}$$

Define $\mathbf{H}$ as the deformation gradient, we have that

$$\mathbf{H} \equiv \text{Grad}\,\mathbf{u}(\mathbf{X},t)$$

so that,

$$\mathbf{H} = \mathbf{F} - \mathbf{1}$$

Small Deformations

Clearly,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}\right)$$
$$= \frac{1}{2}\left(\left(\mathbf{H}^{\mathrm{T}} + \mathbf{1}\right)(\mathbf{H} + \mathbf{1}) - \mathbf{1}\right)$$
$$= \frac{1}{2}\left(\mathbf{H}^{\mathrm{T}} + \mathbf{H} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right)$$

In the limit that $\mathbf{H} \to \mathbf{0}$, we have small deformations. Here, we can write that

$$\mathbf{F} = \mathbf{1} + \mathbf{o}(\mathbf{H})$$

For arbitrary deformation, we can write, The Green-St Venant Strain Tensor in covariant components as,

$$E_{ij} = \frac{1}{2}\left[u_{j},_{i} + u_{i},_{j} + u^{k},_{j}\, u_{k},_{i}\right]$$

Small Deformations

so that products of $\mathbf{H}$ tend to zero faster than $\mathbf{H}$ and we can approximate the Green-St Venant strain by

$$\frac{1}{2}\left(\mathbf{H}^{\mathbf{T}} + \mathbf{H}\right)$$

This quantity, which coincides with the value of Eulerian strain, is called infinitesimal strain.

In this case, we see that the deformation gradient is close to the identity tensor. The deformed state of the material is almost coincident with the undeformed state up to a rigid body motion.

Strain-Displacement Relationships

Given the displacement vector in covariant components, we may write,

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{u} = u_i \mathbf{g}^i$$

$$\text{grad } \mathbf{u} = u_{i,j} \, \mathbf{g}^i \otimes \mathbf{g}^j$$

Where $u_{i,j}$ is the covariant derivative of the displacement. The infinitesimal strain is therefore,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\mathbf{g}^i \otimes \mathbf{g}^j$$

Cartesian

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} + \frac{\partial u_k}{\partial x^i}\frac{\partial u_k}{\partial x^j}\right)$$

or

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right]$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right]$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right]$$

$$E_{xy} = E_{yx} = \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x} + 1\right)\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\left(\frac{\partial u_y}{\partial y} + 1\right) + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right]$$

$$E_{xz} = E_{zx} = \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x} + 1\right)\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\left(\frac{\partial u_z}{\partial z} + 1\right)\right]$$

$$E_{yz} = E_{zy} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \left(\frac{\partial u_y}{\partial y} + 1\right)\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\left(\frac{\partial u_z}{\partial z} + 1\right)\right]$$

Small Strains

Removing all the product terms arising from the contraction of the displacement gradient, we have $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)$
or

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{xy} = E_{yx} = \frac{1}{2}\left[\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right]$$

$$\varepsilon_{xz} = E_{zx} = \frac{1}{2}\left[\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right]$$

$$\varepsilon_{yz} = E_{zy} = \frac{1}{2}\left[\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right]$$

Cylindrical Coordinates

* In Cylindrical Polar coordinates, as in other curvilinear systems the Christoffel symbols summation portion of the covariant derivatives add several additional terms as shown in the following expressions.
* $E_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_z}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_r}{\partial r}\right)^2\right]$

  $$E_{\theta\theta} = \frac{1}{2r^2}\left[{u_r}^2 + 2ru_r + {u_\theta}^2 + \left(\frac{\partial u_r}{\partial \theta}\right)^2 + \left(\frac{\partial u_z}{\partial \theta}\right)^2 + \left(\frac{\partial u_\theta}{\partial \theta}\right)^2 - 2u_\theta \frac{\partial u_r}{\partial \theta}\right.$$

In spherical coordinates, the components of Lagrangian Strain are,

$$E_{\rho\rho} = \frac{\partial u_\rho}{\partial \rho} + \frac{1}{2}\left[\left(\frac{\partial u_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial u_\rho}{\partial \rho}\right)^2 + \left(\frac{\partial u_\phi}{\partial \rho}\right)^2\right]$$

$$E_{\theta\theta} = \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho} + \frac{1}{2\rho^2}\left[u_\theta{}^2 - 2u_\theta \frac{\partial u_\rho}{\partial \theta} + 2u_\rho \frac{\partial u_\theta}{\partial \theta} + \left(\frac{\partial u_\theta}{\partial \theta}\right)^2 + u_\rho{}^2 + \left(\frac{\partial u_\rho}{\partial \theta}\right)^2 + \left(\frac{\partial u_\phi}{\partial \theta}\right)^2\right]$$

$$E_{\phi\phi} = \frac{u_\theta \cot(\theta)}{\rho} + \frac{u_\rho}{\rho} + \frac{1}{\rho \sin(\theta)}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{2\rho^2 \, sin^2\,\theta}\left[\left(u_\theta \cos(\theta) + u_\rho \sin(\theta) + \frac{\partial u_\phi}{\partial \phi}\right)^2 + \left(\frac{\partial u_\theta}{\partial \phi} - u_\phi \cos(\theta)\right)^2 + \left(\frac{\partial u_\rho}{\partial \phi} - u_\phi \sin(\theta)\right)^2\right]$$

$$E_{\rho\theta} = E_{\theta\rho} = \frac{1}{2\rho}\left[\rho \frac{\partial u_\theta}{\partial \rho} - u_\theta + \frac{\partial u_\rho}{\partial \theta} + \frac{\partial u_\theta}{\partial \rho}\left(\frac{\partial u_\theta}{\partial \theta} + u_\rho\right) + \frac{\partial u_\rho}{\partial \rho}\left(\frac{\partial u_\rho}{\partial \theta} - u_\theta\right) + \frac{\partial u_\phi}{\partial \theta}\frac{\partial u_\phi}{\partial \rho}\right]$$

$$E_{\rho\phi} = E_{\phi\rho} = \frac{1}{2\rho}\left[\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} + \rho \frac{\partial u_\phi}{\partial \rho} - u_\phi + \frac{\partial u_\phi}{\partial \rho}\left(u_\theta \cot(\theta) + u_\rho + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + \frac{\partial u_\theta}{\partial \rho}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} - u_\phi \cot(\theta)\right) + \frac{\partial u_\rho}{\partial \rho}\left(\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} - u_\phi\right)\right]$$

$$E_{\theta\phi} = E_{\phi\theta} = \frac{1}{2\rho}\left[\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} - u_\phi \cot(\theta) + \frac{\partial u_\phi}{\partial \theta} + \frac{\partial u_\phi}{\partial \rho}\left(u_\theta \cot(\theta) + u_\rho + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right) + \frac{\partial u_\theta}{\partial \rho}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} - u_\phi \cot(\theta)\right)\right.$$

Small strains in Curvilinear Coordinates

Cylindrical

$$
\begin{pmatrix}
\frac{\partial u_r}{\partial r} & \frac{-u_\theta + \frac{\partial u_r}{\partial \theta} + r\frac{\partial u_\theta}{\partial r}}{2r} & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\
\frac{-u_\theta + \frac{\partial u_r}{\partial \theta} + r\frac{\partial u_\theta}{\partial r}}{2r} & \frac{u_r + \frac{\partial u_\theta}{\partial \theta}}{r} & \frac{\frac{\partial u_z}{\partial \theta} + r\frac{\partial u_\theta}{\partial z}}{2r} \\
\frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & \frac{\frac{\partial u_z}{\partial \theta} + r\frac{\partial u_\theta}{\partial z}}{2r} & \frac{\partial u_z}{\partial z}
\end{pmatrix}
$$

Spherical

$$\begin{pmatrix} \dfrac{\partial u_\rho}{\partial \rho} & \dfrac{1}{2\rho}\dfrac{\partial u_\rho}{\partial \theta} - \dfrac{u_\theta}{2\rho} + \dfrac{1}{2}\dfrac{\partial u_\theta}{\partial \rho} & \dfrac{1}{2\rho\sin\theta}\dfrac{\partial u_\rho}{\partial \phi} - \dfrac{u_\phi}{2\rho} + \dfrac{1}{2}\dfrac{\partial u_\phi}{\partial \rho} \\ \dfrac{1}{2\rho}\dfrac{\partial u_\rho}{\partial \theta} - \dfrac{u_\theta}{2\rho} + \dfrac{1}{2}\dfrac{\partial u_\theta}{\partial \rho} & \dfrac{u_\rho}{\rho} + \dfrac{1}{\rho}\dfrac{\partial u_\theta}{\partial \theta} & \dfrac{1}{2\rho}\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} - u_\phi\cot\theta + \dfrac{\partial u_\phi}{\partial \theta}\right) \\ \dfrac{1}{2\rho\sin\theta}\dfrac{\partial u_\rho}{\partial \phi} - \dfrac{u_\phi}{2\rho} + \dfrac{1}{2}\dfrac{\partial u_\phi}{\partial \rho} & \dfrac{1}{2\rho}\left(\dfrac{1}{\sin\theta}\dfrac{\partial u_\theta}{\partial \phi} - u_\phi\cot\theta + \dfrac{\partial u_\phi}{\partial \theta}\right) & \dfrac{u_\rho}{\rho} + \dfrac{u_\theta\cot\theta}{\rho} + \dfrac{1}{\rho\sin\theta}\dfrac{\partial u_\phi}{\partial \phi} \end{pmatrix}$$